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Application of the Fractal Market Hypothesis for Modelling Macroeconomic Time Series

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Application of the Fractal Market Hypothesis for Macroeconomic Time Series Analysis

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***Abstract*— This paper explores the conceptual background to financial time series analysis and financial signal processing in terms of the Efficient Market Hypothesis. By revisiting the principal conventional approaches to market analysis and the reasoning associated with them, we develop a Fractal Market Hypothesis that is based on the application of non-stationary fractional dynamics using an operator of the type**

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

where σ^{-1} is the fractional diffusivity and q is the Fourier dimension which, for the topology considered, (i.e. the one-dimensional case) is related to the Fractal Dimension $1 < D_F < 2$ by $q = 1 - D_F + 3/2$.

We consider an approach that is based on the signal $q(t)$ and its interpretation, including its use as a macroeconomic volatility index. In practice, this is based on the application of a moving window data processor that utilises Orthogonal Linear Regression to compute q from the power spectrum of the windowed data. This is applied to FTSE close-of-day data between 1980 and 2007 which reveals plausible correlations between the behaviour of this market over the period considered and the amplitude fluctuations of $q(t)$ in terms of a macroeconomic model that is compounded in the operator above.

***Index Terms*— Fractional Diffusion Equation, Time Series Analysis, Macroeconomic Modelling, Volatility Index**

I. INTRODUCTION

THE application of statistical techniques for analysing financial time series is a well established practice. This includes a wide range of stochastic modelling methods and the use of certain partial differential equations for describing financial systems (e.g. the Black-Scholes equation for financial derivatives). Attempts to develop stochastic models for financial time series, which are essentially digital signals composed of 'tick data'[1] [1], [2] can be traced back to the early Twentieth Century when Louis Bachelier [3] proposed that fluctuations in the prices of stocks and shares (which appeared to be yesterday's price plus some random change) could be viewed in terms of random walks in which price changes were entirely independent of each other. Thus, one of the simplest models for price variation is based on the sum of independent random numbers. This is the basis for Brownian motion (i.e. the random walk motion first observed by the Scottish Botanist, Robert Brown [4], who, in 1827, noted that pollen grains suspended in water appear to undergo continuous jittery motion - a result of the random impacts on the pollen grains by water molecules) in which the random numbers are considered to conform to a normal distribution.

With macroeconomic financial systems, the magnitude of a change in price du tends to depend on the price u itself. We therefore need to modify the Brownian random walk model to include this observation. In this case, the logarithm of the price change as a function of time t (which is also assumed to conform to a normal distribution) is modelled according to the equation

$$\frac{du}{u} = \alpha dv + \beta dt \quad \text{or} \quad \frac{d}{dt} \ln u = \beta + \alpha \frac{dv}{dt} \qquad (1)$$

where α is the volatility, dv is a sample from a normal distribution and β is a drift term which reflects the average rate of growth of an asset[2]. Here, the relative price change of an asset is equal to a random value plus an underlying trend component - a 'log-normal random walk', e.g [5] - [8].

Brownian motion models have the following basic properties: (i) statistical stationarity of price increments in which samples of Brownian motion taken over equal time increments can be superimposed onto each other in a statistical sense; (ii) scaling of price where samples of Brownian motion corresponding to different time increments can be suitably re-scaled such that they too, can be superimposed onto each other in a statistical sense. Such models fail to predict extreme behaviour in financial time series because of the intrinsic assumption that such time series conform to a normal distribution, i.e. Gaussian processes that are stationary in which the statistics - the standard deviation, for example - do not change with time.

Random walk models, which underpin the so called Efficient Market Hypothesis (EMH) [9]-[12], have been the basis for financial time series analysis since the work of Bachelier in the late Nineteenth Century. Although the Black-Scholes equation [13], developed in the 1970s for valuing options, is deterministic (one of the first financial models to achieve determinism), it is still based on the EMH, i.e. stationary Gaussian statistics. The EMH is based on the principle that the current price of an asset fully reflects all available information relevant to it and that new information is immediately incorporated into the price. Thus, in an efficient market, the modelling

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[1]Data that provides traders with daily tick-by-tick data - time and sales - of trade price, trade time, and volume traded, for example, at different sampling rates as required.

[2]Note that both α and β may very with time t

of asset prices is concerned with modelling the arrival of new information. New information must be independent and random, otherwise it would have been anticipated and would not be new. The arrival of new information can send 'shocks' through the market (depending on the significance of the information) as people react to it and then to each other's reactions. The EMH assumes that there is a rational and unique way to use the available information and that all agents possess this knowledge. Further, the EMH assumes that this 'chain reaction' happens effectively instantaneously. These assumptions are clearly questionable at any and all levels of a complex financial system.

The EMH implies independence of price increments and is typically characterised by a normal of Gaussian Probability Density Function (PDF) which is chosen because most price movements are presumed to be an aggregation of smaller ones, the sums of independent random contributions having a Gaussian PDF. However, it has long been known that financial time series do not follow random walks. An illustration of this is given in Figure 1 which shows a (discrete) financial signal $u(t)$ (data obtained from [14]), the log derivative of this signal $d\log u(t)/dt$ and a Gaussian distributed random signal. The log derivative is considered in order to: (i) eliminate the characteristic long term exponential growth of the signal; (ii) obtain a signal on the daily price differences[3] in accord with the left hand side term of equation (1). Clearly, there is a marked difference in the characteristics of a real financial signal and a random Gaussian signal. This simple comparison indicates a failure of the statistical independence assumption which underpins the EMH.

The shortcomings of the EMH model (as illustrated in Figure 1) include: failure of the independence and Gaussian distribution of increments assumption, clustering, apparent non-stationarity and failure to explain momentous financial events such as 'crashes' leading to recession and, in some extreme cases, depression. These limitations have prompted a new class of methods for investigating time series obtained from a range of disciplines. For example, Re-scaled Range Analysis (RSRA), e.g. [15], [16], which is essentially based on computing the Hurst exponent [17], is a useful tool for revealing some well disguised properties of stochastic time series such as persistence (and anti-persistence) characterized by non-periodic cycles. Non-periodic cycles correspond to trends that persist for irregular periods but with a degree of statistical regularity often associated with non-linear dynamical systems. RSRA is particularly valuable because of its robustness in the presence of noise. The principal assumption associated with RSRA is concerned with the self-affine or fractal nature of the statistical character of a time-series rather than the statistical 'signature' itself. Ralph Elliott first reported on the fractal properties of financial data in 1938 (e.g. [18] and reference therein). He was the first to observe that segments of financial time series data of different sizes could be scaled in such a way that they were statistically the same producing so called Elliot waves. Since then, many different self-affine models for price variation have been developed, often based on (dynamical)

[3]The gradient is computed using forward differencing.

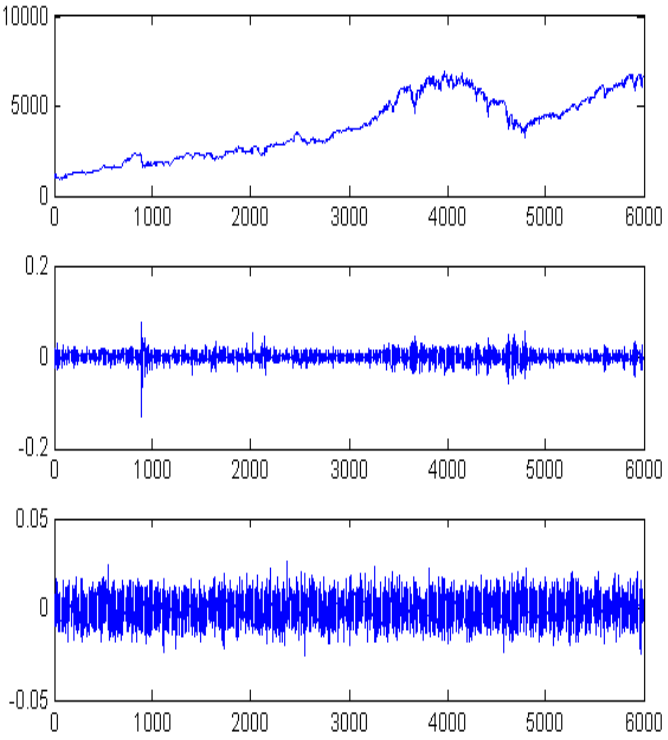

Fig. 1. Financial time series for the FTSE value (close-of-day) from 02-04-1984 to 12-12-2007 (top), the log derivative of the same time series (centre) and a Gaussian distributed random signal (bottom).

Iterated Function Systems (IFS). These models can capture many properties of a financial time series but are not based on any underlying causal theory of the type attempted in this paper.

A good stochastic financial model should ideally consider all the observable behaviour of the financial system it is attempting to model. It should therefore be able to provide some predictions on the immediate future behaviour of the system within an appropriate confidence level. Predicting the markets has become (for obvious reasons) one of the most important problems in financial engineering. Although, at least in principle, it might be possible to model the behaviour of each individual agent operating in a financial market, one can never be sure of obtaining all the necessary information required on the agents themselves and their modus operandi. This principle plays an increasingly important role as the scale of the financial system, for which a model is required, increases. Thus, while quasi-deterministic models can be of value in the understanding of micro-economic systems (with known 'operational conditions'), in an ever increasing global economy (in which the operational conditions associated with the fiscal policies of a given nation state are increasingly open), we can take advantage of the scale of the system to describe its behaviour in terms of functions of random variables.

II. Market Analysis

The stochastic nature of financial time series is well known from the values of the stock market major indices such as the FTSE (Financial Times Stock Exchange) in the UK, the

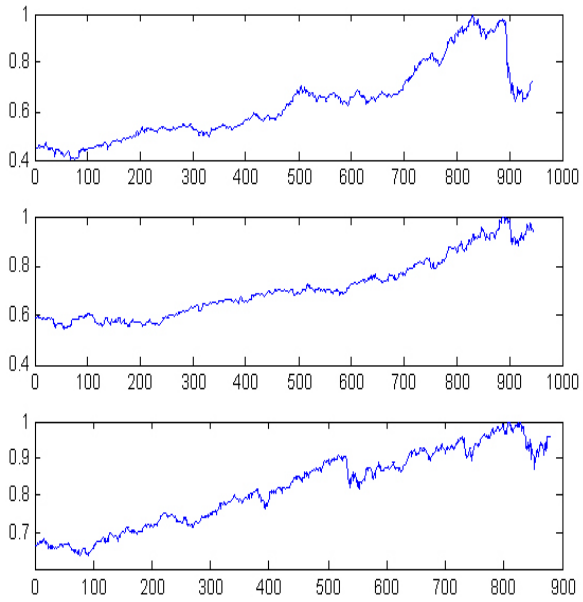

Fig. 2. Evolution of the 1987, 1997 and 2007 financial crashes. Normalised plots (i.e. where the data has been rescaled to values between 0 and 1 inclusively) of the daily FTSE value (close-of-day) for 02-04-1984 to 24-12-1987 (top), 05-04-1994 to 24-12-1997 (centre) and 02-04-2004 to 24-09-2007 (bottom)

Dow Jones in the US which are frequently quoted. A principal aim of investors is to attempt to obtain information that can provide some confidence in the immediate future of the stock markets often based on patterns of the past, patterns that are ultimately based on the interplay between greed and fear. One of the principle components of this aim is based on the observation that there are 'waves within waves' and 'events within events' that appear to permeate financial signals when studied with sufficient detail and imagination. It is these repeating patterns that occupy both the financial investor and the systems modeller alike and it is clear that although economies have undergone many changes in the last one hundred years, the dynamics of market data do not appear to change significantly (ignoring scale). For example, Figure 2 shows the build up to three different 'crashes', the one of 1987 and that of 1997 (both after approximately 900 days) and what may turn out to be a crash of 2007 (at the time of writing this paper). The similarity in behaviour of these signals is remarkable and is indicative of the quest to understand economic signals in terms of some universal phenomenon from which appropriate (macro) economic models can be generated. In an efficient market, only the revelation of some dramatic information can cause a crash, yet post-mortem analysis of crashes typically fail to (convincingly) tell us what this information must have been.

