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Diffusion and Fractional Diffusion Based Models for Multiple Light Scattering and Image Analysis

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Diffusion and Fractional Diffusion based Models for Multiple Light Scattering and Image Analysis

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***Abstract*— This paper considers a fractional light diffusion model as an approach to characterizing the case when intermediate scattering processes are present, i.e. the scattering regime is neither strong nor weak. In order to introduce the basis for this approach, we revisit the elements of formal scattering theory and the classical diffusion problem in terms of solutions to the inhomogeneous wave and diffusion equations respectively. We then address the significance of these equations in terms of a random walk model for multiple scattering. This leads to the proposition of a fractional diffusion equation for modelling intermediate strength scattering that is based on a generalization of the diffusion equation to fractional form. It is shown how, by induction, the fractional diffusion equation can be justified in terms of the generalization of a random walk model to fractional form as characterized by the Hurst exponent. Image processing and analysis methods are proposed that are based on diffusion and fractional diffusion models and some application examples given.**

***Index Terms*— Multiple Scattering, Optical Diffusion, Fractional Optical Diffusion, Random Walk Processes, Intermediate Strength Scattering, Image Processing and Analysis**

I. Introduction

THE use of formal scattering methods for modelling the interaction of light with an inhomogeneous medium together with associated inverse scattering models is well known (e.g. [1]). In applications associated with the processing and analysis of an image, the aim is to develop a model that maps the object plane to the image plane. If the scattering is 'weak' (i.e. based on single scattering events) and the scattered wavefield is measured in the far field, then the map is determined by the Fourier transform which, for a clear aperture, yields the fundamental imaging equation [2]

$$I(x,y) = p(x,y) \otimes_2 f(x,y) + n(x,y)$$

for an image I where p is the point spread function (a characteristic of the imaging system), f is the object function and $\otimes_2$ denotes the two-dimensional convolution operation, i.e.

$$p(x,y) \otimes_2 f(x,y) = \int\int p(x-x',y-y')f(x',y')dx'dy'$$

The noise n is taken to be a stochastic function which at best, can be characterized by a probability density function $\Pr[n(x,y)]$ that conforms to a physically significant statistical model. The function n is taken to include a range of perturbations to the scattered field that is recorded in the image plane. Within the context of the weak scattering approximation used to derive the fundamental imaging equation, this includes multiple scattering.

The object function $f(x,y)$ is related to a three-dimensional scattering function $\gamma(\mathbf{r})$ where $\mathbf{r}$ is the three-dimensional spatial vector. In the far field, the weak scattered wavefield u_s is (ignoring scaling factors) given by the Fourier transform of the scattering function [3]

$$u_s(\mathbf{k}) \sim \hat{F}_3[\gamma(\mathbf{r})]$$

where $\hat{F}_3$ denotes the three-dimensional Fourier transform operator and $\mathbf{k}$ is the spatial frequency vector. The inverse scattering problem is then compounded in the inversion of this result, i.e. the inverse Fourier transform. This weak scattering result can be interpreted in terms of single scattering events generated by a scattering function consisting of an ensemble of localized point-like scatterers, for example. When multiple is present, this simple result is not sufficient to model the scattered field which must be modified to take into account double, triple, quadruple etc. scattering events. This yields results that make the objective of 'engineering' a practically viable imaging and image processing model for various applications rather intractable. In such cases, it can be of value to develop a stochastic model for the scattered field whereby, instead of relating the scattering function to some object function which is then mapped onto the image plane, we attempt to generate a model for the probability density function of a multiple scattered wavefield in order to account for the statistical distribution of the intensity field obtained in the image plane. This involves an approach in which the resultant scattered wavefield (i.e. the wave amplitude) is taken to be a consequence of a random walk where each node in the random walk is taken to be a scattering event.

There is a fundamental connection between a random walk model for describing Brownian motion, for example, and the process of diffusion as defined by the diffusion equation. This 'connectivity' provides an approach for interpreting strong scattering in terms of a diffusive process. But formal scattering methods (including multiple scattering) are based on considering solutions to the (inhomogeneous) wave equation. Now, the essential difference between the wave equation and the diffusion equation is with regard to the order of the time differential. By 'fractionalizing' the time differential and

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considering a fractional diffusion equation of the type [4]

$$\left(\nabla^2 + \frac{1}{D^q}\frac{\partial^q}{\partial t^q}\right) u(\mathbf{r},t) = 0$$

where D is the fractional diffusivity, we consider the role that the fractional exponent q plays in terms of characterizing an image from a fully diffusive (strong scattering) model when $q = 1$ to a propagative (weak scattering) model when $q = 2$. In order to introduce this idea, we review, by way of a short tutorial, the principal formal solutions to the forward and inverse scattering problem in terms of solutions to the inhomogeneous wave equation for both deterministic and random media. We then address the properties and solutions to the (inhomogeneous) diffusion equation and discuss the basis for using this equation to model the propagation of light through an optical diffuser. This provides an inverse solution to the optical diffusion problem that can be cast in terms of appropriate finite impulse response filters, the first order solution providing the well known 'high emphasis filter' [5], [6]. The principles associated with random phase walk models are addressed and the rationale for generalizing some well known results to fractional form considered. Forward and inverse solutions to the fractional diffusion equation are derived and, in the latter case, used to propose of a new metric for segmenting a digital image under the assumption that it has been formed from a fractional diffusive (intermediate scattering) process.

II. Formal Scattering Methods for Scalar Wavefields

Formal scattering methods for scalar electromagnetic wavefields interacting with (non-conductive) dielectric media are based on the inhomogeneous Helmholtz equation [7] which can be derived quite generally from the (inhomogeneous) time dependent wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(\mathbf{r},t) = 0$$

by letting

$$\frac{1}{c^2} = \frac{1}{c_0^2}(1+\gamma)$$

where γ is a dimensionless quantity (the scattering function) and c_0 is a constant (wave speed). With $U(\mathbf{r},t) = u(\mathbf{r},\omega)\exp(i\omega t)$ we have

$$(\nabla^2 + k^2)u(\mathbf{r},k) = -k^2\gamma(\mathbf{r})u(\mathbf{r},k)$$

where

$$k = \frac{\omega}{c_0}.$$

In electromagnetism, u is the scalar electric field, c_0 is the speed of light and the scattering function, $\gamma = \epsilon_r - 1$ where $\epsilon_r(\mathbf{r})$ is the relative permittivity[1], is taken to be of compact support [8], [9], i.e.

$$\gamma(\mathbf{r}) \quad \exists \quad \forall \quad \mathbf{r} \in V$$

[1]The relative permeability is assumed to be constant.

The general (Green's function) solution to this equation at a point $\mathbf{r}_0$ is [10], [11]

$$u(\mathbf{r}_0,k) = k^2 \int_V g\gamma u d^3\mathbf{r} + \oint_S (g\nabla u - u\nabla g)\cdot\hat{\mathbf{n}} d^2\mathbf{r}$$

where g is the 'outgoing' free space Green's function given by

$$g(\mathbf{r} \mid \mathbf{r}_0, k) = \frac{\exp(ik \mid \mathbf{r} - \mathbf{r}_0 \mid)}{4\pi \mid \mathbf{r} - \mathbf{r}_0 \mid}$$

which is the solution to the equation (where δ^3 is the three-dimensional delta function)

$$(\nabla^2 + k^2)g(\mathbf{r} \mid \mathbf{r}_0, k) = -\delta^3(\mathbf{r} - \mathbf{r}_0)$$

and $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface element $d^{\mathbf{r}}$ of a closed surface S. To compute the surface integral, a condition for the behaviour of u on the surface S of γ must be chosen. Consider the case where the incident wavefield u_i is a simple plane wave of unit amplitude

$$\exp(i\mathbf{k}\cdot\mathbf{r})$$

satisfying the homogeneous wave equation

$$(\nabla^2 + k^2)u_i(\mathbf{r},k) = 0.$$

By choosing the condition $u(\mathbf{r},k) = u_i(\mathbf{r},k)$ on the surface of γ, we obtain the result

$$u(\mathbf{r}_0,k) = k^2 \int_V g\gamma u d^3\mathbf{r} + \oint_S (g\nabla u_i - u_i\nabla g)\cdot\hat{\mathbf{n}} d^2\mathbf{r}.$$

Now, using Green's theorem to convert the surface integral back into a volume integral, we have

$$\oint_S (g\nabla u_i - u_i\nabla g)\cdot\hat{\mathbf{n}} d^2\mathbf{r} = \int_V (g\nabla^2 u_i - u_i\nabla^2 g) d^3\mathbf{r}.$$

Noting that

$$\nabla^2 u_i = -k^2 u_i$$

and that

$$\nabla^2 g = -\delta^3 - k^2 g$$

we obtain

$$\int_V (g\nabla^2 u_i - u_i\nabla^2 g) d^3\mathbf{r} = \int_V \delta^3 u_i d^3\mathbf{r} = u_i.$$

Hence, by choosing the field u to be equal to the incident wavefield u_i on the surface of γ, we obtain a solution of the form

$$u = u_i + u_s$$

where

$$u_s = k^2 \int_V g\gamma u d^3\mathbf{r}.$$

The function u_s is the scattered wavefield.

A. The Born Approximation

From the last result it is clear that in order to compute the scattered field u_s, we must define u inside the volume integral. Unlike the surface integral, a boundary condition will not help here because it is not sufficient to specify the behaviour of u at a boundary. In this case, the behaviour of u throughout V needs to be known. In general, it is not possible to do this (i.e. to compute the scattered wavefield exactly) and we are forced to choose a model for u inside V that is compatible with a particular physical problem in the same way that an appropriate set of boundary conditions are required to evaluate the surface integral. The simplest model for the internal field is based on assuming that u behaves like u_i for $\mathbf{r} \in V$. The scattered field is then given by

$$u_s(\mathbf{r}_0, k) = k^2 \int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})u_i(\mathbf{r}, k)d^3\mathbf{r}.$$

This assumption provides an approximate solution for the scattered field and is known as the Born approximation [7] after Max Born who was amongst the first to introduced the approximation in the study of (non-relativistic) quantum scattering when the basic wave equation is the Schrödinger equation[2] [12]

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = \gamma(\mathbf{r})u(\mathbf{r}, k)$$

where γ is a scattering potential (not necesserily of compact support).

There is another way of deriving this result that is instructive and helps to obtain a criteria for the validity of this approximation which will be considered shortly. We start with the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)u = -k^2\gamma u$$

and consider a solution for u in terms of a sum of the incident and scattered fields, i.e.

$$u = u_i + u_s.$$

The wave equation then becomes

$$(\nabla^2 + k^2)u_s + (\nabla^2 + k^2)u_i = -k^2\gamma(u_i + u_s).$$

If the incident field satisfies

$$(\nabla^2 + k^2)u_i = 0,$$

then

$$(\nabla^2 + k^2)u_s = -k^2\gamma(u_i + u_s).$$

Assuming that $u_s << u_i$, then

$$u_i + u_s \simeq u_i$$

and we obtain

$$(\nabla^2 + k^2)u_s \simeq -k^2\gamma u_i.$$

[2]For $k^2 = 2mE/\hbar^2$ where m is the mass of a partical, E is its non-relativistic energy and $\hbar$ is Planck's constant.

Solving for u_s using the Green's function and homogeneous boundary conditions (i.e. $u_s = 0$ on S and $\nabla u_s = 0$ on S) we get

$$u_s = \oint_S (g\nabla u_s - u_s\nabla g)\cdot\hat{\mathbf{n}}d^2\mathbf{r} + k^2\int_V g\gamma u_i d^3\mathbf{r}$$

$$= k^2\int_V g\gamma u_i d^3\mathbf{r}.$$

1) Validity of the Born Approximation: In general, the Born approximation requires that u_s is 'small' or 'weak' compared to u_i. What do we mean by the term 'weak' and how can we quantify it? One way to answer this question is to compute an appropriate measure for both the incident and scattered fields and compare the two results. Consider the case where we compute the root mean square modulus (i.e. the ℓ_2 norm) of each field. We then require

$$\left(\int_V \mid u_s(\mathbf{r}_0, k) \mid^2 d^3\mathbf{r}_0\right)^{\frac{1}{2}} << \left(\int_V \mid u_i(\mathbf{r}_0, k) \mid^2 d^3\mathbf{r}_0\right)^{\frac{1}{2}}$$

or

$$\frac{\|u_s\|}{\|u_i\|} << 1 \qquad (1)$$

Essentially, this condition means that the overall intensity of u_s in V is small compared to that of u_i in V. Let us now look in more detail at the nature of this condition. Ideally, what we want is a version of the condition that can be cast in terms of a set of physical parameters (such as the wavelength and the physical extent of γ for example). The Born scattered field at $\mathbf{r}_0$ is given by

$$u_s(\mathbf{r}_0, k) = k^2\int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})u_i(\mathbf{r}, k)d^3\mathbf{r}.$$

By taking the ℓ_2 norm of this equation we can write

$$\|u_s(\mathbf{r}_0, k)\| = \|k^2\int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})u_i(\mathbf{r}, k)d^3\mathbf{r}\|$$

$$\leq k^2\|u_i(\mathbf{r}_0, k)\| \times \|\int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})d^3\mathbf{r}\|.$$

Using this result, the condition required for the Born approximation to hold, i.e. condition (1), can be written as

$$k^2\|\int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})d^3\mathbf{r}\| << 1, \quad \mathbf{r}_0 \in V. \qquad (2)$$

Here, the norm involves integration over the spatial variable $\mathbf{r}_0$ in the scattering volume V. To emphasize this we write $\mathbf{r}_0 \in V$.

Condition (2) can be written as

$$I(\mathbf{r}_0) << 1$$

where

$$I(\mathbf{r}_0) = k^2\|\int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})d^3\mathbf{r}\|$$

$$\leq k^2 \left(\int_V \mid g(\mathbf{r} \mid \mathbf{r}_0, k) \mid^2 d^3\mathbf{r} \right)^{\frac{1}{2}} \left(\int_V \mid \gamma(\mathbf{r}) \mid^2 d^3\mathbf{r} \right)^{\frac{1}{2}}.$$

Substituting the expression for the three-dimensional Green's function into the above expression, we have

$$I(\mathbf{r}_0) \leq k^2 \left(\frac{1}{16\pi^2} \int_V \frac{1}{\mid \mathbf{r} - \mathbf{r}_0 \mid^2} d^3\mathbf{r} \int_V \mid \gamma(\mathbf{r}) \mid^2 d^3\mathbf{r} \right)^{\frac{1}{2}}.$$

A relatively simple calculation can now be performed if we consider γ to be a sphere of volume V and radius R and resort to calculating its least upper bound which occurs when $\mathbf{r}_0 = \mathbf{0}$. Using spherical polar coordinates (r, θ, ϕ), we have

$$\sup \int_V \frac{1}{\mid \mathbf{r} - \mathbf{r}_0 \mid^2} d^3\mathbf{r} = \int_V \frac{1}{r^2} d^3\mathbf{r} = \int_0^{2\pi} \int_{-1}^{1} \int_0^R dr d(\cos\theta) d\phi$$
$$= 4\pi R$$

where sup denotes the superior value. Using this result, we can write

$$\sup I(\mathbf{r}_0) \leq k^2 \left(\frac{R}{4\pi} \int_V \mid \gamma(\mathbf{r}) \mid^2 d^3\mathbf{r} \right)^{\frac{1}{2}}$$

and noting that

$$V = \int_V d^3\mathbf{r} = \frac{4}{3}\pi R^3$$

we obtain

$$\sup I(\mathbf{r}_0) \leq \frac{1}{\sqrt{3}} k^2 R^2 \bar{\gamma}$$

where

$$\bar{\gamma} = \sqrt{\frac{\int \mid \gamma \mid^2 d^3\mathbf{r}}{\int d^3\mathbf{r}}}.$$

Hence, the condition for the Born approximation to apply becomes (ignoring $\sqrt{3}$)

$$k^2 R^2 \bar{\gamma} << 1$$

or

$$\bar{\gamma} << \frac{1}{k^2 R^2}.$$

This condition demonstrates that in principle, large values of γ can occur so long as its root mean square value over the volume V is small compared to $1/k^2R^2$. In scattering theory, γ is said to be a 'weak scatterer'. Note that when k or R approach zero, this condition is easy to satisfy and that Born scattering is more likely to occur in situations when

$$\frac{\lambda}{R} >> 1$$

where λ is the wavelength (noting that $k = 2\pi/\lambda$). If

$$\frac{\lambda}{R} \sim 1$$

then the value of $\bar{\gamma}$ must be small for Born scattering to occur. We assume that the scattered field is, on average, weak compared to the incident field. We may consider the term 'weak', to imply that the total energy associated with u_s inside the inhomogeneity γ is small compared to u_i outside the scatterer.