In modern economies, the distribution of stock returns and anomalies like market crashes emerge as a result of considerable complex interaction. In the analysis of financial time series, it is inevitable that assumptions need to be made to make the derivation of a model possible. This is the most vulnerable stage of the process. Over simplifying assumptions lead to unrealistic models. There are two main approaches to financial modelling: The first approach is to look at the statistics of market data and to derive a model based on an educated guess of the 'mechanics' of the market. The model can then be tested using real data. The idea is that this process of trial and error helps to develop the right theory of market dynamics. The alternative is to 'reduce' the problem and try to formulate a microscopic model such that the desired behaviour 'emerges', again, by guessing agents' strategic rules. This offers a natural framework for interpretation; the problem is that this knowledge may not help to make statements about the future unless some methods for describing the behaviour can be derived from it. Although individual elements of a system cannot be modelled with any certainty, global behaviour can sometimes be modelled in a statistical sense provided the system is complex enough in terms of its network of interconnection and interacting components.

In complex systems, the elements adapt to the aggregate pattern they co-create. As the components react, the aggregate changes, as the aggregate changes the components react anew. Barring the reaching of some asymptotic state or equilibrium, complex systems keep evolving, producing seemingly stochastic or chaotic behaviour. Such systems arise naturally in the economy. Economic agents, be they banks, firms, or investors, continually adjust their market strategies to the macroscopic economy which their collective market strategies create. It is important to appreciate that there is an added layer of complexity within the economic community: Unlike many physical systems, economic elements (human agents) react with strategy and foresight by considering the implications of their actions (some of the time!). Although we can not be certain whether this fact changes the resulting behaviour, we can be sure that it introduces feedback which is the very essence of both complex systems and chaotic dynamical systems that produce fractal structures.

The link between dynamical systems, chaos and the economy is an important one because it is dynamical systems that illustrate that local randomness and global determinism can co-exist. Global determinism can be considered, at least in a qualitative sense, in terms of broad social issues and the reaction of distinct groups to changing social attitudes, particularly in economies that have traditionally been enhanced by an open and often pro-active policy towards the immigration of peoples from diverse cultural backgrounds. For example, in 1656, Cromwell permitted an open door policy to immigration from continental Europe, partly in an attempt to enhance the economy of England that had been severely compromised by the English Civil wars of 1642-46 and 1648-49 [19]. The long term effect of this was to provide a new financial infrastructure that laid the foundations for future economic development. It is arguable that Cromwell's policy is the principal reason why the 'English revolution' of the Eighteenth Century was primarily an industrial one. Issues concerning the current and future economic welfare of England may then be appreciated in terms of the attitudes and values associated with new waves of immigrants and the policy of appeasement adopted at government level.

Complex systems can be split into two categories: equilibrium and non-equilibrium. Equilibrium complex systems, undergoing a phase transition, can lead to 'critical states' that often exhibit random fractal structures in which the statistics of the 'field' are scale invariant. For example, when ferromagnets are heated, as the temperature rises, the spins of the electrons which contribute to the magnetic field gain energy and begin to change in direction. At some critical temperature, the spins form a random vector field with a zero mean and a phase transition occurs in which the magnetic field averages to zero. But the field is not just random, it is a self-affine random field whose statistical distribution is the same at different scales, irrespective of the characteristics of the distribution. Non-equilibrium complex systems or 'driven' systems give rise to 'self organised critical states', an example is the growing of sand piles. If sand is continuously supplied from above, the sand starts to pile up. Eventually, little avalanches will occur as the sand pile inevitably spreads outwards under the force of gravity. The temporal and spatial statistics of these avalanches are scale invariant.

Financial markets can be considered to be non-equilibrium systems because they are constantly driven by transactions that occur as the result of new fundamental information about firms and businesses. They are complex systems because the market also responds to itself, often in a highly non-linear fashion, and would carry on doing so (at least for some time) in the absence of new information. The 'price change field' is highly non-linear and very sensitive to exogenous shocks and it is probable that all shocks have a long term effect. Market transactions generally occur globally at the rate of hundreds of thousands per second. It is the frequency and nature of these transactions that dictate stock market indices, just as it is the frequency and nature of the sand particles that dictates the statistics of the avalanches in a sand pile. These are all examples of random scaling fractals [20]-[28].

III. Does a Macroeconomy have memory?

When faced with a complex process of unknown origin, it is usual to select an independent process such as Brownian motion as a working hypothesis where the statistics and probabilities can be estimated with great accuracy. However, using traditional statistics to model the markets assumes that they are games of chance. For this reason, investment in securities is often equated with gambling. In most games of chance, many degrees of freedom are employed to ensure that outcomes are random. In the case of a simple dice, a coin or roulette wheel, for example, no matter how hard you may try, it is physically impossible to master your roll or throw such that you can control outcomes. There are too many non-repeatable elements (speeds, angles and so on) and non-linearly compounding errors involved. Although these systems have a limited number of degrees of freedom, each outcome is independent of the previous one. However, there are some games of chance that involve memory. In Blackjack, for example, two cards are dealt to each player and the object is to get as close as possible to 21 by twisting (taking another card) or sticking. In a 'bust' (over 21), the player loses; the winner is the player that stays closest to 21. Here, memory is introduced because the cards are not replaced once they are taken. By keeping track of the cards used, one can assess the shifting probabilities as play progresses. This game illustrates that not all gambling is governed by Gaussian statistics. There are processes that have long-term memory, even though they are probabilistic in the short term. This leads directly to the question, does the economy have memory? A system has memory if what happens today will affect what happens in the future.

Memory can be tested by observing correlations in the data. If the system today has no affect on the system at any future time, then the data produced by the system will be independently distributed and there will be no correlations. A function that characterises the expected correlations between different time periods of a financial signal $u(t)$ is the Auto-Correlation Function (ACF) defined by

$$A(t) = u(t) \odot u(t) = \int_{-\infty}^{\infty} u(\tau)u(\tau - t)d\tau.$$

where $\odot$ denotes that the correlation operation. This function can be computed either directly (evaluation of the above integral) or via application of the power spectrum using the correlation theorem

$$u(t) \odot u(t) \Longleftrightarrow \mid U(\omega) \mid^2$$

where $\Longleftrightarrow$ denotes transformation from real space t to Fourier space ω (the angular frequency), i.e.

$$U(\omega) = \mathcal{F}[u(t)] = \int_{-\infty}^{\infty} u(t) \exp(-i\omega t)dt$$

where $\mathcal{F}$ denotes the Fourier transform operator. The power spectrum $\mid U(\omega) \mid^2$ characterises the amplitude distribution of the correlation function from which we can estimate the time span of memory effects. This also offers a convenient way to calculate the correlation function (by taking the inverse Fourier transform of $\mid U(\omega) \mid^2$). If the power spectrum has more power at low frequencies, then there are long time correlations and therefore long-term memory effects. Inversely, if there is greater power at the high frequency end of the spectrum, then there are short-term time correlations and evidence of short-term memory. White noise, which characterises a time series with no correlations over any scale, has a uniformly distributed power spectrum.

Since prices movements themselves are a non-stationary process, there is no ACF as such. However, if we calculate the ACF of the price increments du/dt, then we can observe how much of what happens today is correlated with what happens in the future. According to the EMH, the economy has no memory and there will therefore be no correlations, except for today with itself. We should therefore expect the power spectrum to be effectively constant and the ACF to be a delta function. The power spectra and the ACFs of log price changes $d \log u/dt$ and their absolute value $\mid d \log u/dt \mid$ for the FTSE 100 index (daily close) from 02-04-1984 to 24-09-2007 is given in Figure 3. The power spectra of the data is not constant with rogue spikes (or groups of spikes) at the intermediate and high frequency portions of the spectrum. For

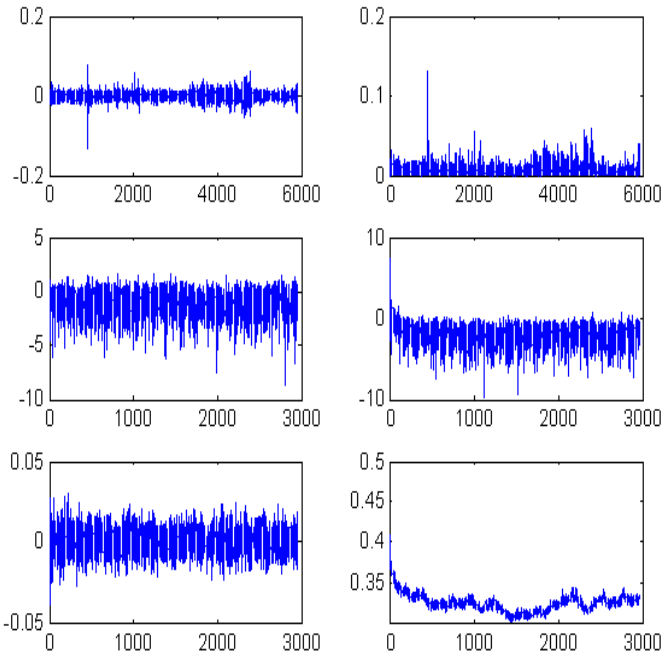

Fig. 3. Log-power spectra and ACFs of log price changes and absolute log price changes for FTSE 100 index (daily close) from 02-04-1984 to 24-09-2007. Top-left: log price changes; top-right: absolute value of log price changes; middle: log power spectra; bottom: ACFs.

the absolute log price increments, there is evidence of a power law at the low frequency end, indicating that there is additional correlation in the signs of the data.

The ACF of the log price changes is relatively featureless, indicating that the excess of low frequency power within the signal has a fairly subtle effect on the correlation function. However, the ACF of the absolute log price changes contains a number of interesting features. It shows that there are a large number of short range correlations followed by an irregular decline up to approximately 1500 days after which the correlations start to develop again, peaking at about 2225 days. The system governing the magnitudes of the log price movements clearly has a better long-term memory than it should. The data used in this analysis contains 5932 daily price movements and it is therefore improbable that these results are coincidental and correlations of this, or any similar type, what ever the time scale, effectively invalidates the independence assumption of the EMH.

IV. Stochastic Modelling of Macroeconomic Data

Developing mathematical models to simulate stochastic processes has an important role in financial analysis and information systems in general where it should be noted that information systems are now one of the most important aspects in terms of regulating financial systems, e.g. [29]-[32]. A good stochastic model is one that accurately predicts the statistics we observe in reality, and one that is based upon some well defined rationale. Thus, the model should not only describe the data, but also help to explain and understand the system.

There are two principal criteria used to define the characteristics of a stochastic field: (i) The PDF or the Characteristic Function (i.e. the Fourier transform of the PDF); the Power Spectral Density Function (PSDF). The PSDF is the function that describes the envelope or shape of the power spectrum of a signal. In this sense, the PSDF is a measure of the field correlations. The PDF and the PSDF are two of the most fundamental properties of any stochastic field and various terms are used to convey these properties. For example, the term ‘zero-mean white Gaussian noise’ refers to a stochastic field characterized by a PSDF that is effectively constant over all frequencies (hence the term ‘white’ as in ‘white light’) and has a PDF with a Gaussian profile whose mean is zero.

Stochastic fields can of course be characterized using transforms other than the Fourier transform (from which the PSDF is obtained) but the conventional PDF-PSDF approach serves many purposes in stochastic systems theory. However, in general, there is no general connectivity between the PSDF and the PDF either in terms of theoretical prediction and/or experimental determination. It is not generally possible to compute the PSDF of a stochastic field from knowledge of the PDF or the PDF from the PSDF. Hence, in general, the PDF and PSDF are fundamental but non-related properties of a stochastic field. However, for some specific statistical processes, relationships between the PDF and PSDF can be found, for example, between Gaussian and non-Gaussian fractal processes [33] and for differentiable Gaussian processes [34].

There are two conventional approaches to simulating a stochastic field. The first of these is based on predicting the PDF (or the Characteristic Function) theoretically (if possible). A pseudo random number generator is then designed whose output provides a discrete stochastic field that is characteristic of the predicted PDF. The second approach is based on considering the PSDF of a field which, like the PDF, is ideally derived theoretically. The stochastic field is then typically simulated by filtering white noise. A ‘good’ stochastic model is one that accurately predicts both the PDF and the PSDF of the data. It should take into account the fact that, in general, stochastic processes are non-stationary. In addition, it should, if appropriate, model rare but extreme events in which significant deviations from the norm occur.

New market phenomenon result from either a strong theoretical reasoning or from compelling experimental evidence or both. In econometrics, the processes that create time series such as the FTSE have many component parts and the interaction of those components is so complex that a deterministic description is simply not possible. As in all complex systems theory, we are usually required to restrict the problem to modelling the statistics of the data rather than the data itself, i.e. to develop stochastic models. When creating models of complex systems, there is a trade-off between simplifying and deriving the statistics we want to compare with reality and simulating the behaviour through an emergent statistical behaviour. Stochastic simulation allows us to investigate the effect of various traders’ behavioural rules on the global statistics of the market, an approach that provides for a natural interpretation and an understanding of how the amalgamation of certain concepts leads to these statistics.

One cause of correlations in market price changes (and

volatility) is mimetic behaviour, known as herding. In general, market crashes happen when large numbers of agents place sell orders simultaneously creating an imbalance to the extent that market makers are unable to absorb the other side without lowering prices substantially. Most of these agents do not communicate with each other, nor do they take orders from a leader. In fact, most of the time they are in disagreement, and submit roughly the same amount of buy and sell orders. This is a healthy non-crash situation; it is a diffusive (random-walk) process which underlies the EMH and financial portfolio rationalization.

One explanation for crashes involves a replacement for the EMH by the Fractal Market Hypothesis (FMH) which is the basis of the model considered in this paper. The FMH proposes the following: (i) The market is stable when it consists of investors covering a large number of investment horizons which ensures that there is ample liquidity for traders; (ii) information is more related to market sentiment and technical factors in the short term than in the long term - as investment horizons increase and longer term fundamental information dominates; (iii) if an event occurs that puts the validity of fundamental information in question, long-term investors either withdraw completely or invest on shorter terms (i.e. when the overall investment horizon of the market shrinks to a uniform level, the market becomes unstable); (iv) prices reflect a combination of short-term technical and long-term fundamental valuation and thus, short-term price movements are likely to be more volatile than long-term trades - they are more likely to be the result of crowd behaviour; (v) if a security has no tie to the economic cycle, then there will be no long-term trend and short-term technical information will dominate. Unlike the EMH, the FMH states that information is valued according to the investment horizon of the investor. Because the different investment horizons value information differently, the diffusion of information will also be uneven. Unlike most complex physical systems, the agents of the economy, and perhaps to some extent the economy itself, have an extra ingredient, an extra degree of complexity. This ingredient is consciousness.

V. Random Walk Processes

The purpose of revisiting random walk processes is that it provides a useful conceptual reference to the model that is introduced later on in this paper and in particular, appreciation of the use of the fractional diffusion equation for describing self-affine stochastic fields, an equation that arises through the unification of coherent and incoherent random walks. We shall consider a random walk in the plane where the amplitude remains constant but where the phase changes, first by a constant factor and then by a random value between 0 and 2π.

A. Coherent (Constant) Phase Walks

Consider a walk in the (real) plane where the length from one step to another is constant - the amplitude a - and where the direction that is taken after each step is the same. In this simple case, the 'walker' continues in a straight line and after n steps the total length of the path the walker has taken will be just an. We define this value as the resultant amplitude A - the total length of the walk - which will change only by account of the number of steps taken. Thus,

$$A = an.$$

If each step takes a set period of time t to complete, then it is clear that

$$A(t) = at.$$

This scenario is limited by the fact that we are assuming that each step is of precisely the same length and takes precisely the same period of time to accomplish. In general, we consider a to be the mean value of all the step lengths and t to be the cumulative time associated with the average time taken to perform all steps. A walk of this type has a coherence from one step or cluster of steps to the next, is entirely predictable and correlated in time.

If the same walk takes place in the complex plane then the phase θ from one step to the next is the same. Thus, the result is given by

$$A\exp(i\theta) = \sum_n a\exp(i\theta) = na\exp(i\theta).$$

The resultant amplitude is given by na as before and the total phase value is θ. We can also define the intensity which is given by

$$I = \mid A\exp(i\theta) \mid^2 = A^2$$

Thus, as a function of time, the intensity associated with this coherent phase walk is given by

$$I(t) = a^2t^2.$$

Suppose we make the walk slightly more complicated and consider the case where the phase increases by a small constant factor of θ at each step. After n steps, the result will be given by the sum of all the steps taken, i.e.

$$A\exp(i\Theta) = \sum_n a\exp(in\theta)$$

$$= a[1 + \exp(i\theta) + \exp(2i\theta) + ... + \exp[i(n-1)\theta]$$

$$= a\frac{[1-\exp(in\theta)]}{[1-\exp(i\theta)]}$$

$$= a\frac{\exp(in\theta/2)[\exp(-in\theta/2) - \exp(in\theta/2)]}{\exp(i\theta/2)[\exp(-i\theta/2) - \exp(i\theta/2)]}$$

$$= a\exp[i(n-1)\theta/2)]\frac{\sin(n\theta/2)}{\sin\theta/2}.$$

Now, after many steps, when n is large,

$$\alpha = (n-1)\theta/2 \simeq n\theta/2$$

and when the phase change θ is small,

$$\sin(\theta/2) \simeq \frac{\theta}{2} \simeq \frac{\alpha}{n}$$

and we obtain the result

$$A\exp(i\Theta) = na\exp[i((n-1)/2)\theta]\mathrm{sinc}\alpha, \quad \mathrm{sinc}\alpha = \frac{\sin\alpha}{\alpha}.$$

For very small changes in the phase $\theta << 1$, $\mathrm{sinc}\alpha \simeq 1$ and the resultant amplitude A is, as before, given by an or as a function of time, by at.

B. Incoherent (Random) Phase Walk

Incoherent or random phase walks are the basis of modelling many kinds of statistical fluctuations. It is also the principle physical model associated with the stochastic behaviour of an ensemble of particles that collectively exhibit the process of diffusion. The first quantitative description of Brownian motion was undertaken by Albert Einstein and published in 1905 [35]. The basic idea is to consider a random walk in which the mean value of each step is a but where there is no correlation in the direction of the walk from one step to the next. That is, the direction taken by the walker from one step to next can be in any direction described by an angle between 0 and 360 degrees or 0 and 2π radians - for a walk in the plane. The angle that is taken at each step is entirely random and all angles are taken to be equally likely. Thus, the PDF of angles between 0 and 2π is given by

$$\Pr[\theta] = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi; \\ 0, & \text{otherwise.} \end{cases}$$

If we consider the random walk to take place in the complex plane, then after n steps, the position of the walker will be determined by a resultant amplitude A and phase angle Θ given by the sum of all the steps taken, i.e.

$$\begin{aligned} A\exp(i\Theta) &= a\exp(i\theta_1) + a\exp(i\theta_2) + ... + a\exp(i\theta_n) \\ &= a\sum_{m=1}^{n}\exp(i\theta_m). \end{aligned}$$

The problem is to obtain a scaling relationship between A and n. Clearly we should not expect A to be proportional to the number of steps n as is the case with a coherent walk. The trick to finding this relationship is to analyse the result of taking the square modulus of $A\exp(i\Theta)$. This provides an expression for the intensity I given by

$$\begin{aligned} I &= a^2\left|\sum_{m=1}^{n}\exp(i\theta_m)\right|^2 \\ &= a^2\sum_{m=1}^{n}\exp(i\theta_m)\sum_{m=1}^{n}\exp(-i\theta_m) \\ &= a^2\left[n + \sum_{j=1,j\neq k}^{n}\exp(i\theta_j)\sum_{k=1}^{n}\exp(-i\theta_k)\right]. \end{aligned}$$

Now, in a typical term

$$\exp(i\theta_j)\exp(-i\theta_k) = \cos(\theta_j - \theta_k) + i\sin(\theta_j - \theta_k)$$

of the double summation, the functions $\cos(\theta_j - \theta_k)$ and $\sin(\theta_j - \theta_k)$ have random values between ± 1. Consequently, as n becomes larger and larger, the double sum will reduces to zero since more and more of these terms cancel each other out. This insight is the basis for stating that for $n >> 1$

$$I = a^2 n$$

and the resulting amplitude is therefore given by

$$A = a\sqrt{n}.$$

In this case, A is proportional to the square root of the number of steps taken and if each step is taken over a mean time period, then we obtain the result

$$A(t) = a\sqrt{t}.$$

With a coherent walk we can state that the resulting amplitude after a time t will be at. This is a deterministic result. However, with an incoherent random walk, the interpretation of the above result is that $a\sqrt{t}$ is the amplitude associated with the most likely position that the random walker will be after time t. If we imagine many random walkers, each starting out on their 'journey' from the origin of the (complex) plane at $t = 0$, record the distances from the origin of this plane after a set period of time t, then the PDF of A will have a maximum value - the 'mode' of the distribution - that occurs at $a\sqrt{t}$. In the case of a perfectly coherent walk, the PDF will consist of a unit spike that occurs at at.

Figure 4 shows coherent and a incoherent phase walks in the plane. Each position of the walk $(x_j, y_j), \quad j = 1, 2, 3, ..., N$ has been computed using (for $a = 1$)

$$x_j = \sum_{i=1}^{j}\cos(\theta_i)$$

$$y_j = \sum_{i=1}^{j}\sin(\theta_i)$$

where $\theta_i \in [0, 2\pi]$ is uniformly distributed and computed using the standard linear congruential pseudo random number generator

$$x_{i+1} = ax_i \mathrm{mod} P, \quad i = 1, 2, ..., N \tag{2}$$

with $a = 7^7$ and $P = 2^{31} - 1$ and an arbitrary value of x_0 - the 'seed'. For the coherent phase walk

$$\theta_i = \frac{2\pi}{16}\frac{x_i}{\|\mathbf{x}\|_\infty}$$

which limits the angle to a small range between 0 and $\pi/8$ radians[4]. For the incoherent phase walk, the range of values is between 0 and 2π radians, i.e.

$$\theta_i = 2\pi\frac{x_i}{\|\mathbf{x}\|_\infty}$$

VI. Physical Interpretation

In the (classical) kinetic theory of matter (including gases, liquids, plasmas and some solids), we consider a to be the average distance a particle travels before it randomly collides and scatters from another particle. The scattering process is taken to be entirely elastic, i.e. the interaction does not affect the particle in any way other than to change the direction in which it travels. Thus, a represents the *mean free path* of a particle. The mean free path is a measure how far a particle can travel before scattering with another particle which in turn, is related to the number of particle per unit volume - the density

[4] $\|\mathbf{x}\|_\infty$ denote the uniform norm, equivalent to the maximum value of the array vector $\mathbf{x}$.