2) Asymptotic Born Scattering: By measuring u_s, we can attempt to invert the relevant integral equation and hence recover or reconstruct γ. This is an inverse scattering problem. The simplest type of inverse scattering problem occurs when a Born scattered wavefield is measured in the far field or Fraunhofer zone and the Green's function has the from

$$g(\mathbf{r} \mid \mathbf{r}_0, k) = \frac{1}{4\pi r_0} \exp(ikr_0) \exp(-ik\hat{\mathbf{n}}_0 \cdot \mathbf{r}), \quad \frac{r}{r_0} << 1$$

where $r = \mid \mathbf{r} \mid$, $r_0 = \mid \mathbf{r}_0 \mid$ and $\hat{\mathbf{n}}_0 = \mathbf{r}_0/r_0$. Thus, when the incident field is a (unit) plane wave

$$u_i = \exp(ik\hat{\mathbf{n}}_i \cdot \mathbf{r})$$

where $\hat{\mathbf{n}}_i$ points in the direction of the incident field, the Born scattered field is given by

$$u_s(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_i, k)$$
$$= \frac{k^2}{4\pi r_0} \exp(ikr_0) \int_V \exp[-ik(\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}_i) \cdot \mathbf{r}] \gamma(\mathbf{r}) d^3\mathbf{r}.$$

From this result, it is clear, that the function γ can be recovered from u_s by three-dimensional Fourier inversion. Observe that when $\hat{\mathbf{n}}_0 = \hat{\mathbf{n}}_i$

$$u_s = \frac{k^2}{4\pi r_0} \exp(ikr_0) \int_V \gamma(\mathbf{r}) d^3\mathbf{r}.$$

This is called the forward-scattered field. In terms of Fourier analysis, it represents the zero frequency or DC level of the spectrum of γ. Another special case arises when $\hat{\mathbf{n}}_s = -\hat{\mathbf{n}}_i$. The scattered field that is produced in this case is called the back-scattered field and is given by

$$u_s(\hat{\mathbf{n}}_0, k) = \frac{k^2}{4\pi r_0} \exp(ikr_0) \int_V \exp(-2ik\hat{\mathbf{n}}_0 \cdot \mathbf{r}) \gamma(\mathbf{r}) d^3\mathbf{r}.$$

B. The Born Series

Let us now consider a natural extension to the Born approximation which is based on generating a series solution to the problem in which

$$u(\mathbf{r}_0, k) = u_i(\mathbf{r}_0, k) + u_s(\mathbf{r}_0, k)$$

where

$$u_s(\mathbf{r}_0, k) = k^2 \int_V g(\mathbf{r} \mid \mathbf{r}_0, k) \gamma(\mathbf{r}) u(\mathbf{r}, k) d^3\mathbf{r}.$$

We have seen that the Born approximation to this equation is given by considering $u \sim u_i, \quad \mathbf{r} \in V$ which is valid provided $\|u_s\| << \|u_i\|$. We then obtain an approximate solution u_1 say of the form

$$u_1(\mathbf{r}_0, k) = u_i(\mathbf{r}_0, k) + k^2 \int_V g(\mathbf{r} \mid \mathbf{r}_0, k) \gamma(\mathbf{r}) u_i(\mathbf{r}, k) d^3\mathbf{r}.$$

This result can be considered to be the first approximation to a series solution, in which the second approximation u_2 say is given by

$$u_2(\mathbf{r}_0,k)=u_i(\mathbf{r}_0,k)+k^2\int\limits_V g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})u_1(\mathbf{r},k)d^3\mathbf{r}$$

and the third approximation u_3 is given by

$$u_3(\mathbf{r}_0,k)=u_i(\mathbf{r}_0,k)+k^2\int\limits_V g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})u_2(\mathbf{r},k)d^3\mathbf{r}$$

and so on. In general, we can consider the iteration

$$u_{n+1}(\mathbf{r}_0,k)=u_i(\mathbf{r}_0,k)+k^2\int\limits_V g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})u_n(\mathbf{r},k)d^3\mathbf{r}$$

$$n=0,1,2,3,..$$

where $u_0=u_i$.

In principle, if this series converges, then it must converge to the solution. To investigate its convergence, it is convenient to use operator notation and write

$$u_{n+1}=u_i+\hat{I}u_n$$

where $\hat{I}$ is the integral operator

$$\hat{I}=k^2\int\limits_V d^3\mathbf{r}g\gamma.$$

At each iteration n we can consider the solution to be given by

$$u_n=u+\epsilon_n$$

where ϵ_n is the error associated with the solution at iteration n and u is the exact solution so that a necessary condition for convergence is that $\epsilon_n\to 0$ as $n\to\infty$. Now,

$$u+\epsilon_{n+1}=u_i+\hat{I}(u+\epsilon_n)=u_i+\hat{I}u+\hat{I}\epsilon_n$$

and therefore we can write

$$\epsilon_{n+1}=\hat{I}\epsilon_n$$

since $u=u_i+\hat{I}u$. Thus

$$\epsilon_1=\hat{I}\epsilon_0;\quad \epsilon_2=\hat{I}\epsilon_1=\hat{I}(\hat{I}\epsilon_0);\quad \epsilon_3=\hat{I}\epsilon_2=\hat{I}[\hat{I}(\hat{I}\epsilon_0)];\quad ...$$

or

$$\epsilon_n=\hat{I}^n\epsilon_0$$

from which it follows that

$$\|\epsilon_n\|=\|\hat{I}^n\epsilon_0\|\leq\|\hat{I}^n\|\times\|\epsilon_0\|\leq\|\hat{I}\|^n\|\epsilon_0\|.$$

The condition for convergence therefore becomes

$$\lim_{n\to\infty}\|\hat{I}\|^n=0.$$

This is only possible if

$$\|\hat{I}\|<1$$

or

$$k^2\|\int\limits_V g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})d^3\mathbf{r}\|<1.$$

Comparing this result with condition (2) and the analysis of this condition given before, then

$$\bar{\gamma}<\frac{1}{k^2R^2}$$

must be satisfied for the series to converge where R is the radius of a sphere of volume V.

This series solution, which can be written out as

$$u(\mathbf{r}_0,k)=u_i(\mathbf{r}_0,k)+k^2\int\limits_V g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})u_i(\mathbf{r},k)d^3\mathbf{r}+$$

$$k^4\int\limits_V g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})\left[\int\limits_V g(\mathbf{r}_1\mid\mathbf{r},k)\gamma(\mathbf{r}_1)u_i(\mathbf{r}_1,k)d^3\mathbf{r}_1\right]d^3\mathbf{r}$$

$$+...$$

$$=u_i(\mathbf{r}_0,k)+k^2\int\limits_V d^3\mathbf{r}g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})u_i(\mathbf{r},k)$$

$$+k^4\int\limits_V\int\limits_V d^3\mathbf{r}d^3\mathbf{r}_1g(\mathbf{r}\mid\mathbf{r}_0,k)\gamma(\mathbf{r})g(\mathbf{r}_1\mid\mathbf{r},k)\gamma(\mathbf{r}_1)u_i(\mathbf{r}_1,k)$$

$$+...$$

is an example of a Neumann series solution to a Fredholm integral equation and is known as the Born series. Note that the scattered field can be written in the form

$$u_s(\mathbf{r},k)=k^2g(r,k)\otimes_3\gamma(\mathbf{r})u_i(\mathbf{r},k)$$

$$+k^4g(r,k)\otimes_3\gamma(\mathbf{r})[g(r)\otimes_3\gamma(\mathbf{r})u_i(\mathbf{r},k)]$$

$$+...$$

where $\otimes_3$ denotes the three-dimensional convolution integral over V which can be interpreted as follows:

$$u(\mathbf{r}_0,k)=\text{incident wavefield}$$
$$+$$
$$\text{wavefield generated by single scattering events}$$
$$+$$
$$\text{wavefield generated by double scattering events}$$
$$+$$
$$\text{wavefield generated by triple scattering events}$$
$$+$$
$$\vdots$$

Each term in this series expresses the effects due to single, double and triple etc. scattering, i.e. the wavefields generated by an increasing number of interactions.

For an incident plane wave, each term in the Born series includes scaling by $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$ etc. so that multiple-scattering gets 'weaker by the term'. This is due to the form of the Green's function in three-dimensions which scales as $1/r$, the intensity of the field being $1/r^2$ - the inverse square law. Thus, if the scattering function is characterized by a number of scattering 'sites' (i.e. isolated positions in space where, for example, γ is non-zero and of compact support) then provided the distance between these sites is large, the effect of multiple scattering will be insignificant. However, if these sites are close

together where the effect of the multiple scattering wavefield falling off as $1/r^2$, $1/r^3$ etc. is not appreciable, then multiple scattering events will contribute significantly to the scattered field. Hence, one way to interpret the meaning of 'weak' and 'strong' scattering is in terms of the 'density' of scattering sites over the volume V being low or high respectively. For $\lambda \sim R$ where R is the characteristic size of the scatterer, the Born approximation holds provided the root mean square of the scattering function over the volume is much less than 1 which is a quantification of the principle that the density of isolated scattering sites from which we can suppose the scattering function is composed, is low.

Another important feature of the Born series for Helmholtz scattering is that the terms are scaled by k^2, k^4, k^6, etc. Thus, for a fixed $k << 1$ (long wavelength waves),

$$u_s(\mathbf{r}_0, k) = k^2 \int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})u_i(\mathbf{r}, k)d^3\mathbf{r}$$

and

$$u_s(\mathbf{r}_0, k) = k^2 \int_V \frac{\gamma(\mathbf{r})d^3\mathbf{r}}{4\pi \mid \mathbf{r} - \mathbf{r}_0 \mid}, \quad k \to 0$$

which describes a very weak (static) field.

C. Inverse Scattering

Inverse scattering aims to reconstruct the scattering function from measurements of the scattered field. The practicability of solving inverse scattering problems analytically and implementing them experimentally varies considerably from one application to another. Thus, an inversion method is usualy based on the approximation that has been applied to solve the forward scattering problem for a given inhomogeneous wave equation.

1) Inverse Scattering for Single Scattering Processes: Under the Born approximation, the scattered field is given by

$$u_s(\mathbf{r}, k) = k^2 g(r, k) \otimes_3 \gamma(\mathbf{r})u_i(\mathbf{r}, k)$$

and it is clear that the inverse solution is compounded in the deconvolution of u_s to recover γ. This problem assumes that the scattered field is measured in the near field which is not always of practical value.

In the far, the scattering amplitude is given by

$$A(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_i, k) = k^2 \int_V \exp[-ik(\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}_i) \cdot \mathbf{r}]\gamma(\mathbf{r})d^3\mathbf{r}.$$

The inverse solution to this problem is therefore compounded in the inverse Fourier transform which provides that essential 'link' between the application of the Born approximation in the far field and the Fourier transform. This 'link' is essential in imaging science and is why the Fourier transform plays such an essential role. Inverse solutions under the Born approximation are in effect the same as implementing Fourier based reconstruction methods in imaging science, at least when the data collected is the result of a scattering event. However, when the scattering processes involved are not as weak as they should be to support application of the Born approximation, Fourier based image reconstructions can become distorted. There is however, a technique for inverting a wavefield that is the result of multiple Born scattering known as the Jost-Kohn method first published in 1952 [14]. A brief overview of this method follows.

2) Inverse Scattering for Multiple Scattering Processes: Using operator notation, the Born series can be written as

$$u = u_i + \hat{I}_i\gamma + \hat{I}_i(\gamma\hat{I}\gamma) + I_i[\gamma\hat{I}(\gamma\hat{I}\gamma)] + ...$$

where γ is either the scattering potential (for Schrödinger scattering) or $k^2\gamma$ (for Helmholtz scattering) and

$$\hat{I}_i = \int d^3\mathbf{r}u_i g, \quad \hat{I} = \int d^3\mathbf{r}g.$$

Now, let $\epsilon U = u - u_i$ and

$$\gamma = \sum_{j=1}^{\infty} \epsilon^j \gamma_j.$$

Then

$$\epsilon U = \hat{I}_i[\epsilon\gamma_1 + \epsilon^2\gamma_2 + \epsilon^3\gamma_3 + ...]$$
$$+\hat{I}_i[(\epsilon\gamma_1 + \epsilon^2\gamma_2 + \epsilon^3\gamma_3 + ...)\hat{I}(\epsilon\gamma_1 + \epsilon^2\gamma_2 + \epsilon^3\gamma_3 + ...)]$$
$$+\hat{I}_i\{(\epsilon\gamma_1 + \epsilon^2\gamma_2 + \epsilon^3\gamma_3 + ...)\hat{I}[(\epsilon\gamma_1 + \epsilon^2\gamma_2 + \epsilon^3\gamma_3 + ...)$$
$$\hat{I}(\epsilon\gamma_1 + \epsilon^2\gamma_2 + \epsilon^3\gamma_3 + ...)]\} + ...$$

Equating terms with common coefficients ϵ, ϵ^2 etc. we have For $j = 1$:

$$U = \hat{I}_i\gamma_1; \quad \gamma_1 = \hat{I}_i^{-1}U.$$

For $j = 2$:

$$0 = \hat{I}_i\gamma_2 + \hat{I}_i(\gamma_1\hat{I}\gamma_1); \quad \gamma_2 = -\hat{I}_i^{-1}[\hat{I}_i(\gamma_1\hat{I}\gamma_1)]$$

and so on. By computing the functions γ_j using this iterative method, the scattering function γ is obtained by summing γ_j for $\epsilon = 1$. This approach provides a formal exact inverse scattering solution but it is not unconditional, i.e. the inverse solution is only applicable when the Born series converges to the exact scattering solution and thus when

$$\| \int_V g(\mathbf{r} \mid \mathbf{r}_0, k)\gamma(\mathbf{r})d^3\mathbf{r}\| < 1.$$

Note that for $j = 1$, the solution for γ_1 is that obtained under the Born approximation.