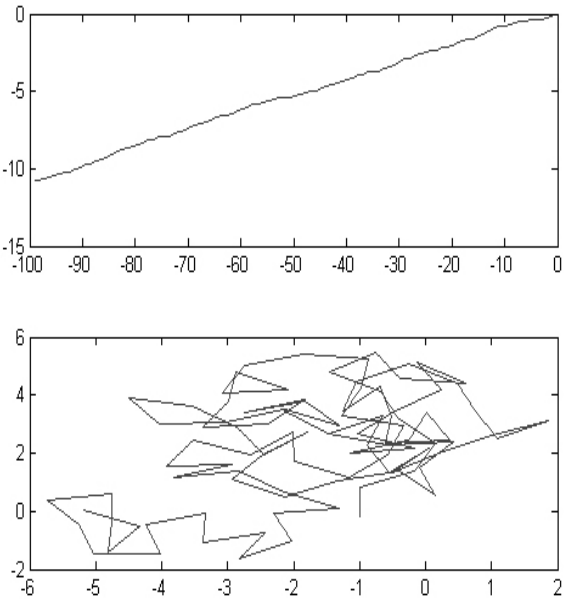

Fig. 4. Examples of a coherent (top) and incoherent (bottom) random walk in the plane for $N = 100$.

of a gas for example. If we imagine a particle 'diffusing' through an ensemble of particles, then the mean free path is a measure of the 'diffusivity' of the medium in which the process of diffusion takes place. This is a feature of all classical diffusion processes which can be formulated in terms of the diffusion equation with diffusivity D. The dimensions of diffusivity are $\mathrm{length}^2/\mathrm{time}$ and may be interpreted in terms of the characteristic distance of a random walk process which varies with the square root of time.

If we consider a wavefront travelling through space and scattering from a site that changes the direction of propagation, then the mean free path can be taken to be the average number of wavelengths taken by the wavefront to propagate from one interaction to another. After scattering from many sites, the wavefront can be considered to have diffused through the 'diffuser'. Here, the mean free path is a measure of the density of scattering sites, which in turn, is a measure of the diffusivity of the material - an optical diffuser, for example.

We can use the random walk model associated with a wavefield to interpret the flow of information through a complex network of 'sites' that are responsible for passing on the information from one site to the next. If a packet of information (e.g. a stream of bits of arbitrary length) travels directly from A to B, then, in terms of the random walk models discussed above, the model associated with this information exchange is 'propagative'; it is a coherent process which is correlated in time and its principal physical characteristic is determined by the speed at which the information flows from A to B. On the other hand, suppose that this information packet is transferred from A to B via information interchange sites C, D,...,Z,... In this case the flow of information is diffusive and is characterised by the diffusivity of the information interchange 'system'. To a first order approximation, the diffusivity will depend on the number of sites that are required to manage the reception and transmission of the information packet. As the number of sites decreases the flow of information becomes more propagative and less diffusive. Thus, we can consider the Internet, for example (albeit a good one) to be a source of information diffusion, not in terms of the diffusion of the information it coveys but in terms of the way in which information packets 'walk through' the network. Further, we can think of the internet itself as being an active medium for the propagation of financial information from one site to another.

A. The Classical Diffusion Equation

The homogeneous diffusion equation is given by (for the one-dimension case x) [36]

$$\left(\frac{\partial^2}{\partial x^2} - \sigma\frac{\partial}{\partial t}\right) u(x,t) = 0$$

for a diffusivity $D = \sigma^{-1}$. The field $u(x,t)$ represents a measurable quantity whose space-time dependence is determined by the random walk of a large ensemble of particles or a multiple scattered wavefield or information flowing through a complex network. We consider an initial value for this field denoted by $u_0 \equiv u(x,0) = u(x,t)$ at $t = 0$. For example, u could be the temperature of a material that starts 'radiating' heat at time $t = 0$ from a point in space x due to a mass of thermally energised particles, each of which undertakes a random walk from the source of heat in which the most likely position of any particle after a time t is proportional to $\sqrt{t}$. In optical diffusion, for example, u denotes the intensity of light. The light wavefield is taken to be composed of an ensemble of wavefronts or rays, each of which undergoes multiple scattering as it propagates through the diffuser. For a single wavefront element, multiple scattering is equivalent to a random walk of that element.

The relationship between a random walk model and the diffusion equation can also be attributed to Einstein [35] who derived the diffusion equation using a random particle model system assuming that the movements of the particles are independent of the movements of all other particles and that the motion of a single particle at some interval of time is independent of its motion at all other times. The derivation is as follows: Let τ be a small interval of time in which a particle moves some distance between λ and $\lambda + d\lambda$ with a probability $P(\lambda)$ where τ is long enough to assume that the movements of the particle in two separate periods of τ are independent. If n is the total number of particles and we assume that $P(\lambda)$ is constant between λ and $\lambda + d\lambda$, then the number of particles which will travel a distance between λ and $\lambda + d\lambda$ in τ is given by

$$dn = nP(\lambda)d\lambda.$$

If $u(x,t)$ is the concentration (number of particles per unit volume) then the concentration at time $t + \tau$ is described by the integral of the concentration of particles which have been

displaced by λ in time τ, as described by the equation above, over all possible λ, i.e.

$$u(x,t+\tau) = \int_{-\infty}^{\infty} u(x+\lambda,t)P(\lambda)d\lambda.$$

Since, τ is assumed to be small, we can approximate $u(x,t+\tau)$ using the Taylor series and write

$$u(x,t+\tau) \simeq u(x,t) + \tau\frac{\partial}{\partial t}u(x,t).$$

Similarly, using a Taylor series expansion of $u(x+\lambda,t)$, we have

$$u(x+\lambda,t) \simeq u(x,t) + \lambda\frac{\partial}{\partial x}u(x,t) + \frac{\lambda^2}{2!}\frac{\partial^2}{\partial x^2}u(x,t)$$

where the higher order terms are neglected under the assumption that if τ is small, then the distance travelled, λ, must also be small. We can then write

$$u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) = u(x,t)\int_{-\infty}^{\infty} P(\lambda)d\lambda$$

$$+\frac{\partial}{\partial x}u(x,t)\int_{-\infty}^{\infty}\lambda P(\lambda)d\lambda + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t)\int_{-\infty}^{\infty}\lambda^2 P(\lambda)d\lambda.$$

For isotropic diffusion, $P(\lambda) = P(-\lambda)$ and so P is an even function with usual normalization condition

$$\int_{-\infty}^{\infty} P(\lambda)d\lambda = 1.$$

As λ is an odd function, the product $\lambda P(\lambda)$ is also an odd function which, if integrated over all values of λ, equates to zero. Thus we can write

$$u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) = u(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t)\int_{-\infty}^{\infty}\lambda^2 P(\lambda)d\lambda$$

so that

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)\int_{-\infty}^{\infty}\frac{\lambda^2}{2\tau}P(\lambda)d\lambda.$$

Finally, defining the diffusivity as

$$D = \int_{-\infty}^{\infty}\frac{\lambda^2}{2\tau}P(\lambda)d\lambda$$

we obtain the diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t).$$

B. The Classical Wave Equation

The wave equation (homogeneous form) is given by (for the one-dimension case) [36]

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(x,t) = 0$$

where c is the wave speed and u denotes the amplitude of the wavefield. A possible solution to this equation is

$$u(x,t) = p(x-ct)$$

which describes a wave with distribution p moving along x at velocity c. For the initial value problem where

$$u(x,0) = v(x), \quad \frac{\partial}{\partial t}u(x,0) = w(x)$$

the (d'Alembert) general solution is given by [36]

$$u(x,t) = \frac{1}{2}[v(x-ct) + v(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} w(\xi)d\xi.$$

This solution is of limited use in that the range of x is unbounded and only applies to the case on an 'infinite string'. For the case when $w = 0$, the solution can be taken to describe two identical waves with amplitude distribution $v(x)$ travelling away from each other. Neither wave is taken to undergo any interaction as it travels along a straight path and thus, after time t the distance travelled will be ct. This is analogous to a walker undertaking a perfectly coherent walk with an average step length of c and after a period of time t reaching a position ct. The point here, is that we can relate the diffusion equation and the wave equation to two types of processes. The diffusion equation describes a field generated by incoherent random processes with no time correlations whereas the wave equation describes a field generated by coherent processes that are correlated in time. One of the aims of this paper is to formulate an equation that models the intermediate case - the fractional diffusion equation - in which random walk process have a directional bias.

VII. Hurst Processes

For a walk in the plane, $A(t) = at$ for a coherent walk and $A(t) = a\sqrt{t}$ for an incoherent walk. However, what would be the result if the walk was neither coherent or incoherent but partial coherent/incoherent? In other words, suppose the random walk exhibited a bias with regard to the distribution of angles used to change the direction. What would be the effect on the scaling law $\sqrt{t}$? Intuitively, one expects that as the distribution of angles reduces, the corresponding walk becomes more and more coherent, exhibiting longer and longer time correlations until the process conforms to a fully coherent walk. A simulation of such an effect is given in Figure 5 which shows a random walk in the (real) plane as the (uniform) distribution of angles decreases. The walk becomes less and less random as the width of the distribution is reduced.

The equivalent effect for a random phase walk in three-dimensions is given in Figure 6. Each position of the walk

$$(x_j, y_j, z_j), \quad j = 1, 2, 3, ..., N$$

has been computed using

$$x_j = \sum_{i=1}^{j} \cos(\theta_i)\cos(\phi_i)$$

$$y_j = \sum_{i=1}^{j} \sin(\theta_i)\cos(\phi_i)$$

$$z_j = \sum_{i=1}^{j} \sin(\phi_i)$$

for $N = 500$. The uniform random number generator used to compute θ_i and ϕ_i is the same - equation (2) - but with different seeds. Conceptually, scaling models associated with the intermediate case(s) should be based on a generalisation of the scaling laws $\sqrt{t}$ and t to the form t^H where $0.5 \leq H < 1$. This reasoning is the basis for generalising the random walk processes considered so far, the exponent H being known as the Hurst exponent or 'dimension'.

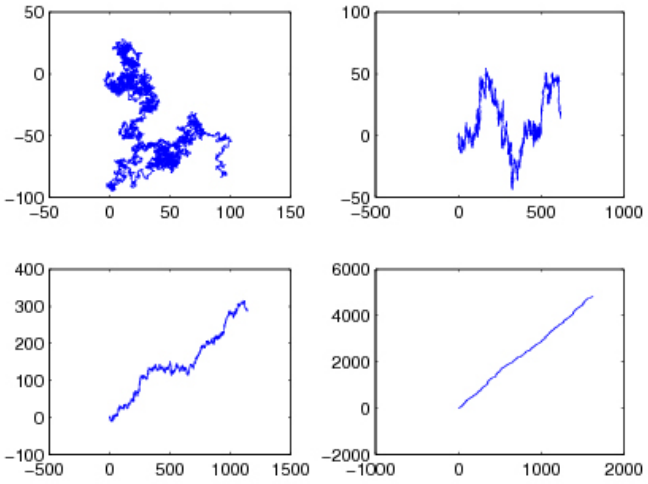

Fig. 5. Random phase walks in the plane for a uniform distribution of angles $\theta_i \in [0, 2\pi]$ (top left), $\theta_i \in [0, 1.9\pi]$ (top right), $\theta_i \in [0, 1.8\pi]$ (bottom left) and $\theta_i \in [0, 1.2\pi]$ (bottom right).

H E Hurst (1900-1978) was an English civil engineer who designed dams and worked on the Nile river dam projects in the 1920s and 1930s. He studied the Nile so extensively that some Egyptians reportedly nicknamed him 'the father of the Nile'. The Nile river posed an interesting problem for Hurst as a hydrologist. When designing a dam, hydrologists need to estimate the necessary storage capacity of the resulting reservoir. An influx of water occurs through various natural sources (rainfall, river overflows etc.) and a regulated amount needs to be released for primarily agricultural purposes, for example, the storage capacity of a reservoir being based on the net water flow. Hydrologists usually begin by assuming that the water influx is random, a perfectly reasonable assumption when dealing with a complex ecosystem. Hurst,

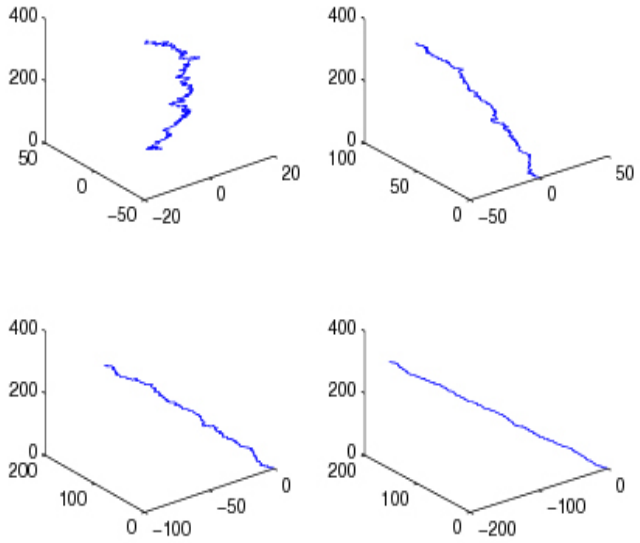

Fig. 6. Three dimensional random phase walks for a uniform distribution of angles $(\theta_i, \phi_i) \in ([0, 2\pi], [0, 2\pi])$ (top left), $(\theta_i, \phi_i) \in ([0, 1.6\pi], [0, 1.6\pi])$ (top right), $(\theta_i, \phi_i) \in ([0, 1.3\pi], [0, 1.3\pi])$ (bottom left) and $(\theta_i, \phi_i) \in ([0, \pi], [0, \pi])$ (bottom right).

however, had studied the 847-year record that the Egyptians had kept of the Nile river overflows, from 622 to 1469. He noticed that large overflows tended to be followed by large overflows until abruptly, the system would then change to low overflows, which also tended to be followed by low overflows. There appeared to be cycles, but with no predictable period. Standard statistical analysis of the day revealed no significant correlations between observations, so Hurst, who was aware of Einstein's work on Brownian motion, developed his own methodology [37] lead to the scaling law t^H. This scaling law makes no prior assumptions about any underlying distributions. It simply tells us how the system is scaling with respect to time. So how do we interpret the Hurst exponent? We know that $H = 0.5$ is consistent with an independently distributed system. The range $0.5 < H \leq 1$, implies a persistent time series, and a persistent time series is characterized by positive correlations. Theoretically, what happens today will ultimately have a lasting effect on the future. The range $0 < H \leq 0.5$ indicates anti-persistence which means that the time series covers less ground than a random process. In other words, there are negative correlations. For a system to cover less distance, it must reverse itself more often than a random process.

VIII. Lévy Processes

The generalisation of Einstein's equation $A(t) = a\sqrt{t}$ by Hurst to the form $A(t) = at^H, 0 < H \leq 1$ was necessary in order for Hurst to analyse the apparent random behaviour of the annual rise and fall of the Nile river for which Einstein's model was inadequate. In considering this generalisation, Hurst paved the way for an appreciation that most natural stochastic phenomena which, at first site, appear random, have certain trends that can be identified over a given period of

time. In other words, many natural random patterns have a bias to them that leads to time correlations in their stochastic behaviour, a behaviour that is not an inherent characteristic of a random walk model and fully diffusive processes in general. This aspect of stochastic field theory was taken up in the late 1930s by the French mathematician Paul Lévy (1886-1971) [38].

Lévy processes are random walks whose distribution has infinite moments. The statistics of (conventional) physical systems are usually concerned with stochastic fields that have PDFs where (at least) the first two moments (the mean and variance) are well defined and finite. Lévy statistics is concerned with statistical systems where all the moments (starting with the mean) are infinite.

Many distributions exist where the mean and variance are finite but are not representative of the process, e.g. the tail of the distribution is significant, where rare but extreme events occur. These distributions include Lévy distributions. Lévy's original approach[5] to deriving such distributions is based on the following question: Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine. The characteristic function (i.e. the Fourier transform) $P(k)$ of such a distribution $p(x)$ was first shown by Lévy to be given by (for symmetric distributions only)

$$P(k) = \exp(-a \mid k \mid^q), \quad 0 < q \leq 2$$

where a is a (positive) constant. If $q = 0$,

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-a) \exp(ikx) dk = \exp(-a)\delta(x)$$

and the distribution is concentrated solely at the origin as described by the delta function $\delta(x)$. When $q = 1$, the Cauchy distribution

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-a \mid k \mid) \exp(ikx) dk = \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

is obtained and when $q = 2$, $p(x)$ is characterized by the Gaussian distribution

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ak^2) \exp(ikx) dk$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{a}} \exp[-x^2/(4a)],$$

whose first and second moments are finite. The Cauchy distribution has a relatively long tail compared with the Gaussian distribution and a stochastic field described by a Cauchy distribution is likely to have more extreme variations when compared to a Gaussian distributed field. For values of q between 0 and 2, Lévy's characteristic function corresponds to a PDF of the form

$$p(x) \sim \frac{1}{x^{1+q}}, x \to \infty.$$

This can be shown as follows[6]: For $0 < q < 1$ and since the characteristic function is symmetric, we have

$$p(x) = \mathrm{Re}[f(x)]$$

where

$$f(x) = \frac{1}{\pi} \int_0^{\infty} e^{ikx} e^{-k^q} dk$$

$$= \frac{1}{\pi} \left(\left[\frac{1}{ix} e^{ikx} e^{-k^q} \right]_{k=0}^{\infty} - \frac{1}{ix} \int_0^{\infty} e^{ikx} (-qk^{q-1} e^{-k^q}) dk \right)$$

$$= \frac{q}{2\pi i x} \int_{-\infty}^{\infty} dk H(k) k^{q-1} e^{-k^q} e^{ikx}, \quad x \to \infty$$

where

$$H(k) = \begin{cases} 1, & k > 0 \\ 0, & k < 0 \end{cases}$$

For $0 < q < 1$, $f(x)$ is singular at $k = 0$ and the greatest contribution to this integral is the inverse Fourier transform of $H(k)k^{q-1}$. Noting that [27]

$$\mathcal{F}^{-1} \left[\frac{1}{(ik)^q} \right] \sim \frac{1}{x^{1-q}}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and that

$$H(k) \Longleftrightarrow \delta(x) + \frac{i}{\pi x} \sim \delta(x), \quad x \to \infty$$

then, using the convolution theorem, we have

$$f(x) \sim \frac{q}{i\pi x} \frac{i^{1-q}}{x^q}$$

and thus

$$p(x) \sim \frac{1}{x^{1+q}}, \quad x \to \infty$$

For $1 < q < 2$, we can integrate by parts twice to obtain

$$f(x) = \frac{q}{i\pi x} \int_0^{\infty} dk k^{q-1} e^{-k^q} e^{ikx}$$

$$= \frac{q}{i\pi x} \left[\frac{1}{ix} k^{q-1} e^{-k^q} e^{ikx} \right]_{k=0}^{\infty}$$

$$+ \frac{q}{\pi x^2} \int_0^{\infty} dk e^{ikx} [(q-1)k^{q-2} e^{-k^q} - q(k^{q-1})^2 e^{-k^q}]$$

$$= \frac{q}{\pi x^2} \int_0^{\infty} dk e^{ikx} [(q-1)k^{q-2} e^{-k^q} - q(k^{q-1})^2 e^{-k^q}], \quad x \to \infty.$$

[5] P Lévy was the research supervisor of B Mandelbrot, the 'inventor' of 'fractal geometry'.

[6] The author acknowledges Dr K I Hopcraft, School of Mathematical Sciences, Nottingham University, England, for his help in deriving this result.

The first term of this result is singular and therefore provides the greatest contribution and thus we can write,

$$f(x) \simeq \frac{q(q-1)}{2\pi x^2} \int_{-\infty}^{\infty} H(k) e^{ikx} (k^{q-2} e^{-k^q}) dk.$$

In this case, for $1 < q < 2$, the greatest contribution to this integral is the inverse Fourier transform of k^{q-2} and hence,

$$f(x) \sim \frac{q(q-1)}{\pi x^2} \frac{i^{2-q}}{x^{q-1}}$$

so that

$$p(x) \sim \frac{1}{x^{1+q}}, \quad x \to \infty$$

which maps onto the previous asymptotic as $q \to 1$ from the above.

For $q \geq 2$, the second moment of the Lévy distribution exists and the sums of large numbers of independent trials are Gaussian distributed. For example, if the result were a random walk with a step length distribution governed by $p(x), \quad q > 2$, then the result would be normal (Gaussian) diffusion, i.e. a Brownian process. For $q < 2$ the second moment of this PDF (the mean square), diverges and the characteristic scale of the walk is lost. This type of random walk is called a Levy flight.

IX. The Fractional Diffusion Equation

We can consider a Hurst process to be a form of fractional Brownian motion based on the generalization

$$A(t) = at^H, \quad H \in (0, 1].$$

Given that incoherent random walks describe processes whose macroscopic behaviour is characterised by the diffusion equation, then, by induction, Hurst processes should be characterised by generalizing the diffusion operator

$$\frac{\partial^2}{\partial x^2} - \sigma \frac{\partial}{\partial t}$$

to the fractional form

$$\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}$$

where $q \in (0, 2]$ and $D = 1/\sigma$ is the fractional diffusivity. Fractional diffusive processes can therefore be interpreted as intermediate between classical diffusive (random phase walks with $H = 0.5$; diffusive processes with $q = 1$) and 'propagative process' (coherent phase walks for $H = 1$; propagative processes with $q = 2$), e.g. [39], [40] and [38] - references therein. Fractional diffusion equations can also be used to model Lévy distributions [41] and fractal time random walks [42], [43]. However, it should be noted that the fractional diffusion operator given above is the result of a phenomenology. It is no more (and no less) than a generalisation of a well known differential operator to fractional form which follows from a physical analysis of a fully incoherent random process and it generalisation to fractional form in terms of the Hurst exponent. Note that the diffusion and wave equations can be derived rigorously from a range of fundamental physical laws (conservation of mass, the continuity equation, Fourier's law of thermal conduction, Newton's laws of motion and so on) and that, in comparison, our approach to introducing a fractional differential operator is based on postulation alone. It is therefore similar to certain other differential operators, a notable example being Schrödinger's operator.