3) Exact Inverse Scattering Solutions: Unconditional or exact solutions to the inverse scattering problem are usually rare and/or of little practical significance due to the incompatibility of the solution with the experimental conditions under which the data is obtained (e.g. near field .v. far field) and/or the ill-conditioned nature of the solution, i.e. a solution that is unacceptably sensitive to data error. An example of such an exact inverse scattering 'transform' follows: Given that

$$(u - u_i) = k^2 g \otimes_3 \gamma u,$$

then for an arbitrary function q

$$q \otimes_3 (u - u_i) = k^2 q \otimes_3 (g \otimes_3 \gamma u).$$

Taking the Laplacian of this equation, we have

$$\nabla^2[q \otimes_3 (u - u_i)] = k^2\nabla^2(q \otimes_3 g \otimes_3 \gamma u)$$

$$= k^2\nabla^2(q \otimes_3 g) \otimes_3 \gamma u = -k^2\delta^3 \otimes_3 \gamma u = -k^2\gamma u$$

provided

$$\nabla^2(q \otimes_3 g) = -\delta^3.$$

But

$$\nabla^2(q \otimes_3 g)$$

$$= q \otimes_3 \nabla^2 g = q \otimes_3 (-k^2 g - \delta^3) = -k^2 q \otimes_3 g - q = -\delta^3$$

and hence

$$q = \delta^3 - k^2 q \otimes_3 g$$

so that

$$\nabla^2[q \otimes_3 (u - u_i)] = \nabla^2[\delta^3 \otimes_3 (u - u_i) - k^2 q \otimes_3 g \otimes_3 (u - u_i)]$$
$$= \nabla^2[(u - u_i) - k^2 q \otimes_3 g \otimes_3 (u - u_i)] = -k^2\gamma u.$$

Thus,

$$\gamma = \frac{1}{u}\nabla^2\left[q \otimes_3 g \otimes_3 (u - u_i) - \frac{1}{k^2}(u - u_i)\right]$$

where q is given by the solution to

$$\nabla^2(q \otimes_3 g) = -\delta^3$$

or

$$q \otimes_3 g = \frac{1}{4\pi r}$$

so that

$$\gamma = \frac{1}{u}\nabla^2\left[\frac{1}{4\pi r} \otimes_3 (u - u_i) - \frac{1}{k^2}(u - u_i)\right].$$

Finally since, $u = u_i + u_s$, we can write

$$\gamma = \frac{1}{u_i + u_s}\nabla^2\left[\frac{1}{4\pi r} \otimes_3 u_s - \frac{1}{k^2}u_s)\right]$$
$$= \frac{u^*}{|\, u \,|^2}\nabla^2\left[\frac{1}{4\pi r} \otimes_3 u_s - \frac{1}{k^2}u_s)\right], \quad |\, u \,| > 0. \qquad (4)$$

Note that this result reduces to the inhomogeneous Helmholtz equation since, given that $(\nabla^2 + k^2)u_i = 0$,

$$\begin{aligned}
\gamma &= \frac{1}{u}\nabla^2\left[\frac{1}{4\pi r} \otimes_3 (u - u_i) - \frac{1}{k^2}(u - u_i)\right] \\
&= \frac{1}{u}\left[-\delta^3 \otimes_3 (u - u_i) - \frac{1}{k^2}(\nabla^2 u - \nabla^2 u_i)\right] \\
&= \frac{1}{u}\left[-(u - u_i) - \frac{1}{k^2}\nabla^2 u + \frac{1}{k^2}\nabla^2 u_i)\right] \\
&= \frac{1}{k^2 u}\left[-(\nabla^2 u + k^2 u) + \nabla^2 u_i + k^2 u_i\right] \\
&= -\frac{1}{k^2 u}(\nabla^2 + k^2)u.
\end{aligned}$$

Equation (4) assumes that u is measured in the near field and that the measurements are complete, neither of which may be practicable. Further, equation (4) is ill-conditioned with regard to errors in the computation of $\nabla^2 u_s$ and $|\, u \,|$. The condition $|\, u \,| > 0$ implies that the amplitude of the sum of the incident and scattered wavefields is always greater than zero. Near-zero values of $|\, u \,|$ that are in error will therefore be 'amplified' in the computation of γ. The derivation of equation (4) therefore needs to be modified in terms of the application of an appropriate regularisation method which lies beyond the scope of this publication.

D. Scattering from Random Media

Analysis of scattering from a random medium ideally requires a model for the physical behaviour of the random variable(s) that is derived from basic principles. Ideally, this involves modelling the scattered field in terms of its interaction with an ensemble of 'scattering sites' based on an assumed stochastic process. If the density of these scattering sites is low enough so that multiple scattering is minimal, then we can apply Born scattering to develop a model for the intensity of a wavefield interacting with a random Born scatterer.

In the far field, the Born scattered field (i.e. the scattering amplitude) is given by the Fourier transform of the scattering function. If this function is known *a priori*, then the scattering amplitude can be determined. This is an example of a deterministic model. If the scattering function is stochastic (i.e. a randomly distributed scatterer) such that it can only be quantified in terms of a statistical distribution (i.e. the probability density function (PDF) - denoted by Pr) then we can simulate the (Born) scattered field by designing a random number generator that outputs deviates that conform to this distribution. The Fourier transform of this stochastic field then provides the Born scattering amplitude. Thus, given a three dimensional Helmholtz scattering function $\gamma(\mathbf{r}), \quad \mathbf{r} \in V$ with $\Pr[\gamma(\mathbf{r})]$ known *a priori*, the scattering amplitude A is given by

$$A(\hat{\mathbf{N}}, k) = k^2 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r})\gamma(\mathbf{r})d^3\mathbf{r}$$

where $\hat{\mathbf{N}} = \hat{\mathbf{n}}_s - \hat{\mathbf{n}}_i$ and $\gamma(\mathbf{r})$ is a stochastic function whose deviates conform to the PDF $\Pr[\gamma(\mathbf{r})]$.

If we consider the object function f (i.e. a two-dimensional map of the three-dimensional scattering function) to be a stochastic function, then we can model this function in terms of a random distribution of amplitudes using an appropriate random number generator. A coherent image of this function is then given by (e.g. [1], [2], [5], [6])

$$I(x, y) = |\, p(x, y) \otimes_2 f(x, y) \,|^2$$

and an incoherent image by

$$I(x, y) = |\, p(x, y) \,|^2 \otimes_2 |\, f(x, y) \,|^2$$

where p is the point spread function for a coherent image and $|\, p \,|^2$ is the intensity point spread function for an incoherent image. An example of simulating such images is given in Figure 1 which is based on the application of a zero mean Gaussian distributed random field for the object function f and point spread functions for a square aperture. There is a striking difference between these images. The coherent image yields 'speckle' which is a feature of all coherent images and is due to the 'phase mixing' of the functions p and f associated with the convolution operation given above.

The intensity of the scattering amplitude is given by

$$I(\hat{\mathbf{N}}, k) = |\, A(\hat{\mathbf{N}}, k) \,|^2 = A(\hat{\mathbf{N}}, k)A^*(\hat{\mathbf{N}}, k)$$

$$= k^4 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r})\gamma(\mathbf{r})d^3\mathbf{r} \int_V \exp(ik\hat{\mathbf{N}} \cdot \mathbf{r}')\gamma^*(\mathbf{r}')d^3\mathbf{r}'.$$

Using the autocorrelation theorem, we have

$$I(\hat{\mathbf{N}},k) = k^4 \int_V \exp(-ik\hat{\mathbf{N}}\cdot\mathbf{r})\Gamma(\mathbf{r})d^3\mathbf{r}$$

where Γ is the autocorrelation function given by

$$\Gamma(\mathbf{r}) = \int_V \gamma(\mathbf{r}')\gamma^*(\mathbf{r}'+\mathbf{r})d^3\mathbf{r}'.$$

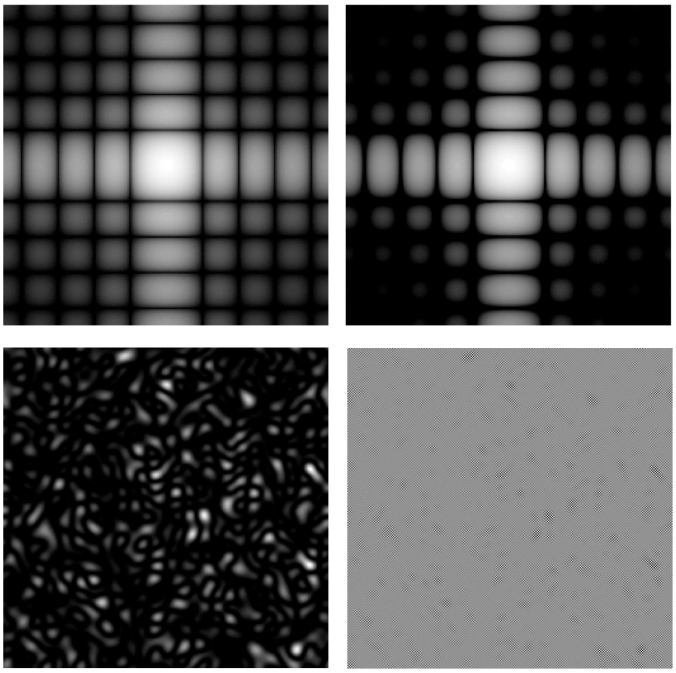

Fig. 1. Simulation of the coherent (bottom-left) and incoherent (bottom-right) images associated with light scattering from a random medium imaged through a square aperture with coherent (top-left) and incoherent (top-right) point spread functions whose absolute values are shown using a logarithmic grey-scale.

This result allows us to evaluate the intensity of the Born scattered amplitude by computing the Fourier transform of the autocorrelation function of the scattering function which is taken to be composed of a number of scatterers distributed at random throughout V. This requires the autocorrelation function to be defined for a particular type of random scatterer. Thus, a random medium can be characterized via its autocorrelation function by measuring the scattered intensity and inverse Fourier transforming the result.

From the autocorrelation theorem, the characteristics of the autocorrelation function can be formulated by considering its expected spectral properties since

$$\Gamma(\mathbf{r}) \Longleftrightarrow \mid \widetilde{\gamma}(\mathbf{k}) \mid^2$$

where $\widetilde{\gamma}$ is the Fourier transform of γ, $\mathbf{k}$ is the spatial frequency vector and $\Longleftrightarrow$ denotes the transformation from real space $\mathbf{r}$ to Fourier space $\mathbf{k}$. Hence, in order to evaluate the most likely form of the autocorrelation function we can consider the properties of the power spectrum of the scattering function. If this function is 'white' noise, for example (i.e. its Power Spectral Density Function or PSDF is a constant), then the autocorrelation function is a delta function whose Fourier transform is a constant. However, in practice, we can expect that few scattering functions have a PSDF characterized by white noise, rather, the PSDF will tend to decay as the frequency increases. We can consider a model for the PSDF based on the Gaussian function

$$\mid \widetilde{\gamma}(\mathbf{k}) \mid^2 = \widetilde{\gamma}_0^2 \exp\left(-\frac{k^2}{k_0^2}\right),$$

for example, where $\widetilde{\gamma}_0 = \widetilde{\gamma}(0)$, $k = \mid \mathbf{k} \mid$ and k_0 is the standard deviation which is a measure of the correlation length. This form yields an autocorrelation function which is of the same type, i.e. a Gaussian function. If the geometry of the scattering function is self-affine, then we can model the scattering function as a random scattering fractal whose PSDF is characterized by [15]

$$\mid \widetilde{\gamma}(\mathbf{k}) \mid^2 \sim \frac{1}{k^{2q}}$$

where $q > 0$, the autocorrelation function being characterized by

$$\Gamma(\mathbf{r}) \sim \frac{1}{r^{3-q}}.$$

Other issues in determining the nature of the autocorrelation function are related to the physical conditions imposed on the stochastic characteristics of the scatterer.

The method discussed above can be used to model the (Born) scattered intensity from a random medium which requires an estimate of the autocorrelation of the scattering function to be known. However, this approach assumes that the density of scattering sites from which the scatterer is composed is low so that the Born approximation is valid. When the density of scattering sites increases and multiple scattering is present, the problem become progressively intractable. One approach to overcoming this problem is to resort to a purely stochastic approach which involves developing a statistical model, not for the scattering function, but for the scattered field itself which is discussed later.

III. The Diffusion Equation

The homogeneous diffusion equation [7]

$$\nabla^2 u(\mathbf{r},t) = \sigma\frac{\partial}{\partial t}u(\mathbf{r},t), \quad \sigma = \frac{1}{D}$$

where D is the 'Diffusivity', differs in many aspects from the scalar wave equation. The most important single feature is the asymmetry of the diffusion equation with respect to time. For the wave equation, if $u(\mathbf{r},t)$ is a solution, so is $u(\mathbf{r},-t)$. However, if $u(\mathbf{r},t)$ is a solution of

$$\nabla^2 u = \sigma\frac{\partial u}{\partial t}$$

the function $u(\mathbf{r},-t)$ is not; it is a solution of the quite different equation,

$$\nabla^2 u(\mathbf{r},-t) = -\sigma\frac{\partial}{\partial t}u(\mathbf{r},-t).$$

Thus, unlike the wave equation, the diffusion equation differentiates between past and future. This is because the diffusing

field u represents the behaviour of some average property of an ensemble (e.g. of particles) which cannot in general go back to an original state. Causality must therefore be considered in the solution to the diffusion equation. This in turn leads to the use of the one-sided Laplace transform (i.e. a causal transform) for solving the equation with respect to t (compared to the Fourier transform - a non-causal transform - used to solve the wave equation with respect to t).