The fractional diffusion operator given above is appropriate for modelling fractional diffusive processes that are stationary. For non-stationary fractional diffusion, we could consider the case where the diffusivity is time variant as defined by the function $\sigma(t)$. However, a more interesting case arises when the characteristics of the diffusion processes change over time becoming less or more diffusive. This is illustrated in terms of the random walk in the plane given in Figure 7. Here, the walk starts off being fully diffusive (i.e. $H = 0.5$ and $q = 1$), changes to being fractionally diffusive ($0.5 < H < 1$ and $1 < q < 2$) and then changes back to being fully diffusive. The result given in Figure 7 shows a transition from two episodes that are fully diffusive which has been generated using uniform phase distributions whose width changes from 2π to 1.8π and back to 2π. In terms of fractional diffusion, this is equivalent to having an operator

$$\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}$$

where $q = 1, t \in (0, T_1]$; $q > 1, t \in (T_1, T_2]$; $q = 1, t \in (T_2, T_3]$ where $T_3 > T_2 > T_1$. If we want to generalise such processes over arbitrary periods of time, then we should consider q to be a function of time. We can then introduce a non-stationary fractional diffusion operator given by

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}.$$

This operator is the theoretical basis for the Fractal Market Hypothesis considered in this paper.

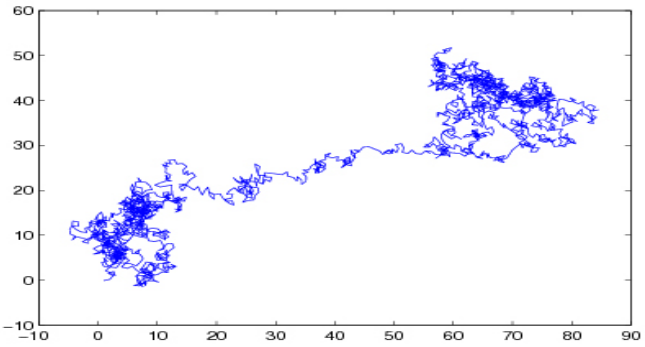

Fig. 7. Non-stationary random phase walk in the plane.

X. Fractional Dynamic Model

We consider an inhomogeneous non-stationary fractional diffusion equation of the form

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}} \right] u(x, t) = F(x, t)$$

where F is a stochastic source term with some PDF and u is the stochastic field whose solution we require. Specifying

q to be in the range $0 \leq q \leq 2$, leads to control over the basic physical characteristics of the equation so that we can define an anti-persistent field $u(x,t)$ when $q < 1$, a diffusive field when $q = 1$ and a propagative field when $q = 2$. In this case, non-stationarity is introduced through the use of a time varying fractional derivative whose values modify the physical characteristics of the equation.

The range of values of q is based on deriving an equation that is a generalisation of both diffusive and propagative processes using, what is fundamentally, a phenomenology. When $q = 0\ \forall t$, the time dependent behaviour is determined by the source function alone; when $q = 1\ \forall t$, u describes a diffusive process where $D = \sigma^{-1}$ is the 'diffusivity'; when $q = 2$ we have a propagative process where σ is the 'slowness' (the inverse of the wave speed). The latter process should be expected to 'propagate information' more rapidly than a diffusive process leading to transients or 'flights' of some type. We refer to q as the 'Fourier Dimension' which is related to the Hurst Exponent by $q = H + D_T/2$ where D_T is the Topological Dimension and to the Fractal Dimension D_F by $q = 1 - D_F + 3D_T/2$ as shown in Appendix I.

Since $q(t)$ 'drives' the non-stationary behaviour of u, the way in which we model $q(t)$ is crucial. It is arguable that the changes in the statistical characteristics of u which lead to its non-stationary behaviour should also be random. Thus, suppose that we let the Fourier dimension at a time t be chosen randomly, a randomness that is determined by some PDF. In this case, the non-stationary characteristics of u will be determined by the PDF (and associated parameters) alone. Also, since q is a dimension, we can consider our model to be based on the 'statistics of dimension'. There are a variety of PDFs that can be applied which will in turn affect the range of q. By varying the exact nature of the distribution considered, we can 'drive' the non-stationary behaviour of u in different ways. However, in order to apply different statistical models for the Fourier dimension, the range of q can not be restricted to any particular range, especially in the case of a normal distribution. We therefore generalize further and consider the equation

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^{q(t)}\frac{\partial^{q(t)}}{\partial t^{q(t)}}\right] u(x,t) = F(x,t), -\infty < q(t) < \infty, \forall t.$$

which allows us to apply different PDFs for q covering arbitrary ranges. For example, suppose we consider a system which is assumed to be primarily diffusive; then a 'normal' PDF of the type

$$\Pr[q(t)] = \frac{1}{\sigma\sqrt{2\pi}}\exp[-(q-1)^2/2\sigma^2], \quad -\infty < q < \infty$$

where σ is the standard deviation, will ensure that u is entirely diffusive when $\sigma \to 0$. However, as σ is increased in value, the likelihood of $q = 2$ (and $q = 0$) becomes larger. In other words, the standard deviation provides control over the likelihood of the process becoming propagative.

Irrespective of the type of distribution that is considered, the equation

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^{q(t)}\frac{\partial^{q(t)}}{\partial t^{q(t)}}\right] u(x,t) = F(x,t)$$

poses a fundamental problem which is how to define and work with the term

$$\frac{\partial^{q(t)}}{\partial t^{q(t)}}u(x,t).$$

Given the result (for constant q)

$$\frac{\partial^q}{\partial t^q}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)^q U(x,\omega)\exp(i\omega t)d\omega$$

we might generalize as follows:

$$\frac{\partial^{q(\tau)}}{\partial t^{q(\tau)}}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)^{q(\tau)} U(x,\omega)\exp(i\omega t)d\omega.$$

However, if we consider the case where the Fourier dimension is a relatively slowly varying function of time, then we can legitimately consider $q(t)$ to be composed of a sequence of different states $q_i = q(t_i)$. This approach allows us to develop a stationary solution for a fixed q over a fixed period of time. Non-stationary behaviour can then be introduced by using the same solution for different values of q over fixed (or varying) periods of time and concatenating the solutions for all q.

XI. Green's Function Solution

We consider a Green's function solution to the equation

$$\left(\frac{\partial^2}{\partial x^2} - \sigma^q\frac{\partial^q}{\partial t^q}\right)u(x,t) = F(x,t), \quad -\infty < q < \infty$$

when $F(x,t) = f(x)n(t)$ where $f(x)$ and $n(t)$ are both stochastic functions. Applying a separation of variables here is not strictly necessary. However, it yields a solution in which the terms affecting the temporal behaviour of $u(x,t)$ are clearly identifiable. Thus, we require a general solution to the equation

$$\left(\frac{\partial^2}{\partial x^2} - \sigma^q\frac{\partial^q}{\partial t^q}\right)u(x,t) = f(x)n(t).$$

Let

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}U(x,\omega)\exp(i\omega t)d\omega$$

and

$$n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}N(\omega)\exp(i\omega t)d\omega.$$

Then, using the result

$$\frac{\partial^q}{\partial t^q}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}U(x,\omega)(i\omega)^q\exp(i\omega t)d\omega$$

we can transform the fractional diffusion equation to the form

$$\left(\frac{\partial^2}{\partial x^2} + \Omega_q^2\right)U(x,\omega) = f(x)N(\omega)$$

where we shall take

$$\Omega_q = i(i\omega\sigma)^{\frac{q}{2}}$$

and ignore the case for $\Omega_q = -i(i\omega\sigma)^{\frac{q}{2}}$. Defining the Green's function g to be the solution of [44], [45]

$$\left(\frac{\partial^2}{\partial x^2} + \Omega_q^2\right) g(x \mid x_0, \omega) = \delta(x - x_0)$$

where δ is the delta function, we obtain the following solution:

$$U(x_0, \omega) = N(\omega) \int_{-\infty}^{\infty} g(x \mid x_0, \omega) f(x) dx \tag{3}$$

where [36]

$$g(x \mid x_0, k) = \frac{i}{2\Omega_q} \exp(i\Omega_q \mid x - x_0 \mid)$$

under the assumption that u and $\partial u/\partial x \to 0$ as $x \to \pm\infty$. This result reduces to conventional solutions for cases when $q = 1$ (diffusion equation) and $q = 2$ (wave equation) as shall now be shown.

A. Wave Equation Solution

When $q = 2$, the Green's function defined above provides a solution for the outgoing Green's function. Thus, with $\Omega_2 = -\omega\sigma$, we have

$$U(x_0, \omega) = \frac{N(\omega)}{2i\omega\sigma} \int_{-\infty}^{\infty} \exp(-i\omega\sigma \mid x - x_0 \mid) f(x) dx.$$

Fourier inverting and using the convolution theorem for the Fourier transform, we get

$$u(x_0, t) = \frac{1}{2\sigma} \int_{-\infty}^{\infty} dx f(x) ...$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N(\omega)}{i\omega} \exp(-i\omega\sigma \mid x - x_0 \mid) \exp(i\omega t) d\omega$$

$$= \frac{1}{2\sigma} \int_{-\infty}^{\infty} dx f(x) \int_{-\infty}^{t} n(t - \sigma \mid x - x_0 \mid) dt$$

which describes the propagation of a wave travelling at velocity $1/\sigma$ subject to variations in space and time as defined by $f(x)$ and $n(t)$ respectively. For example, when f and n are both delta functions,

$$u(x_0, t) = \frac{1}{2\sigma} H(t - \sigma \mid x - x_0 \mid).$$

This is a d'Alembertian type solution to the wave equation where the wavefront occurs at $t = \sigma \mid x - x_0 \mid$ in the causal case.

B. Diffusion Equation Solution

When $q = 1$ and $\Omega_1 = i\sqrt{i\omega\sigma}$,

$$u(x_0, t) = \frac{1}{2} \int_{-\infty}^{\infty} dx f(x) ...$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-\sqrt{i\omega\sigma} \mid x - x_0 \mid)}{\sqrt{i\omega\sigma}} N(\omega) \exp(i\omega t) d\omega.$$

For $p = i\omega$, we can write this result in terms of a Bromwich integral (i.e. an inverse Laplace transform) and using the convolution theorem for Laplace transforms with the result

$$\int_{c-i\infty}^{c+i\infty} \frac{\exp(-a\sqrt{p})}{\sqrt{p}} \exp(pt) dp = \frac{1}{\sqrt{\pi t}} \exp[-a^2/(4t)],$$

we obtain

$$u(x_0, t) =$$

$$\frac{1}{2\sqrt{\sigma}} \int_{-\infty}^{\infty} dx f(x) \int_{0}^{t} \frac{\exp[-\sigma(x_0 - x)^2/(4t_0)]}{\sqrt{\pi t_0}} n(t - t_0) dt_0.$$

Now, if for example, we consider the case when n is a delta function, the result reduces to

$$u(x_0, t) =$$

$$\frac{1}{2\sqrt{\pi\sigma t}} \int_{-\infty}^{\infty} f(x) \exp[-\sigma(x_0 - x)^2/(4t)] dx, \quad t > 0$$

which describes classical diffusion in terms of the convolution of an initial source $f(x)$ (introduced at time $t = 0$) with a Gaussian function.

C. General Series Solution

The evaluation of $u(x_0, t)$ via direct Fourier inversion for arbitrary values of q is not possible due to the irrational nature of the exponential function $\exp(i\Omega_q \mid x - x_0 \mid)$ with respect to ω. To obtain a general solution, we use the series representation of the exponential function and write

$$U(x_0, \omega) = \frac{iM_0 N(\omega)}{2\Omega_q} \left[1 + \sum_{m=1}^{\infty} \frac{(i\Omega_q)^m}{m!} \frac{M_m(x_0)}{M_0}\right] \tag{4}$$

where

$$M_m(x_0) = \int_{-\infty}^{\infty} f(x) \mid x - x_0 \mid^m dx.$$

We can now Fourier invert term by term to develop a series solution. Given that we consider $-\infty < q < \infty$, this requires us to consider three distinct cases.