A. Green's Function for the Diffusion Equation

To obtain a general solution to the diffusion equation, we need to evaluate the Green's function G for the diffusion equation subject to the causality condition

$$G(\mathbf{r} \mid \mathbf{r}_0, t \mid t_0) = 0 \quad \text{if} \quad t < t_0.$$

This can be accomplished for one-, two- and three-dimension simultaneously [13]. With $R = \mid \mathbf{r} - \mathbf{r}_0 \mid$ and $\tau = t - t_0$ we require the solution of the equation

$$\left(\nabla^2 - \sigma\frac{\partial}{\partial\tau}\right) G(R,\tau) = -\delta^n(R)\delta(\tau), \quad \tau > 0$$

where n is 1, 2 or 3 depending on the number of dimensions. One way of solving this equation is to first take the Laplace transform with respect to τ, then solve for G (in Laplace space) and inverse Laplace transform. This requires an initial condition to be specified (the value of G at $\tau = 0$). Another way to solve this equation is to take its Fourier transform with respect to R, solve for G (in Fourier space) and then inverse Fourier transform. Here, we adopt the latter approach. Let

$$G(R,\tau) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \widetilde{G}(\mathbf{k},\tau)\exp(i\mathbf{k}\cdot\mathbf{R})d^n\mathbf{k}$$

and

$$\delta^n(R) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \exp(i\mathbf{k}\cdot\mathbf{R})d^n\mathbf{k}.$$

Then the equation for G reduces to

$$\sigma\frac{\partial\widetilde{G}}{\partial\tau} + k^2\widetilde{G} = \delta(\tau)$$

which has the solution

$$\widetilde{G} = \frac{1}{\sigma}\exp(-k^2\tau/\sigma)H(\tau)$$

where $H(\tau)$ is the step function

$$H(\tau) = \begin{cases} 1, & \tau > 0; \\ 0, & \tau < 0. \end{cases}$$

Hence, the Green's functions are given by

$$G(R,\tau) = \frac{1}{\sigma(2\pi)^n}H(\tau)\int_{-\infty}^{\infty}\exp(i\mathbf{k}\cdot\mathbf{R})\exp(-k^2\tau/\sigma)d^n\mathbf{k}$$

$$= \frac{1}{\sigma(2\pi)^n}H(\tau)\left(\int_{-\infty}^{\infty}\exp(ik_xR_x)\exp(-k_x^2\tau/\sigma)dk_x\right)...$$

By rearranging the exponent in the integral, it becomes possible to evaluate each integral exactly. Thus, with

$$ik_xR_x - k_x^2\frac{\tau}{\sigma} = -\left(k_x\sqrt{\frac{\tau}{\sigma}} - i\frac{R_x}{2}\sqrt{\frac{\sigma}{\tau}}\right)^2 - \left(\frac{\sigma R_x^2}{4\tau}\right)$$

$$= -\frac{\tau}{\sigma}\xi^2 - \left(\frac{\sigma R_x^2}{4\tau}\right)$$

where

$$\xi = k_x - i\frac{\sigma R_x}{2\tau},$$

the integral over k_x becomes

$$\int_{-\infty}^{\infty}\exp\left[-\left(\frac{\tau}{\sigma}\xi^2\right) - \left(\frac{\sigma R_x}{4\tau}\right)\right]d\xi$$

$$= e^{-(\sigma R_x^2/4\tau)}\int_{-\infty}^{\infty} e^{-(\tau\xi^2/\sigma)}d\xi = \sqrt{\frac{\pi\sigma}{\tau}}\exp\left[-\left(\frac{\sigma R_x^2}{4\tau}\right)\right]$$

with similar results for the integrals over k_y and k_z giving the result

$$G(R,\tau) = \frac{1}{\sigma}\left(\frac{\sigma}{4\pi\tau}\right)^{\frac{n}{2}}\exp\left[-\left(\frac{\sigma R^2}{4\tau}\right)\right]H(\tau).$$

The function G satisfies an important property which is valid for all n:

$$\int_{-\infty}^{\infty} G(R,\tau)d^n\mathbf{r} = \frac{1}{\sigma}, \quad \tau > 0.$$

This is the expression for the conservation of the Green's function associated with the diffusion equation. For example, if we consider the diffusion of heat, then if at a time t_0 and at a point in space $\mathbf{r}_0$ a source of heat is introduced instantaneously (i.e. a heat impulse), then the heat diffuses out through the medium characterized by σ in such a way that the total heat energy is unchanged.

B. Green's Function Solution to the Diffusion Equation

Working in three dimensions, let us consider the general solution to the equation

$$\left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right)u(\mathbf{r},t) = -f(\mathbf{r},t)$$

where f is a source function of compact support ($\mathbf{r} \in V$) and define the Green's function as the solution to the equation

$$\left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right)G(\mathbf{r} \mid \mathbf{r}_0, t \mid t_0) = -\delta^3(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)$$

It is convenient to first take the Laplace transform of these equations with respect to $\tau = t - t_0$ to obtain

$$\nabla^2\bar{u} - \sigma[-u_0 + p\bar{u}] = -\bar{f}$$

and

$$\nabla^2\bar{G} + \sigma[-G_0 + p\bar{G}] = -\delta^3$$

where

$$\bar{u}(\mathbf{r},p) = \int_0^{\infty} u(\mathbf{r},\tau)\exp(-p\tau)d\tau,$$

$$\bar{G}(\mathbf{r} \mid \mathbf{r}_0, p) = \int_0^\infty G(\mathbf{r} \mid \mathbf{r}_0, \tau) \exp(-p\tau) d\tau,$$

$$\bar{f}(\mathbf{r}, p) = \int_0^\infty f(\mathbf{r}, \tau) \exp(-p\tau) d\tau.$$

$$u_0 \equiv u(\mathbf{r}, \tau = 0) \quad \text{and} \quad G_0 \equiv G(\mathbf{r} \mid \mathbf{r}_0, \tau = 0) = 0.$$

Pre-multiplying the equation for $\bar{u}$ by $\bar{G}$ and the equation for $\bar{G}$ by $\bar{u}$, subtracting the two results and integrating over V we obtain

$$\int_V (\bar{G}\nabla^2\bar{u} - \bar{u}\nabla^2\bar{G}) d^3\mathbf{r} + \sigma \int_V u_0 \bar{G} d^3\mathbf{r} = -\int_V \bar{f}\bar{G} d^3\mathbf{r} + \bar{u}(\mathbf{r}_0, p).$$

Using Green's theorem and rearranging the result gives

$$\bar{u}(\mathbf{r}_0, p) = \int_V \bar{f}(\mathbf{r}, p)\bar{G}(\mathbf{r} \mid \mathbf{r}_0, p) d^3\mathbf{r} + \sigma \int_V u_0(\mathbf{r})\bar{G}(\mathbf{r} \mid \mathbf{r}, p) d^3\mathbf{r} + \oint_S (\bar{g}\nabla\bar{u} - \bar{u}\nabla\bar{g}) \cdot \mathbf{n} d^2\mathbf{r}.$$

Finally, taking the inverse Laplace transform and using the convolution theorem for Laplace transforms, we can write

$$u(\mathbf{r}_0, \tau) = \int_0^\tau \int_V f(\mathbf{r}, \tau') G(\mathbf{r} \mid \mathbf{r}_0, \tau - \tau') d^3\mathbf{r} d\tau' + \sigma \int_V u_0(\mathbf{r}) G(\mathbf{r} \mid \mathbf{r}_0, \tau) d^3\mathbf{r} + \int_0^\tau \oint_S [G(\mathbf{r} \mid \mathbf{r}_0, \tau')\nabla u(\mathbf{r}, \tau - \tau') - u(\mathbf{r}, \tau')\nabla G(\mathbf{r} \mid \mathbf{r}_0, \tau - \tau')] \cdot \hat{\mathbf{n}} d^2\mathbf{r} d\tau'.$$

The first two terms are convolutions of the Green's function with the source function and the initial field $u(\mathbf{r}, \tau = 0)$ respectively.

By way of a simple example, suppose we consider the source term to be zero and the volume of interest is the infinite domain, so that the surface integral is zero. Then we have

$$u(\mathbf{r}_0, \tau) = \sigma \int_V u_0(\mathbf{r}) G(\mathbf{r} \mid \mathbf{r}_0, \tau) d^3\mathbf{r}.$$

In one dimension, this reduces to

$$u(x_0, \tau) = \sqrt{\frac{\sigma}{4\pi\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{\sigma(x_0 - x)^2}{4\tau}\right] u_0(x)\, dx, \quad \tau > 0.$$

Observe that the field u at a time $t > 0$ is given by the convolution of the field at time $t = 0$ with the (Gaussian) function

$$\sqrt{\frac{\sigma}{4\pi t}} \exp\left(-\frac{\sigma x^2}{4t}\right).$$

In two-dimensions, the equivalent result is

$$u(x, y, t) = \frac{\sigma}{4\pi t} \exp\left[-\left(\frac{\sigma(x^2 + y^2)}{4t}\right)\right] \otimes_2 u_0(x, y). \quad (5)$$

IV. Derivation of the Diffusion Equation for the Intensity of Light

Consider the three-dimensional homogeneous time dependent wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u = 0$$

where c is taken to be a constant (light speed). Let

$$u(x, y, z, t) = \phi(x, y, z, t) \exp(i\omega t)$$

where it is assumed that field ϕ varies significantly slowly in time compared with $\exp(i\omega t)$ and note that

$$u^*(x, y, z, t) = \phi^*(x, y, z, t) \exp(-i\omega t)$$

is also a solution to the wave equation. Differentiating

$$\nabla^2 u = \exp(i\omega t)\nabla^2\phi,$$

and

$$\frac{\partial^2}{\partial t^2} u = \exp(i\omega t)\left(\frac{\partial^2}{\partial t^2}\phi + 2i\omega\frac{\partial\phi}{\partial t} - \omega^2\phi\right) \simeq \exp(i\omega t)\left(2i\omega\frac{\partial\phi}{\partial t} - \omega^2\phi\right)$$

when

$$\left|\frac{\partial^2\phi}{\partial t^2}\right| << 2\omega \left|\frac{\partial\phi}{\partial t}\right|.$$

Under this condition, the wave equation reduces to

$$(\nabla^2 + k^2)\phi = \frac{2ik}{c}\frac{\partial\phi}{\partial t}$$

where $k = \omega/c$. However, since u^* is also a solution,

$$(\nabla^2 + k^2)\phi^* = -\frac{2ik}{c}\frac{\partial\phi^*}{\partial t}$$

and thus,

$$\phi^*\nabla^2\phi - \phi\nabla^2\phi^* = \frac{2ik}{c}\left(\phi^*\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi^*}{\partial t}\right)$$

which can be written in the form

$$\nabla^2 I - 2\nabla \cdot (\phi\nabla\phi^*) = \frac{2ik}{c}\frac{\partial I}{\partial t}$$

where $I = \phi\phi^* = \mid \phi \mid^2$. Let ϕ be given by

$$\phi(\mathbf{r}, t) = A(\mathbf{r}, t) \exp(ik\hat{\mathbf{n}} \cdot \mathbf{r})$$

where $\hat{\mathbf{n}}$ is a unit vector and A is the amplitude function. Differentiating, and noting that $I = A^2$, we obtain

$$\hat{\mathbf{n}} \cdot \nabla A = \frac{2}{c}\frac{\partial A}{\partial t}$$

or

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) A(x, y, z, t) = \frac{2}{c}\frac{\partial}{\partial t} A(x, y, z, t)$$

which is the unconditional continuity equation for the amplitude A of a wavefield

$$u(\mathbf{r}, t) = A(\mathbf{r}, t) \exp[i(k\hat{\mathbf{n}} \cdot \mathbf{r} + \omega t)]$$

where A varies slowly with time.

The equation

$$\nabla^2 I - 2\nabla\cdot(\phi\nabla\phi^*) = \frac{2ik}{c}\frac{\partial I}{\partial t}$$

is valid for $k = k_0 - i\kappa$ (i.e. $\omega = \omega_0 - i\kappa c$) and so, by equating the real and imaginary parts, we have

$$D\nabla^2 I + 2\mathrm{Re}[\nabla\cdot(\phi\nabla\phi^*)] = \frac{\partial I}{\partial t}$$

and

$$\mathrm{Im}[\nabla\cdot(\phi\nabla\phi^*)] = -\frac{k_0}{c}\frac{\partial I}{\partial t}$$

respectively where $D = c/2\kappa$, so that under the condition

$$\mathrm{Re}[\nabla\cdot(\phi\nabla\phi^*)] = 0$$

we obtain

$$D\nabla^2 I = \frac{\partial I}{\partial t}.$$

This is the diffusion equation for the intensity of light I. The condition required to obtain this result can be justified by applying a boundary condition on the surface S of a volume V over which the equation is taken to conform. Using the divergence theorem

$$\mathrm{Re}\int_V \nabla\cdot(\phi\nabla\phi^*)d^3\mathbf{r} = \mathrm{Re}\oint_S \phi\nabla\phi^*\cdot\hat{\mathbf{n}}d^2\mathbf{r}$$

$$= \oint_S (\phi_r\nabla\phi_r + \phi_i\nabla\phi_i)\cdot\hat{\mathbf{n}}d^2\mathbf{r}.$$

Now, if

$$\phi_r(\mathbf{r},t)\nabla\phi_r(\mathbf{r},t) = -\phi_i(\mathbf{r},t)\nabla\phi_i(\mathbf{r},t), \quad \mathbf{r}\in S$$

then the surface integral is zero and

$$D\nabla^2 I(\mathbf{r},t) = \frac{\partial}{\partial t}I(\mathbf{r},t), \quad \mathbf{r}\in V.$$

This boundary condition can be written as

$$\frac{\nabla\phi_r}{\nabla\phi_i} = -\tan\theta$$

where θ is the phase of the field ϕ which implies that the amplitude A of ϕ is constant on the boundary (i.e. $A(\mathbf{r},t) = A_0, \quad \mathbf{r}\in S, \quad \forall t$), since

$$\frac{\nabla A_0\cos\theta(\mathbf{r},t)}{\nabla A_0\sin\theta(\mathbf{r},t)} = -\frac{A_0\sin\theta(\mathbf{r},t)\nabla\theta(\mathbf{r},t)}{A_0\cos\theta(\mathbf{r},t)\nabla\theta(\mathbf{r},t)}$$

$$= -\tan\theta(\mathbf{r},t), \quad \mathbf{r}\in S.$$

Suppose we record the intensity I of a light field in the xy-plane for a fixed value of z. Then for $z = z_0$ say,

$$I(x,y,t) \equiv I(x,y,z_0,t)$$

so that

$$\frac{\partial}{\partial t}I(x,y,t) = D\nabla^2 I(x,y,t).$$

Let this two-dimensional diffusion equation be subject to the initial condition

$$I(x,y,0) = I_0(x,y).$$

Then, at any time $t > 0$, it can be assumed that light diffusion is responsible for blurring the image I_0 and that as time increases, the image becomes progressively more (Gaussian) blurred. By comparing this model with equation (5) it is clear that

$$I(x,y,t) = \frac{1}{4\pi Dt}\exp\left[-\left(\frac{(x^2+y^2)}{4Dt}\right)\right]\otimes_2 I_0(x,y).$$

This result can, for example, be used to model the diffusion of light through an optical diffuser. An example of such an effect is given in Figure 2 which shows a light source (the ceiling light of a steam room) imaged through air and then through steam. Steam effects light by scattering it a large number of times through the complex of small water droplets from which (low temperature) steam is composed. The high degree of multiple scattering that takes place allows us to model the transmission of light through steam in terms of a 'diffusive' rather than a 'propagative' process. The initial condition I_0 denotes the initial image which is, in effect, and with regard to Figure 2, the image of the light source obtained in air.

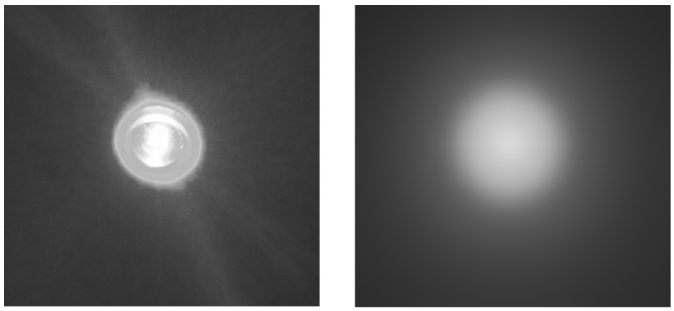

Fig. 2. Image of an optical source (left) and the same source imaged through steam (right).

As observed in Figure 2, the details associated with the light source are blurred through the convolution of the object function I_0 with the Gaussian point spread function, a function that is characteristic of diffusion processes in general.

V. De-diffusion

The problem is to find I_0 from I at some time $t > 0$. Consider the case in which we record the diffused image I at a time $t = T$. The Taylor series for I at $t = 0$ may then be written as

$$I(x,y,0) = I(x,y,T) - T\left[\frac{\partial}{\partial t}I(x,y,t)\right]_{t=T}$$

$$+\frac{T^2}{2!}\left[\frac{\partial^2}{\partial t^2}I(x,y,t)\right]_{t=T} + ...$$

For $T << 1$, we can approximate this function be neglecting all terms after the second term. Using the diffusion equation, we then obtain

$$I(x,y,0) \simeq I(x,y,T) - T\left[\frac{\partial}{\partial t}I(x,y,t)\right]_{t=T}$$

$$= I(x,y,T) - DT\nabla^2 I(x,y,T).$$

Now, since

$$I(x,y,0) = I_0(x,y)$$

we have

$$I_0(x,y) = I(x,y,T) - DT\nabla^2 I(x,y,T).$$

A. The High Emphasis Filter

The high emphasis filter [6] is based on computing an output image I_0 from the input image I via application of the result

$$I_0(x,y) = I(x,y) - \nabla^2 I(x,y)$$

which is the case when $DT = 1$.

This filter can be implemented by computing the digital Laplacian in order to design an appropriate Finite Impulse Response (FIR) filter [3]. Applying a centre differencing scheme, i.e.

$$\nabla^2 I_{ij} = I_{(i+1)j} + I_{(i-1)j} + I_{i(j+1)} + I_{i(j-1)} - 4I_{ij}$$

we have

$$I^0_{ij} = I_{ij} - \nabla^2 I_{ij} = 5I_{ij} - I_{(i+1)j} - I_{(i-1)j} - I_{i(j+1)} - I_{i(j-1)}.$$

where

$$I^0_{ij} \equiv I_0(i,j).$$

The digital Laplacian is a shift invariant linear operation. Applying this operation to a digital image I_{ij} is the same as convolving the image with the two-dimensional array (the FIR filter)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, computing I^0_{ij} is the same as convolving I_{ij} with the FIR filter

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

An example of the application of this filter is given in Figure 3. Given the simplicity of the process (i.e. application of a 3×3 FIR filter), the method provides an effective image enhancement technique providing the degradation of the image conforms to a light diffusion (strong scattering) model.

B. General Solution

If we record an image at a time $t = T$ then by Taylor expanding I at $t = 0$ we can write

$$I(x,y,0) = I(x,y,T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n \left[\frac{\partial^n}{\partial t^n} I(x,y,t)\right]_{t=T}$$

The high emphasis filter derived earlier is obtained by neglecting terms in the series above for $n > 1$ giving an approximate solution for the de-diffused image I_0. If we include all the terms in this series, then an exact solution for I_0 can be obtained. This can be done by noting that (from the diffusion equation)

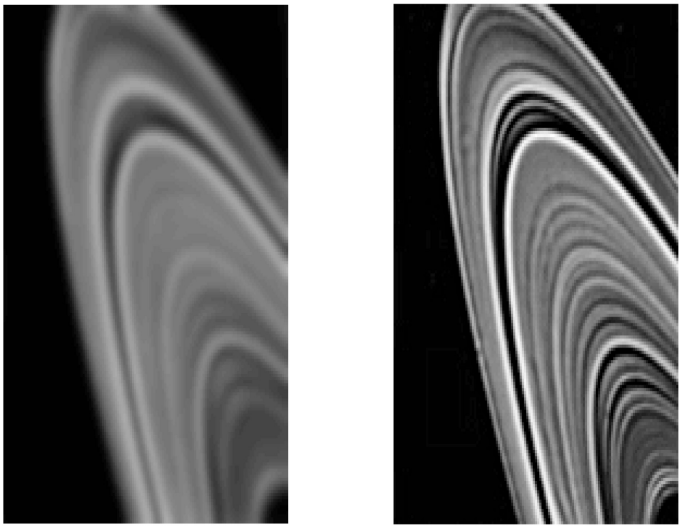

Fig. 3. Original image (left) - rings of Saturn - and an enhanced image (right) using the high emphasis filter.

$$\frac{\partial^2 I}{\partial t^2} = D\nabla^2 \frac{\partial I}{\partial t} = D^2 \nabla^4 I$$

$$\frac{\partial^3 I}{\partial t^3} = D\nabla^2 \frac{\partial^2 I}{\partial t^2} = D^3 \nabla^6 I$$

and so on. In general we can write

$$\left[\frac{\partial^n}{\partial t^n} I(x,y,t)\right]_{t=T} = D^n \nabla^{2n} I(x,y,T).$$

Substituting this result into the series for I_0 given above, we get

$$I_0(x,y) = I(x,y,T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (DT)^n \nabla^{2n} I(x,y,T)$$

and for $DT = 1$

$$I_0 = I - \nabla^2 I + \frac{1}{2!}\nabla^4 I - \frac{1}{3!}\nabla^6 I + ...$$

From this result, we can design FIR filters for the higher order terms. Since

$$\nabla^2 I_{ij} = I_{(i+1)j} + I_{(i-1)j} + I_{i(j+1)} + I_{i(j-1)} - 4I_{ij} = J_{ij}$$

then

$$\begin{aligned}
\nabla^4 I_{ij} = \nabla^2 J_{ij} &= J_{(i+1)j} + J_{(i-1)j} + J_{i(j+1)} + J_{i(j-1)} - 4J_{ij} \\
&= I_{(i+2)j} + I_{ij} + I_{(i+1)(j+1)} + I_{(i+1)(j-1)} - 4I_{(i+1)j} \\
&+ I_{ij} + I_{(i-2)j} + I_{(i-1)(j+1)} + I_{(i-1)(j-1)} - 4I_{(i-1)j} \\
&+ I_{(i+1)(j+1)} + I_{(i-1)(j+1)} + I_{i(j+2)} + I_{ij} - 4I_{i(j+1)} \\
&+ I_{(i+1)(j-1)} + I_{(i-1)(j-1)} + I_{ij} + I_{i(j-2)} - 4I_{i(j-1)} \\
&- 4I_{(i+1)j} - 4I_{(i-1)j} - 4I_{i(j+1)} + 4I_{i(j-1)} + 16I_{ij} \\
&= 20I_{ij} + I_{(i+2)j} + 2I_{(i+1)(j+1)} + 2I_{(i+1)(j-1)} - 8I_{(i+1)j} \\
&+ I_{(i-2)j} + 2I_{(i-1)(j+1)} + 2I_{(i-1)(j-1)} - 8I_{(i-1)j} + I_{i(j+2)}
\end{aligned}$$

$$-8I_{i(j+1)} + I_{i(j-2)} - 8I_{i(j-1)}.$$

In terms of a convolution kernel, the result above can be written as

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, given the convolution kernel associated with the first order solution $I-\nabla^2 I$, the convolution kernel associated with the second order solution $I-\nabla^2 I+\frac{1}{2}\nabla^4 I$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 5 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ \frac{1}{2} & -4 & 10 & -4 & \frac{1}{2} \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -10 & 2 & 0 \\ 1 & -10 & 30 & -10 & 1 \\ 0 & 2 & -10 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

To compute the convolution kernel associated with the third order solution $f-\nabla^2 f+\frac{1}{2}\nabla^4 f-\frac{1}{6}\nabla^6 f$, we use the same method as above to evaluate $\nabla^6 I_{ij}$ to obtain

$$\frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 15 & -3 & 0 & 0 \\ 0 & -3 & 24 & -87 & 24 & -3 & 0 \\ -1 & 15 & -87 & 202 & -87 & 15 & -1 \\ 0 & -3 & 24 & -87 & 24 & -3 & 0 \\ 0 & 0 & -3 & 15 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

An example of the application of these filters is given in Figure 4 which shows the result of diffusing a image by applying a Gaussian low-pass filter and then restoring the image using the first (high emphasis) and second order FIR filter given above.

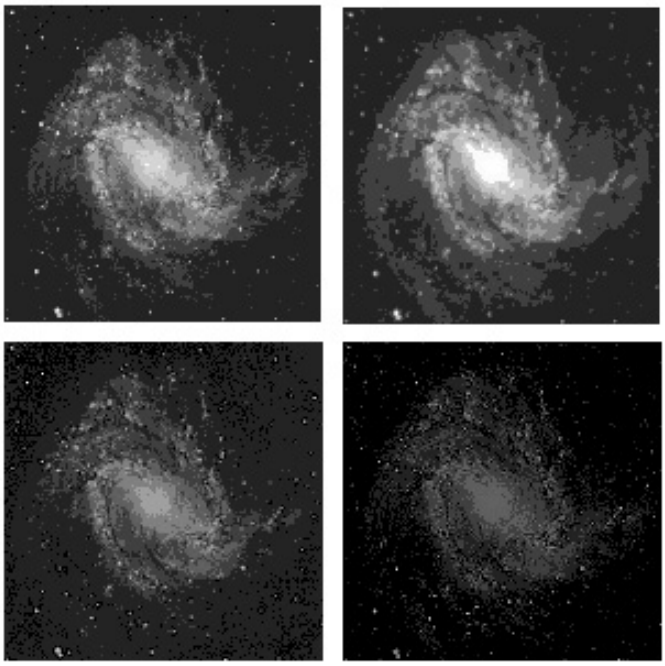

Fig. 4. Original 256×256 image (top-left) - M83 galaxy; result after applying a Gaussian low-pass filter (top-right); output after application of the first order (high emphasis) FIR filter (bottom-left); output after application of the second order FIR filter (bottom-right).

VI. Fractional Diffusion

A. Random Walk Processes

The purpose of revisiting random walk processes is that it provides a useful conceptual reference for introducing fractional diffusion and an appreciation of the use of the fractional diffusion equation, an equation that arises through the generalisation of coherent and incoherent random walk processes into a single model.

In the Nineteenth Century, the Scottish botanist, Robert Brown, discovered (observing through a microscope) the motion exhibited by small particles (pollen grains) that is immersed in a liquid. Each particle follows a random walk as a result of the elastic collisions it has with ensembles of liquid molecules which are them selves in a state of random motion. Brownian motion is the basis of modelling all kinds of statistical fluctuations, most prominently in the field of gambling. However, it was many years after Brown's discovery that work was undertaken to provide a quantitative description associated with this motion. The first work of its type was undertaken by Albert Einstein and published in 1905. The basic idea is to consider a random walk in which the mean value of each step is a but where there is no correlation in the direction of the walk from one step to the next. That is, the direction taken by the walker from one step to next can be in any direction described by an angle between 0 and 2π radians - for a walk in the plane. The angle that is taken at each step is entirely random and all angles are taken to be equally likely. Thus, the PDF of angles between 0 and 2π is given by

$$\Pr[\theta] = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi; \\ 0, & \text{otherwise.} \end{cases}$$

If we consider the random walk to take place in the complex plane, then after n steps, the position of the walker will be determined by a resultant amplitude A and angle Θ given by the sum of all the steps taken, i.e.

$$A\exp(i\Theta) = a\exp(i\theta_1) + a\exp(i\theta_2) + ... + a\exp(i\theta_n)$$

$$= a\sum_{m=1}^{n}\exp(i\theta_m).$$

The problem is to obtain a scaling relationship between A and n. The trick to finding this relationship is to analyse the result of taking the square modulus of $A\exp(i\Theta)$. This provides an expression for the intensity I given by

$$I = a^2\left|\sum_{m=1}^{n}\exp(i\theta_m)\right|^2$$

$$= a^2 \sum_{m=1}^{n} \exp(i\theta_m) \sum_{m=1}^{n} \exp(-i\theta_m)$$

$$= a^2 \left[n + \sum_{j=1, j\neq k}^{n} \exp(i\theta_j) \sum_{k=1}^{n} \exp(-i\theta_k) \right].$$

Now, in a typical term

$$\exp(i\theta_j)\exp(-i\theta_k) = \cos(\theta_j - \theta_k) + i\sin(\theta_i - \theta_k)$$

of the double summation, the functions $\cos(\theta_j - \theta_k)$ and $\sin(\theta_j - \theta_k)$ have random values between ± 1. Consequently, as n becomes larger and larger, the double sum will reduces to zero since more and more of these terms cancel each other out. This insight is the basis for stating that for $n >> 1$

$$I = na^2$$

and the resulting amplitude is therefore given by

$$A = \sqrt{n}a.$$

Thus, A is proportional to the square root of the number of steps taken and if each step is taken over a mean time period, then we obtain the result

$$A(t) = a\sqrt{t}.$$

Clearly, if each step in the walk is in the same direction, then the resulting amplitude after a time t will be at. This is a deterministic result. However, with a random walk, the interpretation of the above result is that $a\sqrt{t}$ is the amplitude associated with the most likely position that the random walker will be after time t. If we imagine many random walkers, each starting out on their 'journey' from the origin of the (complex) plane at $t = 0$, record the distances from the origin of this plane after a set period of time t, then the PDF of A will have a maximum value - the 'mode' of the distribution - that occurs at $a\sqrt{t}$. In the case of a non-random walk, the PDF will consist of a unit spike that occurs at at.

In the (classical) kinetic theory of matter (including gases, liquids, plasmas and some solids), we consider a to be the average distance a particle travels before it randomly collides and scatters off another particle. The scattering process is taken to be entirely elastic, i.e. the interaction does not affect the particle in any way other than to change the direction in which it travels. Thus, a represents the *mean free path* of a particle. The mean free path is a measure how far a particle can travel before scattering with another particle which in turn, is related to the number of particle per unit volume - the density of a gas, for example. If we imagine a particle 'diffusing' through an ensemble of particles, then the mean free path is a measure of the 'diffusivity' of the medium in which the process of diffusion takes place. This is a feature of all classical diffusion processes which can be formulated in terms of the diffusion equation with diffusivity D. The dimensions of diffusivity are $\mathrm{length}^2/\mathrm{time}$ and must be interpreted in terms of a characteristic distance of the process which varies with the square root of time.

Suppose we now consider the three-dimensional diffusion of light to be based on a three-dimensional random walk. Each scattering event is taken to be a point of the random walk in which a ray of light changes its direction randomly (any direction between 0 and 4π radians). The light field is taken to be composed of a complex of rays, each of which propagates through the diffuser in a way that is incoherent and uncorrelated in time. If this is the case, then the propagation of light can be considered to analogous to a process of (classical) diffusion and instead of modelling the process in terms of the (inhomogeneous) wave equation

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})} \frac{\partial^2}{\partial t^2} \right) u(\mathbf{r}, t) = 0$$

with intensity given by $I(\mathbf{r}, t) =\mid u(\mathbf{r}, t) \mid^2$ we can consider the intensity to be given by the solution of the homogeneous diffusion equation

$$\left(\nabla^2 - \frac{1}{D} \frac{\partial}{\partial t} \right) I(\mathbf{r}, t) = 0$$

with initial condition $I(\mathbf{r}, t) = I_0(\mathbf{r})$ at $t = 0$. This assumes that the diffusivity D is constant throughout the diffuser which in turn assumes that $\Pr[c(\mathbf{r})]$ for a random scattering model (based on a solution to the wave equation) is the same throughout the diffuser and thus, the autocorrelation function $\Gamma(\mathbf{r})$ required to compute the intensity.