1) Solution for $q = 0$: Evaluation of $u(x_0, t)$ in this case is trivial since, from equation (3)

$$U(x_0, \omega) = \frac{M(x_0)}{2} N(\omega) \quad \text{or} \quad u(x_0, t) = \frac{M(x_0)}{2} n(t)$$

where

$$M(x_0) = \int_{-\infty}^{\infty} \exp(- \mid x - x_0 \mid) f(x) dx.$$

2) Solution for $q > 0$: Fourier inverting, the first term in equation (4) becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{iN(\omega) M_0}{2\Omega_q} \exp(i\omega t) d\omega =$$

$$\frac{M_0}{2\sigma^{\frac{q}{2}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N(\omega)}{(i\omega)^{\frac{q}{2}}} \exp(i\omega t) d\omega$$

$$= \frac{M_0}{2\sigma^{\frac{q}{2}}} \frac{1}{(2i)^q \sqrt{\pi}} \frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi.$$

The second term is

$$-\frac{M_1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) \exp(i\omega t) d\omega = -\frac{M_1}{2} n(t).$$

The third term is

$$-\frac{iM_2}{2.2!} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) i (i\omega\sigma)^{\frac{q}{2}} \exp(i\omega t) d\omega = \frac{M_2 \sigma^{\frac{q}{2}}}{2.2!} \frac{d^{\frac{q}{2}}}{dt^{\frac{q}{2}}} n(t)$$

and the fourth and fifth terms become

$$\frac{M_3}{2.3!} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) i^2 (i\omega\sigma)^q \exp(i\omega t) d\omega = -\frac{M_3 \sigma^q}{2.3!} \frac{d^q}{dt^q} n(t)$$

and

$$i\frac{M_4}{2.4!} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) i^3 (i\omega\sigma)^{\frac{3q}{2}} \exp(i\omega t) d\omega = \frac{M_4 \sigma^{\frac{3q}{2}}}{2.4!} \frac{d^{\frac{3q}{2}}}{dt^{\frac{3q}{2}}} n(t)$$

respectively with similar results for all other terms. Thus, through induction, we can write $u(x_0, t)$ as a series of the form

$$u(x_0, t) =$$

$$\frac{M_0(x_0)}{2\sigma^{q/2}} \frac{1}{(2i)^q \sqrt{\pi}} \frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi$$

$$-\frac{M_1(x_0)}{2} n(t) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} M_{k+1}(x_0) \sigma^{kq/2} \frac{d^{kq/2}}{dt^{kq/2}} n(t).$$

Observe that the first term involves a fractional integral (the Riemann-Liouville integral), the second term is composed of the source function $n(t)$ alone (apart from scaling) and the third term is an infinite series composed of fractional differentials of increasing order $kq/2$. Also note that the first term is scaled by a factor involving $\sigma^{-q/2}$ whereas the third term is scaled by a factor that includes $\sigma^{kq/2}$.

3) Solution for $q < 0$: In this case, the first term becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{iN(\omega) M_0}{2\Omega_q} \exp(i\omega t) d\omega$$

$$= \frac{M_0}{2} \sigma^{\frac{q}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) (i\omega)^{\frac{q}{2}} \exp(i\omega t) d\omega = \frac{M_0}{2} \sigma^{\frac{q}{2}} \frac{d^{\frac{q}{2}}}{dt^{\frac{q}{2}}} n(t).$$

The second term is the same is in the previous case (for $q > 0$) and the third term is

$$-\frac{iM_2}{2.2!} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N(\omega) i}{(i\omega\sigma)^{\frac{q}{2}}} \exp(i\omega t) d\omega$$

$$= \frac{M_2}{2.2!} \frac{1}{\sigma^{q/2}} \frac{1}{(2i)^q \sqrt{\pi}} \frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi.$$

Evaluating the other terms, by induction we obtain

$$u(x_0, t) = \frac{M_0(x_0) \sigma^{q/2}}{2} \frac{d^{q/2}}{dt^{q/2}} n(t) - \frac{M_1(x_0)}{2} n(t)$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \frac{M_{k+1}(x_0)}{\sigma^{kq/2}} \frac{1}{(2i)^{kq} \sqrt{\pi}} \frac{\Gamma\left(\frac{1-kq}{2}\right)}{\Gamma\left(\frac{kq}{2}\right)} \cdots$$

$$\int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(kq/2)}} d\xi$$

where $q \equiv \mid q \mid, \quad q < 0$. Here, the solution is composed of three terms: a fractional differential, the source term and an infinite series of fractional integrals of order $kq/2$. Thus, the roles of fractional differentiation and fractional integration are reversed as q changes from being greater than to less than zero. All fractional differential operators associated with the equations above and hence forth should be considered in terms of the definition for a fractional differential given by

$$\hat{D}^q f(t) = \frac{d^n}{dt^n} [\hat{I}^{n-q} f(t)], \quad n - q > 0$$

where $\hat{I}$ is the fractional integral operator (the Riemann-Liouville transform),

$$\hat{I}^p f(t) = \frac{1}{\Gamma(p)} \int_{-\infty}^{t} \frac{f(\xi)}{(t-\xi)^{1-p}} d\xi, \quad p > 0 \tag{5}$$

The reason for this is that direct fractional differentiation can lead to divergent integrals. However, there is a deeper interpretation of this result that has a synergy with the issue over whether a macroeconomic system has 'memory' and is based on observing that the evaluation of a fractional differential operator depends on the history of the function in question. Thus, unlike an integer differential operator of order n, a fractional differential operator of order q has 'memory'

because the value of $\hat{I}^{q-n}f(t)$ at a time t depends on the behaviour of $f(t)$ from $-\infty$ to t via the convolution with $t^{(n-q)-1}/\Gamma(n-q)$. The convolution process is of course dependent on the history of a function $f(t)$ for a given kernel and thus, in this context, we can consider a fractional derivative defined via the result above to have memory. In this sense, the operator

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)}\frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

decribes a process, compounded in a field $u(x,t)$, that has a non-stationary memory association with the temporal characteristics of the system it is attempting to model. This is not an intrinsic charcteristic of systems that are purely diffusive $q=1$ or propagative $q=2$.

D. Asymptotic Solutions for an Impulse

We consider a special case in which the source function $f(x)$ is an impulse so that

$$M_m(x_0) = \int_{-\infty}^{\infty} \delta(x) \mid x - x_0 \mid^m dx = \mid x_0 \mid^m .$$

This result immediately suggests a study of the asymptotic solution

$$u(t) = \lim_{x_0 \to 0} u(x_0, t) \tag{6}$$

$$= \begin{cases} \frac{1}{2\sigma^{q/2}} \frac{1}{(2i)^q\sqrt{\pi}} \frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi, & q > 0; \\ \frac{n(t)}{2}, & q = 0; \\ \frac{\sigma^{q/2}}{2} \frac{d^{q/2}}{dt^{q/2}} n(t), & q < 0. \end{cases}$$

The solution for the time variations of the stochastic field u for $q > 0$ are then given by a fractional integral alone and for $q < 0$ by a fractional differential alone. In particular, for $q > 0$, we see that the solution is based on the convolution integral (ignoring scaling)

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t), \quad q > 0$$

where $\otimes$ denotes convolution and in ω-space (ignoring scaling)

$$U(\omega) = \frac{N(\omega)}{(i\omega)^{q/2}}.$$

This result is the conventional random fractal noise model for Fourier dimension q. Table I quantifies the results for different values of q with conventional name associations[7]. The field u has the following fundamental property for $q \in (0,2)$:

$$\lambda^{q/2}\Pr[u(t)] = \Pr[u(\lambda t)].$$

This property describes the statistical self-affinity of u. Thus, the asymptotic solution considered here, yields a result that describes a random scaling fractal field characterized by a PSDF of the form $1/\mid \omega \mid^q$ which is a measure of the time correlations in the signal.

[7]Note that Brown noise conventionally refers to the integration of white noise but that Brownian motion is a form of pink noise because it classifies diffusive processes identified by the case when $q = 1$.

q-value	t-space	ω-space (PSDF)	Name
$q=0$	$\frac{1}{t} \otimes n(t)$	1	White noise
$q=1$	$\frac{1}{\sqrt{t}} \otimes n(t)$	$\frac{1}{\mid\omega\mid}$	Pink noise
$q=2$	$\int^t n(t)dt$	$\frac{1}{\omega^2}$	Brown noise
$q>2$	$t^{(q/2)-1} \otimes n(t)$	$\frac{1}{\mid\omega\mid^q}$	Black noise

TABLE I
NOISE CHARACTERISTICS FOR DIFFERENT VALUES OF q. NOTE THAT THE RESULTS GIVEN ABOVE IGNORE SCALING FACTORS.

Note that $q = 0$ defines the Hilbert transform of $n(t)$ whose spectral properties in the positive half space are identical to $n(t)$ because

$$\frac{1}{t} \otimes n(t) \Longleftrightarrow -i\pi \mathrm{sign}(\omega) N(\omega)$$

where

$$\mathrm{sign}(\omega) = \begin{cases} 1, & \omega > 0; \\ -1, & \omega < 0. \end{cases}$$

The statistical properties of the Hilbert transform of $n(t)$ are therefore the same as $n(t)$ so that

$$\Pr[t^{-1} \otimes n(t)] = \Pr[n(t)].$$

Hence, as $q \to 0$, the statistical properties of $u(t)$ will 'reflect' those of n, i.e.

$$\Pr\left[\frac{1}{t^{1-q/2}} \otimes n(t)\right] = \Pr[n(t)], \quad q \to 0.$$

However, as $q \to 2$ we can expect the statistical properties of $u(t)$ to be such that the width of the PDF of $u(t)$ is reduced. This reflects the greater level of coherence (persistence in time) associated with the stochastic field $u(t)$ for $q \to 2$.

E. Other Asymptotic Solutions

A similar result to the asymptotic solution for $x_0 \to 0$ is obtained when the diffusivity is large, i.e.

$$\lim_{\sigma \to 0} u(x_0, t)$$

$$= \frac{M_0(x_0)}{2\sigma^{q/2}} \frac{1}{(2i)^q\sqrt{\pi}} \frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi$$

$$- \frac{M_1(x_0)}{2} n(t), \quad q > 0. \tag{7}$$

Here, the solution is the sum of fractal noise and white noise. Further, by relaxing the condition $\sigma \to 0$ we can consider the approximation

$$u(x_0, t) \simeq \frac{M_0(x_0)}{2\sigma^{q/2}} \frac{1}{(2i)^q\sqrt{\pi}} \frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \int_{-\infty}^{\infty} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi$$

$$-\frac{M_1(x_0)}{2}n(t)+\frac{M_2(x_0)}{2.2!}\sigma^{q/2}\frac{d^{q/2}}{dt^{q/2}}n(t),\quad q>0,\quad \sigma<<1 \tag{8}$$

in which the solution is expressed in terms of the sum of fractal noise, white noise and the fractional differentiation[8] of white noise.