Although the discussion above has been presented for the case of light, the principle remains the same for the case of any form of electromagnetic wavefield, for example, or indeed for the propagation/diffusion of information in general. Thus, for some random walk process whose macroscopic characteristic are defined by a field u, if the process is diffusive, then the field u is characterised by the operator

$$\nabla^2 - \frac{1}{D} \frac{\partial}{\partial t}$$

and, if the process is propagative, then it is characterised by the operator

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}.$$

In multiple wave scattering theory, we consider a wavefront travelling through space and scattering from a site that changes the direction of propagation. The mean free path is taken to be the average number of wavelengths taken by the wavefront to propagate from one interaction to another as described by the free space Green's function. After scattering from many sites, the wavefront can be considered to have diffused through the 'diffuser'. Here, the mean free path is a measure of the density of scattering sites, which in turn, is a measure of the diffusivity of the material - an optical diffuser for example.

B. Hurst Processes

We have considered random processes that characterise fully coherent (propagative) and fully incoherent (diffusive) behaviour and through the physical interpretation of such processes we have related them to differential operators associated with the corresponding macroscopic behaviour. For a random walk model in the plane, $A(t) = at$ for a coherent walk and $A(t) = a\sqrt{t}$ for an incoherent walk. What would be the result if the walk is neither coherent or incoherent but partially

coherent/incoherent? In other words, suppose the random walk exhibited a bias with regard to the distribution of angles used to change the direction. What would be the effect on the scaling law $\sqrt{t}$? Intuitively, one expects that as the distribution of angles reduces, the corresponding walk becomes more and more coherent, exhibiting longer and longer time correlations until the process conforms to the scaling law t. Conceptually, scaling models associated with the intermediate case(s) should be based on a generalisation of the scaling laws $\sqrt{t}$ and t to the form t^H where $0.5 \leq H < 1$. This reasoning is the basis for generalising the random walk processes considered so far, the exponent H being known as the Hurst exponent or 'dimension'.

H E Hurst (1900-1978) was an English civil engineer who designed dams and worked on the Nile river dam projects in the 1920s and 1930s. He studied the Nile so extensively that some Egyptians reportedly nicknamed him 'the father of the Nile.' The Nile river posed an interesting problem for Hurst as a hydrologist. When designing a dam, hydrologists need to estimate the necessary storage capacity of the resulting reservoir. An influx of water occurs through various natural sources (rainfall, river overflows etc.) and a regulated amount needs to be released for primarily agricultural purposes, for example, the storage capacity of a reservoir being based on the net water flow. Hydrologists usually begin by assuming that the water influx is random, a perfectly reasonable assumption when dealing with a complex ecosystem. Hurst, however, had studied the 847-year record that the Egyptians had kept of the Nile river overflows, from 622 to 1469. He noticed that large overflows tended to be followed by large overflows until abruptly, the system would then change to low overflows, which also tended to be followed by low overflows. There appeared to be cycles, but with no predictable period. Standard statistical analysis of the day revealed no significant correlations between observations, so Hurst developed his own methodology.

Hurst was aware of Einstein's (1905) work on Brownian motion (the erratic path followed by a particle suspended in a fluid) who observed that the distance R the particle covers increased with the square root of time, i.e.

$$R(t) \propto \sqrt{t}$$

where R is the range (equivalent to the amplitude for a walk in the complex plane) covered in time t. It results, from the fact that increments are identically and independently distributed random variables. Hurst's idea was to use this property to test the Nile River's overflows for randomness. His method was as follows: Begin with a time series x_i (with $i = 1, 2, ..., n$) which in Hurst's case was annual discharges of the Nile River. Next, create the adjusted series, $y_i = x_i - \bar{x}$ (where $\bar{x}$ is the mean of x_i). Cumulate this time series to give

$$Y_i = \sum_{j=1}^{i} y_j$$

such that the start and end of the series are both zero and there is some curve in between. (The final value, Y_n has to be zero if the mean is zero.) Then, define the range to be the maximum minus the minimum value of this time series,

$$R_n = \max(Y_i) - \min(Y_i).$$

This adjusted range, R_n is the distance the systems travels for the time index n, i.e. the distance covered by a random walker if the data set y_i were the set of steps. Einstein's equation $R_n = a\sqrt{n}$ will apply provided that the time series x_i is independent for increasing values of n. However, Einstein's equation only applies to series that are in Brownian motion. Hurst's contribution was to generalize this equation to

$$(R/S)_n = an^H$$

where S is the standard deviation for the same n observations and a is a constant. We define a Hurst process to be a process with a (fairly) constant H value. The quotient R/S is referred to as the 'rescaled range' because it has zero mean and is expressed in terms of local standard deviations. In general, the value of R/S increases according to a power law value equal to H known as the Hurst exponent.

Rescaling the adjusted range was a major innovation. Hurst originally performed this operation to enable him to compare diverse phenomenon. Rescaling, fortunately, also allows us to compare time periods many years apart in a range of time series. It is the relative change and not the change itself that is of interest. Rescaled range analysis can also describe time series that have no characteristic scale. By considering the logarithmic version of Hurst's equation, i.e.

$$\log(R/S)_n = \log a + H\log(n)$$

it is clear that the Hurst exponent can be estimated by plotting $\log(R/S)_n$ against the $\log(n)$ and solving for the gradient with a least squares fit, for example. If the system were independently distributed, then $H = 0.5$. Hurst found that the exponent for the Nile River was $H = 0.91$, i.e. the rescaled range increases at a faster rate than the square root of time. This meant that the system was covering more distance than a random process would and therefore the annual discharges of the Nile had to be correlated.

It is important to appreciate that this method makes no prior assumptions about any underlying distributions, it simply tells us how the system is scaling with respect to time. So how do we interpret the Hurst exponent? We know that $H = 0.5$ is consistent with an independently distributed system. The range $0.5 < H \leq 1$, implies a persistent time series, and a persistent time series is characterized by positive correlations. Theoretically, what happens today will ultimately have a lasting effect on the future. The range $0 < H \leq 0.5$ indicates anti-persistence which means that the time series covers less ground than a random process. In other words, there are negative correlations. For a system to cover less distance, it must reverse itself more often than a random process.

Hurst analysed all the data he could including rainfall, sunspots, mud sediments, tree rings and others. In all cases, Hurst found H to be greater than 0.5. He was intrigued that H often took a value of about 0.7 and Hurst suspected that some universal phenomenon was taking place. He carried out some experiments using numbered cards. The values of the cards

were chosen to simulate a probability density function with finite moments, i.e. $0, \pm 1, \pm 3, \pm 5, \pm 7 \text{and} \pm 9$. He first verified that the time series generated by summing the shuffled cards gave $H = 0.5$. To simulate a bias random walk, he carried out the following steps.

1) Shuffle the deck and cut it once, noting the number, say n.
2) Replace the card and re-shuffle the deck.
3) Deal out 2 hands of 26 cards, A and B.
4) Replace the lowest n cards of deck B with the highest n cards of deck A, thus biasing deck B to the level n.
5) Place a joker in deck B and shuffle.
6) Use deck B as a time series generator until the joker is cut, then create a new biased hand.

Hurst undertook 1000 trials of 100 hands and calculated $H = 0.72$. We can think of the process as follows: we first bias each hand, which is determined by a random cut of the pack; then, we generate the time series itself, which is another series of random cuts; then, the joker appears, which again occurs at random. Despite all of these random events $H = 0.72$ would always appear. This is called the 'joker effect'. The joker effect, as described above, demonstrates a tendency for data of a certain magnitude to be followed by more data of approximately the same magnitude, but only for a fixed and random length of time. A natural example of this phenomenon is in weather systems. Good weather and bad weather tend to come in waves or cycles (as in a heat wave for example). This does not mean that weather is periodic, which it is clearly not. We use the term 'non-periodic cycle' to describe cycles of this kind (with no fixed period). Thus, Hurst processes exhibit trends that persist until the equivalent of the joker comes along to change that bias in magnitude and/or direction. In other words rescaled range analysis can be used to characterise a time series that contains within it, many different short-lived trends or biases (both in size and direction). The process continues in this way giving a constant Hurst exponent, sometimes with flat episodes that correspond to the average periods of the non-periodic cycles, depending on the distribution of actual periods.

The generalisation of Einstein's equation $A(t) = a\sqrt{t}$ by Hurst to the form $A(t) = at^H, 0 < H \leq 1$ was necessary in order for Hurst to analyse the apparent random behaviour of the annual rise and fall of the Nile river for which Einstein's model was inadequate. In considering this generalisation, Hurst paved the way for an appreciation that most natural stochastic phenomena which, at first site, appear random, have certain trends that can be identified over a given period of time. In other words, many natural random patterns have a bias to them that leads to time correlations in their stochastic behaviour, a behaviour that is not an inherent characteristic of a random walk model and fully diffusive processes in general.

C. The Fractional Diffusion Equation

Given that incoherent random walks, where $A(t) = a\sqrt{t}$, describe processes whose macroscopic behaviour is characterised by the diffusion equation, then, by induction, Hurst processes, where $A(t) = at^H$, $H \in (0, 1]$, should be characterised by generalizing the diffusion operator

$$\nabla^2 - \sigma \frac{\partial}{\partial t}$$

to the fractional form

$$\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q}$$

where $q \in [1, 2]$ and $D^q = 1/\sigma^q$ is the fractional diffusivity. Fractional diffusive processes can therefore be interpreted as intermediate between diffusive processes proper (random phase walks with $H = 0.5$; diffusive processes with $q = 1$) and 'propagative process' (coherent phase walks for $H = 1$; propagative processes with $q = 2$). For non-stationary processes, we consider the operator

$$\nabla^2 - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}.$$

It should be noted that the fractional diffusion operator given above is the result of a phenomenology. It is no more (and no less) than a generalisation of a well known differential operator to fractional form which follows from a physical analysis of a fully incoherent random process and it generalisation to fractional form in terms of the Hurst exponent. Unlike the diffusion operator (which is based on accepted and experimentally verifiable physical laws - Fourier's law of thermal condition, for example) this approach to introducing a fractional differential operator is based on postulation alone. It is therefore similar to certain other operators, a notable example being Schrödinger's operator in quantum mechanics, i.e.

$$\frac{\hbar^2}{2m}\nabla^2 - i\hbar\frac{\partial}{\partial t}.$$

In order to work with fractional derivatives, it is necessary to briefly review the fractional calculus which for completeness, is provided in Appendix I.

D. Solution to the Fractional Diffusion Equation

Consider the fractional diffusion equation for the intensity I of a wavefield given by

$$D^q \nabla^2 I(\mathbf{r}, t) = \frac{\partial^q}{\partial t^q} I(\mathbf{r}, t)$$

where D is the fractional diffusivity and $I_0(\mathbf{r}) = I(\mathbf{r}, t = 0)$ (the initial condition). For $q = 1$, the solution to this equation in the infinite domain (see Section III) for dimensions $n = 1, 2$ and 3 is (with $\sigma = 1/D$)

$$I(\mathbf{r}_0, \tau) = \sigma \int I_0(\mathbf{r}) G(\mathbf{r} \mid \mathbf{r}_0, \tau) d^n \mathbf{r}.$$

where

$$G(R, \tau) = \frac{1}{\sigma}\left(\frac{\sigma}{4\pi\tau}\right)^{\frac{n}{2}} \exp\left[-\left(\frac{\sigma R^2}{4\tau}\right)\right] H(\tau).$$

which is the solution to

$$\left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right) G(R, \tau) = -\delta^n(R)\delta(\tau).$$

For the fractional diffusion equation, we consider the same basic solution but where the Green's function is given by the solution of

$$\left(\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q}\right) G(R,\tau) = -\delta^n(R)\delta(\tau)$$

where $\sigma^q = 1/D^q$. Using the Fourier based operator for a fractional derivative (see Appendix I), we can transform this equation into the form

$$(\nabla^2 + \Omega_q^2) g(\mathbf{r} \mid \mathbf{r}_0, \omega) = -\delta^n(\mathbf{r} - \mathbf{r}_0)$$

where

$$g(\mathbf{r} \mid \mathbf{r}_0, \omega) = \int_{-\infty}^{\infty} G(\mathbf{r} \mid \mathbf{r}_0, \tau) \exp(i\omega\tau) d\tau,$$

$$\Omega_q^2 = -i\omega\sigma, \quad \Omega_q = \pm i(i\omega\sigma)^{q/2}.$$

Note that for $q = 2$, this equation becomes

$$(\nabla^2 + k^2) g(\mathbf{r} \mid \mathbf{r}_0, \omega) = \delta^n(\mathbf{r} - \mathbf{r}_0)$$

where $k = \pm\omega\sigma$. This equation defines the Green's function for the time independent wave operator in n dimensions, the 'out going' Green's functions being given by [19], [20]

$n = 1$:

$$g(r \mid r_0, k) = \frac{i}{2k} \exp(ik \mid r - r_0 \mid);$$

$n = 2$:

$$g(\mathbf{r} \mid \mathbf{r}_0, k) = \frac{i}{4} H_0(k \mid \mathbf{r} - \mathbf{r}_0 \mid)$$

$$\simeq \frac{1}{\sqrt{8\pi}} \exp(i\pi/4) \frac{\exp(ik \mid \mathbf{r} - \mathbf{r}_0 \mid)}{\sqrt{k \mid \mathbf{r} - \mathbf{r}_0 \mid}}, \quad k \mid \mathbf{r} - \mathbf{r}_0 \mid >> 1$$

where H_0 is the Hankel function, and

$n = 3$:

$$g(\mathbf{r} \mid \mathbf{r}_0, k) = \frac{1}{4\pi \mid \mathbf{r} - \mathbf{r}_0 \mid} \exp(ik \mid \mathbf{r} - \mathbf{r}_0 \mid), \quad n = 3.$$

Generalizing these results, for $q \in [1, 2]$, by writing the exponential function in its series form, with $R = \mid \mathbf{r} - \mathbf{r}_0 \mid$ we have, for $\Omega_q = i(i\omega\sigma)^{q/2}$,

$n = 1$:

$$G(R,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{2\Omega_q} \exp(i\Omega_q R) \exp(i\omega\tau) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega i \frac{\exp(i\omega\tau)}{2(i\omega\sigma)^{q/2}} \left(1 - R(i\omega\sigma)^{q/2} + \frac{R^2}{2!}(i\omega\sigma)^q - ...\right)$$

$$= \frac{1}{2\sigma^{q/2}} \frac{1}{\tau^{1-(q/2)}} - \frac{1}{2} R\delta(\tau)$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2(n+1)!} R^{n+1} \sigma^{nq/2} \delta^{qn/2}(\tau);$$

$n = 2$:

$$G(R,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega\tau) \frac{\exp(i\pi/4)}{\sqrt{8\pi}} \frac{\exp[-(i\omega\sigma)^{q/2} R]}{\sqrt{iR}(i\omega\sigma)^{q/4}}$$

$$= \frac{1}{\sqrt{8\pi R}} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega\tau) ...$$

$$... \left(\frac{1}{(i\omega\sigma)^{q/4}} - (i\omega\sigma)^{q/4} R + \frac{1}{2!}(i\omega\sigma)^{3q/4} R^2 - ...\right)$$

$$= \frac{1}{\sqrt{8\pi R}} \frac{1}{\sigma^{q/4}\tau^{1-q/4}} - \sqrt{\frac{R}{8\pi}} \sigma^{q/4} \delta^{q/4}(\tau)$$

$$+ \frac{1}{\sqrt{8\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} R^{(2n+1)/2} \sigma^{3nq/4} \delta^{3nq/4}(\tau);$$

$n = 3$:

$$G(R,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega\tau) \frac{\exp[-(i\omega\sigma)^{q/2} R]}{4\pi R}$$

$$= \frac{1}{4\pi R} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega\tau) [1 - (i\omega\sigma)^{q/2} R + \frac{1}{2!}(i\omega\sigma)^q R^2 - ...]$$

$$= \frac{\delta(\tau)}{4\pi R} - \frac{1}{4\pi} \sigma^{q/2} \delta^{q/2}(\tau)$$

$$+ \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} R^n \sigma^{(n+1)q/2} \delta^{(n+1)q/2}.$$

These are the Green's functions for the fractional diffusion equation in one-, two- and three-dimensions. Simplification of these infinite sums can be addressed be considering suitable asymptotics, the most significant of which (for arbitrary values of R) is the case when the (fractional) diffusivity D is large. In particular, we note that as $\sigma \to 0$,

$$G(R,\tau) = \frac{1}{2\sigma^{q/2}\tau^{1-(q/2)}} - \frac{1}{2} R\delta(\tau), \quad n = 1;$$

$$G(R,\tau) = \frac{1}{\sqrt{8\pi R}\sigma^{q/4}\tau^{1-(q/4)}}, \quad n = 2;$$

$$G(R,\tau) = \frac{\delta(\tau)}{4\pi R}, \quad n = 3.$$

Thus, in two-dimensions, we can consider a solution to the fractional diffusion equation

$$\left(D^q \nabla^2 - \frac{\partial^q}{\partial t^q}\right) I(\mathbf{r}, t) = 0, \quad I(\mathbf{r}, t = 0) = I_0(\mathbf{r})$$

of the form (for $t_0 = 0$ and at time $t = T$)

$$I(x,y) = \frac{1}{2\sqrt{2\pi}} \frac{1}{(DT)^{1-q/4}} \frac{1}{(x^2+y^2)^{\frac{1}{4}}} \otimes\otimes I_0(x,y),$$

$$D \to \infty$$

which should be compared to the solution to the two-dimensional diffusion equation, i.e.

$$I(x,y) = \frac{1}{4\pi DT} \exp\left[-\left(\frac{x^2+y^2}{4DT}\right)\right] \otimes_2 I_0(x,y).$$

Observe that when the diffusivity is large and the diffusion time $t = T$ is small such that $DT = 1$, the difference between an image obtained by a full two-dimensional diffuser and a fractional diffuser is compounded in the difference between

the convolution of the initial image with (ignoring scaling) the functions $\exp(-R^2/4)$ and $1/\sqrt{R}$. Compared with the Gaussian, the function $R^{-1/2}$ decays more rapidly and hence will have broader spectral characteristics leading to an output that is less blurred than that produced by the convolution of the input with a Gaussian which, in the context of the fractional diffusion model introduced, is to be expected.

E. Optical Fractional Diffusers

Optical diffusers are used in a range of applications including the de-pixelation of Liquid Crystal Displays (LCDs) which becomes especially important when the LCD is composed of relatively few elements and is viewed at close range, e.g. LCD goggles. A common technique is to produce a thin film that is composed of a randomly distributed complex of scatterers (micro-spheroids whose relative permittivity is a weak perturbation of the body of the film) that is over-layed onto the LCD. The goal is to produce a diffuser that 'manages' the light in such a way that it de-pixelates the LCD while minimizing the angular distribution of light. This requires the manufacture of a fractional optical diffuser, an example of which is given in Figure 5 which shows the effect of a 'light management film' manufactured by Microsharp Corporation Limited (http://www.microsharp.co.uk).

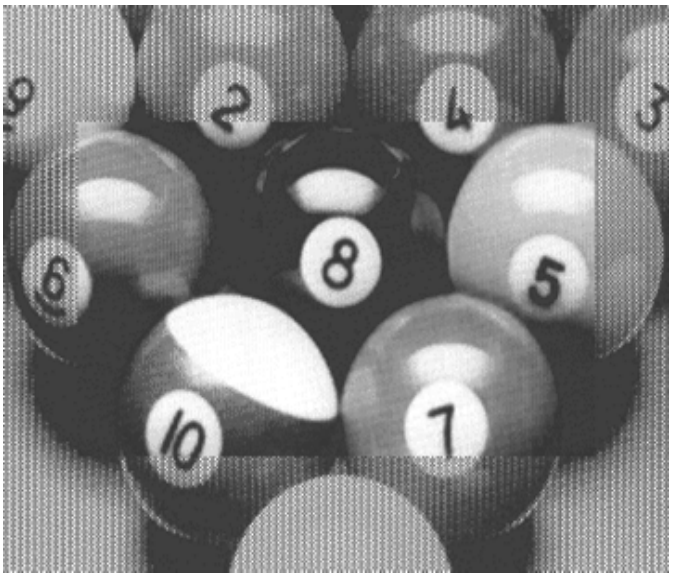

Fig. 5. Illustration of the application of a fractional optical diffuser to a low resolution LCD. The effect of the diffuser is to eliminate the pixelation (central area) generated by the regular LCD lattice (edges) while minimizing the angular field of view.

VII. Fractional De-Diffusion

Let I_0 be represented as a Taylor series at some time $T > 0$, i.e.

$$I(\mathbf{r},0) = I(\mathbf{r},T)+T\left[\frac{\partial}{\partial t}I(\mathbf{r},t)\right]_{t=T} - \frac{T^2}{2!}\left[\frac{\partial^2}{\partial t^2}I(\mathbf{r},t)\right]_{t=T}+...$$

Now, since

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-q}}{\partial t^{1-q}}\frac{\partial^q}{\partial t^q}u$$

then from the fractional diffusion equation

$$\frac{\partial u}{\partial t} = D^q\frac{\partial^{1-q}}{\partial t^{1-q}}\nabla^2 u$$

and

$$\frac{\partial^2}{\partial t^2}u$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(D^q\frac{\partial^{1-q}}{\partial t^{1-q}}\nabla^2 u\right) = D^q\frac{\partial^{1-q}}{\partial t^{1-q}}\nabla^2\frac{\partial u}{\partial t}$$

$$= D^q\frac{\partial^{1-q}}{\partial t^{1-q}}\nabla^2\left(D^q\frac{\partial^{1-q}}{\partial t^{1-q}}\nabla^2 u\right) = D^{2q}\frac{\partial^{1-q}}{\partial t^{1-q}}\left(\frac{\partial^{1-q}}{\partial t^{1-q}}\nabla^4 u\right)$$

so that in general,

$$\frac{\partial^n u}{\partial t^n} = D^{nq}\frac{\partial^{n(1-q)}}{\partial t^{n(1-q)}}\nabla^{2n}u.$$

Now, since (see Appendix I)

$$\frac{\partial^{-q}}{\partial t^{-q}}I(\mathbf{r},t) = \frac{1}{\Gamma(q)t^{1-q}}\otimes I(\mathbf{r},t)$$

we can write the Taylor series for the field at $t = 0$ in terms of the field at $t = T$ as

$$I(\mathbf{r},0) = I(\mathbf{r},T) + \frac{TD^q}{\Gamma(q)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t^{1-q}}\otimes\nabla^2 I(\mathbf{r},t)\right)\right]_{t=T}$$

$$-\frac{T^2D^{2q}}{2!\Gamma(2q)}\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{t^{1-2q}}\otimes\nabla^4 I(\mathbf{r},t)\right)\right]_{t=T}$$

$$+\frac{T^3D^{3q}}{3!\Gamma(3q)}\left[\frac{\partial^3}{\partial t^3}\left(\frac{1}{t^{1-3q}}\otimes\nabla^6 I(\mathbf{r},t)\right)\right]_{t=T} - ...$$

For the case when $T << 1$,

$$I(\mathbf{r},0) = I(\mathbf{r},T) + \frac{TD^q}{\Gamma(q)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t^{1-q}}\otimes\nabla^2 I(\mathbf{r},t)\right)\right]_{t=T}$$

and under the condition that

$$\left[\frac{\partial}{\partial t}\left(\frac{1}{t^{1-q}}\otimes I(\mathbf{r},t)\right)\right]_{t=T} = I(\mathbf{r},T)$$

we can write

$$I(\mathbf{r},0) = I(\mathbf{r},T) + \frac{TD^q}{\Gamma(q)}\nabla^2 I(\mathbf{r},T).$$

Thus, for an image $I(x,y)$ recorded in the image plane at $z = 0$ say, after the image I_0 has been fractionally diffused over a period of time T, we have

$$I_0(x,y) = I(x,y) + \frac{TD^q}{\Gamma(q)}\nabla^2 I(x,y).$$

VIII. Image Segmentation Metric

The result above provides us with an approach to estimating q given I and I_0 as follows: Let

$$P(x,y) = \mid I_0(x,y) - I(x,y) \mid, \quad \text{and} \quad Q(x,y) = \mid \nabla^2 I(x,y) \mid$$

then with $R(x,y) = P(x,y)/Q(x,y)$,

$$\langle R(x,y) \rangle = \frac{TD^q}{\Gamma(q)}$$

where

$$\langle R(x,y) \rangle = \frac{\int\int R(x,y) dxdy}{\int\int dxdy}.$$

Hence,

$$\ln T - \ln \Gamma(q) + q \ln D = M$$

where M is the metric (i.e. a measure of q) given by

$$M = \ln \langle R \rangle \leq \ln \left(\frac{\langle P \rangle}{\langle Q \rangle} \right)$$

This metric can be used effectively as a quality control measure for the manufacture of fractional optical diffusers (see Figure 5). For an image I which has been formed by the fractional diffusion of a uniform light source in which I_0 is a constant,

$$I - I_0 = \frac{TD^q}{\Gamma(q)} \nabla^2 (I - I_0)$$

and with $J = I - I_0$,

$$M = \ln \left(\frac{\langle J(x,y) \rangle}{\langle \mid \nabla^2 J(x,y) \mid \rangle} \right)$$

which can be applied on a moving window W basis in order to segment an image formed through short time fractional diffusion with variable q, the computation of $\langle I \rangle_{(x,y) \in W}$ (the moving average filter) and $\langle \mid \nabla^2 I \mid \rangle_{(x,y) \in W}$ (moving average of the second order edge detector) being relatively simple.

IX. Conclusions

The use of a fully diffusive process for modelling strong (multiple) scattering has been considered and then extended to model intermediate scattering by generalizing the diffusion equation to fractional order $q \in (1,2)$. The rationale for this approach follows that of a random walk model in which diffusive processes characterized by a $t^{\frac{1}{2}}$ scaling law and propagative processes characterized by a t^1 scaling law are generalized to a scaling law of the form t^H where $\frac{1}{2} < H < 1$ is the Hurst exponent.

The homogeneous diffusion equation provides a series solution to the inverse problem in which a Gaussian blurred image can be restored using appropriate FIR filters that depend on the order of the solution that is considered (i.e. the number of terms in the Taylor series). This approach has been extended to include fractional diffusion as defined by the equation (for an image I)

$$D^q \nabla^2 I(x,y,t) = \frac{\partial^q}{\partial t^q} I(x,y,t)$$

where D is the fractional diffusivity and $I_0(x,y) = I(x,y,t=0)$. By computing the appropriate Green's function for this equation, we have shown that the point spread function of the image I is determined by $R^{-1/2}$, $D >> 1$. An FIR filter (a fractional high emphasis filter) has been designed which scales as $TD^q/\Gamma(q)$ compared with TD for the fully diffusive case when $T << 1$. This has provided the basis for the proposition of a new algorithm for segmenting an image into regions of similarity based on a measure of the parameter q (the metric M) in contrast to those algorithms published in, [15] for example.

Appendix I
Overview of Fractional Calculus

In a famous letter from l'Hospital to Leibnitz written in 1695, l'Hospital asked the following question: 'Given that $d^n f/dt^n$ exists for all integer n, what if n be $\frac{1}{2}$'. The reply from Leibnitz was all the more interesting: 'It will lead to a paradox ... From this paradox, one day useful consequences will be drawn'.

Fractional calculus (e.g. [21], [22], [23] and [24]) has been studied for many years by some of the great names of mathematics since the development of (integer) calculus in the late seventeenth century. Relatively few papers and books exist on such a naturally important subject. However, a study of the works in this area of mathematics clearly show that the ideas used to define a fractional differential and a fractional integral are based on definitions which are in effect, little more than generalizations of results obtained using integer calculus. The classical fractional integral operators are the Riemann-Liouville transform [21]

$$\hat{I}^q f(t) = \frac{1}{\Gamma(q)} \int_{-\infty}^{t} \frac{f(\tau)}{(t-\tau)^{1-q}} d\tau, \quad q > 0$$

and the Weyl transform

$$\hat{I}^q f(t) = \frac{1}{\Gamma(q)} \int_{t}^{\infty} \frac{f(\tau)}{(t-\tau)^{1-q}} d\tau, \quad q > 0$$

where

$$\Gamma(q) = \int_0^\infty t^{q-1} \exp(-t) dt.$$

For integer values of q (i.e. when $q = n$ where n is a non-negative integer), the Riemann-Liouville transform reduces to the standard Riemann integral. This transform is just a (causal) convolution of the function $f(t)$ with $t^{q-1}/\Gamma(q)$. For fractional differentiation, we can perform a fractional integration of appropriate order and then differentiate to an appropriate integer order. The reason for this is that direct fractional differentiation can lead to divergent integrals. Thus, the fractional differential operator $\hat{D}^q$ for $q > 0$ is given by

$$\hat{D}^q f(t) \equiv \frac{d^q}{dt^q} f(t) = \frac{d^n}{dt^n} [\hat{I}^{n-q} f(t)].$$

Another (conventional) approach to defining a fractional differential operator is based on using the formula for n^{th} order

differentiation obtained by considering the definitions for the first, second, third etc. differentials using backward and then generalising the formula by replacing n with q. This approach provides us with the result [21]

$$\hat{D}^q f(t) = \lim_{N\to\infty}\left[\frac{(t/N)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(t-j\frac{t}{N}\right)\right].$$

A review of this result shows that for $q = 1$, this is a point process but for other values it is not, i.e. the evaluation of a fractional differential operator depends on the history of the function in question. Thus, unlike an integer differential operator, a fractional differential operator has 'memory'. Although the memory of this process fades, it does not do so quickly enough to allow truncation of the series in order to retain acceptable accuracy. The concept of memory association can also be seen from the result

$$\hat{D}^q f(t) = \frac{d^n}{dt^n}[\hat{I}^{n-q} f(t)]$$

where

$$\hat{I}^{q-n} f(t) = \frac{1}{\Gamma(n-q)}\int_{-\infty}^{t}\frac{f(\tau)}{(t-\tau)^{1+q-n}}d\tau, \quad n-q>0$$

in which the value of $\hat{I}^{q-n} f(t)$ at a point t depends on the behaviour of $f(t)$ from $-\infty$ to t via a convolution with the kernel $t^{n-q}/\Gamma(q)$. The convolution process is of course dependent on the history of the function $f(t)$ for a given kernel and thus, in this context, we can consider a fractional derivative defined via the result above to have memory.

A. The Laplace Transform and the Half Integrator

It informative at this point to consider the application of the Laplace transform to identify an ideal integrator and then a half integrator. The Laplace transform is given by

$$\hat{L}[f(t)] \equiv F(p) = \int_0^\infty f(t)\exp(-pt)dt$$

and from this result we can derive the transform of a derivative given by

$$\hat{L}[f'(t)] = pF(p) - f(0)$$

and the transform of an integral given by

$$\hat{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{p}F(p).$$

Now, suppose we have a standard time invariant linear system whose input is $f(t)$ and whose output is given by

$$s(t) = f(t)\otimes g(t)$$

where the convolution is causal, i.e.

$$s(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Suppose we let

$$g(t) = H(t) = \begin{cases}1, & t>0;\\ 0, & t<0.\end{cases}$$

Then, $G(p) = 1/p$ and the system becomes an ideal integrator:

$$s(t) = f(t)\otimes H(t) = \int_0^t f(t-\tau)d\tau = \int_0^t f(\tau)d\tau.$$

Now, consider the case when we have a time invariant linear system with an impulse response function by given by

$$g(t) = \frac{H(t)}{\sqrt{t}} = \begin{cases}|\,t\,|^{-1/2}, & t>0;\\ 0, & t<0.\end{cases}$$

The output of this system is $f\otimes g$ and the output of such a system with input $f\otimes g$ is $f\otimes g\otimes g$. Now

$$g(t)\otimes g(t) = \int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \int_0^{\sqrt{t}}\frac{2xdx}{x\sqrt{t-x^2}}$$

$$= 2\left[\sin^{-1}\left(\frac{x}{\sqrt{t}}\right)\right]_0^{\sqrt{t}} = \pi.$$

Hence,

$$\frac{H(t)}{\sqrt{\pi t}}\otimes\frac{H(t)}{\sqrt{\pi t}} = H(t)$$

and the system defined by the impulse response function $H(t)/\sqrt{\pi t}$ represents a 'half-integrator' with a Laplace transform given by

$$\hat{L}\left[\frac{H(t)}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{p}}.$$

This result provides an approach to working with fractional integrators and/or differentiators using the Laplace transform. Fractional differential and integral operators can be defined and used in a similar manner to those associated with conventional or integer order calculus and we now provide an overview of such operators.

B. Operators of Integer Order

The following operators are all well-defined, at least with respect to all test functions $u(t)$ say which are (i) infinitely differentiable and (ii) of compact support (i.e. vanish outside some finite interval).

Integral Operator:

$$\hat{I}u(t) \equiv \hat{I}^1 u(t) = \int_{-\infty}^t u(\tau)d\tau.$$

Differential Operator:

$$\hat{D}u(t) \equiv \hat{D}^1 u(t) = u'(t).$$

Identify Operator:

$$\hat{I}^0 u(t) = u(t) = \hat{D}^0 u(t).$$

Now,

$$\hat{I}[\hat{D}u](t) = \int_{-\infty}^{t} u'(\tau)d\tau = u(t)$$

and

$$\hat{D}[\hat{I}u](t) = \frac{d}{dt}\int_{-\infty}^{t} u(\tau)d\tau = u(t)$$

so that

$$\hat{I}^1\hat{D}^1 = \hat{D}^1\hat{I}^1 = \hat{I}^0.$$

For n (integer) order:

$$\hat{I}^n u(t) = \int_{-\infty}^{t} d\tau_{n-1} ... \int_{-\infty}^{\tau_2} d\tau_1 \int_{-\infty}^{\tau_1} u(\tau)d\tau,$$

$$\hat{D}^n u(t) = u^{(n)}(t)$$

and

$$\hat{I}^n[\hat{D}^n u](t) = u(t) = \hat{D}^n[\hat{I}^n u](t).$$

C. Convolution Representation

Consider the function

$$t_+^{q-1}(t) \equiv \mid t \mid^{q-1} H(t) = \begin{cases} \mid t \mid^{q-1}, & t > 0; \\ 0, & t < 0. \end{cases}$$

which, for any $q > 0$ defines a function that is locally integrable. We can then define an integral of order n in terms of a convolution as

$$\hat{I}^n u(t) = \left(u \otimes \frac{1}{(n-1)!} t_+^{n-1} \right)(t)$$

$$= \frac{1}{(n-1)!}\int_{-\infty}^{t} (t-\tau)^{n-1} u(\tau)d\tau$$

$$= \frac{1}{(n-1)!}\int_{-\infty}^{t} \tau^{n-1} u(t-\tau)d\tau$$

In particular,

$$\hat{I}^1 u(t) = (u \otimes H)(t) = \int_{-\infty}^{t} u(\tau)d\tau.$$

These are classical (absolutely convergent) integrals and the identity operator admits a formal convolution representation, using the delta function, i.e.

$$\hat{I}^0 u(t) = \int_{-\infty}^{\infty} \delta(\tau) u(t-\tau)d\tau$$

where

$$\delta(t) = \hat{D}H(t).$$

Similarly,

$$\hat{D}^n u(t) \equiv \hat{I}^{-n} u(t) = \int_{-\infty}^{\infty} \delta^{(n)}(\tau) u(t-\tau)d\tau = u^{(n)}(t).$$

On the basis of the material discussed above, we can now formally extend the integral operator to fractional order and consider the operator

$$\hat{I}^q u(t) = \frac{1}{\Gamma(q)}\int_{-\infty}^{\infty} u(\tau) t_+^{q-1}(t-\tau)d\tau$$

$$= \frac{1}{\Gamma(q)}\int_{-\infty}^{t} u(\tau) t_+^{q-1}(t-\tau)d\tau$$

where

$$\Gamma(q) = \int_0^{\infty} t^{q-1}\exp(-t)dt, \quad q > 0$$

with the fundamental property that

$$\Gamma(q+1) = q\Gamma(q).$$

Here, I^q is an operator representing a time invariant linear system with impulse response function $t_+^{q-1}(t)$ and transfer function $1/p^q$. For the cascade connection of I^{q_1} and I^{q_2} we have

$$\hat{I}^{q_1}[\hat{I}^{q_2} u(t)] = \hat{I}^{q_1+q_2} u(t).$$

This classical convolution integral representation holds for all real $q > 0$ (and formally for $q = 0$, with the delta function playing the role of an impulse function and with a transfer function equal to the constant 1).

D. Fractional Differentiation

For $0 < q < 1$, if we define the (Riemann-Liouville) derivative of order q as

$$\hat{D}^q u(t) \equiv \frac{d}{dt}[\hat{I}^{1-q} u](t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{-\infty}^{t} (t-\tau)^{-q} u(\tau)d\tau,$$

then,

$$\hat{D}^q u(t) = \frac{1}{\Gamma(1-q)}\int_{-\infty}^{t} (t-\tau)^{-q} u'(\tau)d\tau \equiv \hat{I}^{1-q} u'(t).$$

Hence,

$$\hat{I}^q[\hat{D}^q u] = \hat{I}^q[\hat{I}^{1-q} u'] = \hat{I}^1 u' = u$$

and $\hat{D}^q$ is the formal inverse of the operator $\hat{I}^q$. Given any $q > 0$, we can always write $\lambda = n - 1 + q$ and then define

$$\hat{D}^\lambda u(t) = \frac{1}{\Gamma(1-q)}\frac{d^n}{dt^n}\int_{-\infty}^{t} u(\tau)(t-\tau)^{-q}d\tau.$$

D^q is an operator representing a time invariant linear system consisting of a cascade combination of an ideal differentiator

and a fractional integrator of order $1-q$. For D^{λ} we replace the single ideal differentiator by n such that

$$\hat{D}^0 u(t) = \frac{1}{\Gamma(1)} \frac{d}{dt} \int_{-\infty}^{t} u(\tau) d\tau = u(t) \equiv \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$$

and

$$\hat{D}^n u(t) = \frac{1}{\Gamma(1)} \frac{d^{n+1}}{dt^{n+1}} \int_{-\infty}^{t} u(\tau) d\tau$$

$$= u^{(n)}(t) \equiv \int_{-\infty}^{\infty} u(\tau)\delta^{(n)}(t-\tau)d\tau.$$

In addition to the conventional and classical definitions of fractional derivatives and integrals, more general definitions are available including the Erdélyi-Kober fractional integral [25]

$$\frac{t^{-p-q+1}}{\Gamma(q)} \int_{0}^{t} \frac{\tau^{p-1}}{(t-\tau)^{1-q}} f(\tau)d\tau, \quad q>0, \quad p>0$$

which is a generalisation of the Riemann-Liouville fractional integral and the integral

$$\frac{t^{p}}{\Gamma(q)} \int_{t}^{\infty} \frac{\tau^{-q-p}}{(\tau-t)^{1-q}} f(\tau)d\tau, \quad q>0, \quad p>0$$

which is a generalization of the Weyl integral. Further definitions exist based on the application of hypergeometric functions and operators involving other special functions such as the Maijer G-function and the Fox H-function [26]. Moreover, all such operators leading to a fractional integral of the Riemann-Liouville type and the Weyl type to have the general forms (through induction)

$$\hat{I}^q f(t) = t^{q-1} \int_{-\infty}^{t} \Phi\left(\frac{\tau}{t}\right) \tau^{-q} f(\tau) d\tau$$

and

$$\hat{I}^q f(t) = t^{-q} \int_{t}^{\infty} \Phi\left(\frac{t}{\tau}\right) \tau^{q-1} f(\tau) d\tau$$

respectively, where the kernel Φ is an arbitrary continuous function so that the integrals above make sense in sufficiently large functional spaces. Although there are a number of approaches that can be used to define a fractional differential/integral, there is one particular definition, which in terms of its 'ease of use' and wide ranging applications, is of significant value and is based on the Fourier transform, i.e.

$$\frac{d^q}{dt^q} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^q F(\omega) \exp(i\omega t) d\omega$$

where $F(\omega)$ is the Fourier transform of $f(t)$. When $q = 1, 2, 3...$, this definition reduces to a well known result that is trivial to derive in which, for example, the 'filter' $i\omega$ (for the case when $q = 1$) is referred to as a 'differentiator'. When $q < 0$, we have a definition for the fractional integral where, in the case of $q = -1$, for example, the filter $(i\omega)^{-1}$ is an 'integrator'. When $q = 0$ we just have $f(t)$ expressed in terms of its Fourier transform $F(\omega)$. This Fourier based definition of a fractional derivative can be extended further to include a definition for a 'fractional Laplacian' ∇^q where for n dimensions

$$\nabla^q \equiv -\frac{1}{(2\pi)^n} \int d^n\mathbf{k} k^q \exp(i\mathbf{k}\cdot\mathbf{r}), \quad k = \mid \mathbf{k} \mid$$

and $\mathbf{r}$ is an n-dimensional vector. This is the fractional Riesz operator. It is designed to provide a result that is compatible with the case of $q = 2$ for $n > 1$, i.e. $\nabla^2 \Longleftrightarrow -k^2$ (which is the reason for introducing the negative sign). Another equally valid generalization is

$$\nabla^q \equiv \frac{1}{(2\pi)^n} \int d^n\mathbf{k} (ik)^q \exp(i\mathbf{k}\cdot\mathbf{r}), \quad k = \mid \mathbf{k} \mid$$

which introduces a q dependent phase factor of $\pi q/2$ into the operator.

E. Fractional Dynamics

Mathematical modelling using (time dependent) fractional Partial Differential Equations (PDEs) is generally known as fractional dynamics [27], [28]. A number of works have shown a close relationship between fractional diffusion equations of the type (where p is the space-time dependent PDF and σ is the generalized coefficient of diffusion)

$$\nabla^2 p - \sigma \frac{\partial^q}{\partial t^q} p = 0, \quad 0 < q \leq 1$$

and

$$\nabla^q p - \sigma \frac{\partial}{\partial t} p = 0, \quad 0 < q \leq 2$$

and continuous time random walks with either temporal or spatial scale invariance (fractal walks). Fractional diffusion equations of this type have been shown to produce a framework for the description of anomalous diffusion phenomena and Lévy-type behaviour. In addition, certain classes of fractional differential equations are known to yield Lévy-type distributions. For example, the normalized one-sided Lévy-type PDF

$$p(x) = \frac{a^q}{\Gamma(q)} \frac{\exp(-a/x)}{x^{1+q}}, \quad a>0, \quad x>0$$

is a solution of the fractional integral equation

$$x^{2q} p(x) = a^q \hat{I}^{-q} p(x)$$

where

$$\hat{I}^{-q} p(x) = \frac{1}{\Gamma(q)} \int_{0}^{x} \frac{p(y)}{(x-y)^{1-q}} dy, \quad q>0.$$

Another example involves the solution to the anomalous diffusion equation

$$\nabla^q p - \tau \frac{\partial}{\partial t} p = 0, \quad 0 < q \leq 2.$$

Fourier transforming this equation and using the fractional Riesz operator defined previously, we have

$$\frac{\partial}{\partial t}P(k,t) = -\frac{1}{\tau}k^q P(k,t)$$

which has the general solution

$$P(k,t) = \exp(-t \mid k \mid^q /\tau), \quad t > 0.$$

which is the characteristic function of a Lévy distribution. This analysis can be extended further by considering a fractal based generalization of the Fokker-Planck-Kolmogorov (FPK) equation [29]

$$\frac{\partial^q}{\partial t^q}p(x,t) = \frac{\partial^\beta}{\partial x^\beta}[s(x)p(x,t)]$$

where s is an arbitrary function and $0 < q \leq 1$, $0 < \beta \leq 2$. This equation is referred to as the fractal FPK equation; the standard FPK equation is of course recovered for $q = 1$ and $\beta = 2$. The characteristic function associated with $p(x,t)$ is given by

$$P(k,t) = \exp(-ak^\beta t^q)$$

where a is a constant which again, is a characteristic of a Lévy distribution. Finally, d-dimensional fractional master equations of the type [30], [31]

$$\frac{\partial^q}{\partial t^q}p(\mathbf{r},t) = \sum_{\mathbf{s}} w(\mathbf{r}-\mathbf{s})p(\mathbf{s},t), \quad 0 < q \leq 1$$

can be used to model non-equilibrium phase transitions where p denotes the probability of finding the diffusing entity at a position $\mathbf{r} \in R^d$ at time t (assuming that it was at the origin $\mathbf{r} = \mathbf{0}$ at time $t = 0$) and w are the fractional transition rates which measure the propensity for a displacement $\mathbf{r}$ in units of $1/(\text{time})^q$. These equations conform to the general theory of continuous time random walks and provide models for random walks of fractal time.

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