F. Equivalence with a Wavelet Transform

The wavelet transform is defined in terms of projections of $f(t)$ onto a family of functions that are all normalized dilations and translations of a prototype 'wavelet' function w [47], i.e.

$$\mathcal{W}[f(t)]=F_L(t)=\int_{-\infty}^{\infty}f(\tau)w_L(\tau,t)d\tau$$

where

$$w_L(\tau,t)=\frac{1}{\sqrt{L}}w\left(\frac{\tau-t}{L}\right),\quad L>0.$$

The independent variables L and t are continuous dilation and translation parameters respectively. The wavelet transformation is essentially a convolution transform where $w_L(t)$ is the convolution kernel with dilation variable L. The introduction of this factor provides dilation and translation properties into the convolution integral that gives it the ability to analyse signals in a multi-resolution role (the convolution integral is now a function of L), i.e.

$$F_L(t)=w_L(t)\otimes f(t),\quad L>0.$$

In this sense, the asymptotic solution (ignoring scaling)

$$u(t)=\frac{1}{t^{1-q/2}}\otimes n(t),\quad q>0\quad x\to 0$$

is compatible with the case of a wavelet transform where

$$w_1(t)=\frac{1}{t^{1-q/2}}$$

for the stationary case and where, for the non-stationary case,

$$w_1(t,\tau)=\frac{1}{t^{1-q(\tau)/2}}.$$

XII. FTSE Analysis using OLR

We consider the basic model for a financial signal to be given by

$$u(t)=\frac{1}{t^{1-q/2}}\otimes n(t),\quad q>0$$

which has characteristic spectrum

$$U(\omega)=\frac{N(\omega)}{(i\omega)^{q/2}}$$

and is a solution to the fractional diffusion equation

$$\left(\frac{\partial^2}{\partial x^2}-\sigma^q\frac{\partial^q}{\partial t^q}\right)u(x,t)=\delta(x)n(t),\quad x\to 0$$

The PSDF is thus characterised by $\omega^{-q},\omega\geq 0$ and our problem is thus, to compute q from the data $P(\omega)=\mid U(\omega)\mid^2,\omega\geq 0$. For this data, we consider the PSDF

$$\hat{P}(\omega)=\frac{c}{\omega^q}$$

or

$$\ln\hat{P}(\omega)=C+q\ln\omega$$

where $C=\ln c$. The problem is therefore reduced to implementing an appropriate method to compute q (and C) by finding a best fit of the line $\ln\hat{P}(\omega)$ to the data $\ln P(\omega)$. Application of the least squares method for computing q, which is based on minimizing the error

$$e(q,C)=\|\ln P(\omega)-\ln\hat{P}(\omega,q,C)\|_2^2$$

with regard to q and C, leads to errors in the estimates for q which are not compatible with market data analysis. The reason for this is that relative errors at the start and end of the data $\ln P$ may vary significantly especially because any errors inherent in the data P will be 'amplified' through application of the logarithmic transform required to linearise the problem. In general, application of a least squares approach is very sensitive to statistical heterogeneity [48] and in this application, may provide values of q that are not compatible with the rationale associated with the FMH (i.e. values of $1<q<2$ that are intermediate between diffusive and propagative processes). For this reason, an alternative approach must be considered which, in this paper, is based on Orthogonal Linear Regression (OLR).

Applying a standard moving window, $q(t)$ is computed by repeated application of OLR based on the m-code available from [49]. Since q is, in effect, a statistic, its computation is only as good as the quantity (and quality) of data that is available for its computation. For this reason, a relatively large window is required whose length is compatible with: (i) the number of samples available; (ii) the autocorrelation function and long-term memory effects as discussed in Section III. An example of the $q(t)$ signal obtained using a 1000 element window is given in Figure 8 which includes $q(t)$ after it has been smoothed using a Gaussian low-pass filtered to reveal the underlying trends in q. Inspection of the data (i.e. closer inspection of the time series than is shown in Figure 8) clearly illustrates a qualitative relationship between trends in the financial data and $q(t)$ in accordance with the theoretical model considered. In particular, over periods of time in which q increases in value, the amplitude of the financial signal $u(t)$ decreases. Moreover, and more importantly, an upward trend in q appears to be a pre-cursur to a downward trend in $u(t)$. A more detailed example of this behviour is shown in Figure 9 for close of day FTSE data over a smaller period of time (i.e. from 1994 to 1997), a correlation that is compatible with the idea that a rise in the value of q relates to the 'system' becoming more propagative, which in stock market terms, indicates the likelihood for the markets becoming 'bear' dominant in the future.

The results of using the method discussed above not only provides for a general appraisal of different macroeconomic financial time series, but, with regard to the size of selected window used, an analysis of data at any point in time. The output can be interpreted in terms of 'persistence' and

[8] As defined by equation (5).

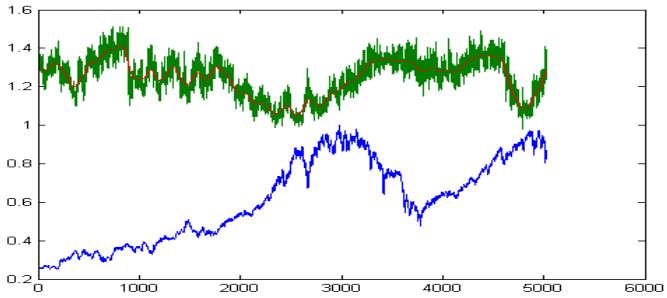

Fig. 8. Application of OLR using a 1000 element window for analysing financial time series composed of FTSE values (close-of-day) from 02-04-1984 to 13-02-2008. The plot shows the time varying Fourier Dimension $q(t)$ (green) onto which is superimposed a Gaussian low-pass filtered version of the signal (red) and the FTSE time series after normalisation.

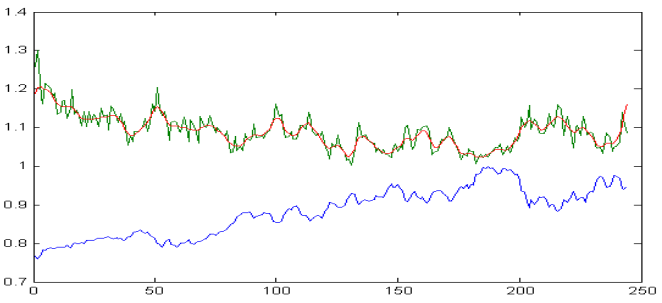

Fig. 9. Application of OLR using a 1000 element window for analysing financial time series composed of FTSE values (close-of-day) from 05-04-1994 to 24-12-1997. The plot shows the time varying Fourier Dimension $q(t)$ (green) onto which is superimposed a Gaussian low-pass filtered version of the signal (red) and the FTSE time series after normalisation.

'anti-persistence' and in terms of the existence or absence of after-effects (macroeconomic memory effects). For those periods in time when $q(t)$ is relatively constant, the existing market tendencies usually remain. Changes in the existing trends tend to occur just after relatively sharp changes in $q(t)$ have developed. This behaviour indicates the possibility of using the time series $q(t)$ for identifying the behaviour of a macroeconomic financial system in terms of both inter-market and between-market analysis. These results support the possibility of using $q(t)$ as an independent macroeconomic volatility predictor. It is noted that, at the time of writing this paper, the value of $q(t)$ associated with those days after approximately day 4800 in Figure 8 (representing the latter half of 2007) indicate the growth of propagative behaviour and thus the macroeconomic instability compounded in the term 'Credit Crunch'. This is not surprising if it is assumed that the downward trend from approximately day 3000 to day 3700 shown in Figure 8 is a natural consequence of the effect of a higher inflationary global economy resulting from the end of the cold war and that the upward trend from approximately day 3700 to 5000 is a consequence of credit policies adopted by banks in an attempt to compensate for this natural inflationary pressure. Under this assumption, the 'Credit Crunch' of 2007 represents a transition that is compounded in a reappraisal of the definition of poverty, namely, that poverty is not a measure of how little one has but a measure of how much one owes.

XIII. Discussion

This paper is concerned with the introduction and theoretical analysis (in terms of general a solution) associated with the non-stationary fractional diffusion operator

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}$$

in the context of a macroeconomic model. By considering a source function of the type $\delta(x)n(t)$ where $n(t)$ is white noise, we have shown that, for $x \to 0$, the fractional diffusive field $u(t)$ at time τ is given by (ignoring scaling)

$$u(t,\tau) = \frac{1}{t^{1-q(\tau)/2}} \otimes n(t)$$

which has Power Spectral Density Function characterised by $\mid \omega \mid^{-q(\tau)/2}$ - a random scaling fractal. It should be noted, that the data analysis reported in this paper is based on an asymptotic solution (i.e. $x \to 0$) used to obtain equation (6) and is thus, limited in the extent to which it 'reflects' the physical principles upon which the model has been established. However, it is noted that the computation of $q(t)$ in the presence of additive white noise is equivalent to the inversion of equation (7) for q (and for arbitrary values of x_0) when $\sigma \to 0$. In this sense, the power spectrum method used to compute $q(t)$ is valid under the assumption that a fractional diffusive process occurs with high diffusivity and a high signal-to-noise ratio (i.e. $\|M_1(x_0)\| \to 0$). For the case when $\sigma << 1$, the inversion of equation (8) to compute q from u might be possible using an iterative approach which can be extended to solve the general case as required.

The non-stationary nature of this model is taken to account for stochastic processes that can vary in time and are intermediate between diffusive and propagative or persistent behaviour. Application of Orthogonal Linear Regression to macroeconomic time series data provides an accurate and robust method to compute $q(t)$ when compared to other statistical estimation techniques such as the least squares method. As a result of the physical interpretation associated with the fractional diffusion equation and the 'meaning' of $q(t)$, we can, in principal, use the signal $q(t)$ as a predictive measure in the sense that as the value of $q(t)$ continues to increases, there is a greater likelihood for volatile behaviour of the markets. This is reflected in the data analysis that is compounded in Figure 8 for the FTSE close-of-day between 1980 to 2007 and in other financial data, the results of which lie beyond the scope of this paper [9]. It should be noted that because financial time series data is assumed to be self-affine, the computation of $q(t)$ can be applied over any time scale, and that the FTSE close-of-day is only one example that has been used in this paper as an illustrative case study.

In a statistical sense, $q(t)$ is just another measure that may, or otherwise, be of value to market traders. In comparison with other statistical measures, this can only be assessed through its practical application in a live trading environment. However, in terms of its relationship to a stochastic model for macroeconomic data, $q(t)$ does provide a measure that

[9] Similar results being observed for other major stock markets.

is consistent with the physical principles associated with a random walk that includes a directional bias, i.e. fractional Brownian motion. The model considered, and the signal processing algorithm proposed, has a close association with re-scaled range analysis for computing the Hurst exponent H since for $D_T = 1$, $q = H + 1/2$ (see Appendix I) [48]. In this sense, the principal contribution of this paper has been to consider a model that is quantified in terms of a physically significant (but phenomenological) model that is compounded in a specific (fractional) partial differential equation. As with other financial time series, their derivatives, transforms etc., a range of statistical measures can be used to characterise $q(t)$, an example being given in Figure 8 and Figure 9 where $q(t)$ has been smoothed to provide a measure of the underlying trends.

In terms of the non-stationary fractional diffusive model considered in this work, the time varying Fourier dimension $q(t)$ can be interpreted in terms of a 'gauge' on the characteristics of a dynamical system. This includes the management processes from which all modern economies may be assumed to be derived. In this sense, the FMH is based on three principal considerations: (i) the non-stationary behaviour associated with any system undergoing continuous change that is driven by a management infrastructure; (ii) the cause and effect that is inherent at all scales (i.e. all levels of management hierarchy); (iii) the self-affine nature of outcomes relating to points (i) and (ii). In a modern economy, the principal issue associated with any form of financial management is based on the flow of information and the assessment of this information at different points connecting a large network. In this sense, a macroeconomy can be assessed in terms of its information network which consists of a distribution of nodes from which information can flow in and out. The 'efficiency' of the system is determined by the level of randomness associated with the direction of flow of information to and from each node. The nodes of the system are taken to be individuals or small groups of individuals whose assessment of the information they acquire together with their remit, responsibilities and initiative, determines the direction of the information flow from one node to the next. The determination of the efficiency of a system in terms of randomness is the most critical in terms of the model developed. It suggests that the performance of a business is related to how well information flows through an organisation. If the information flow is entirely random, then we might surmise that the decisions made which 'drive' the direction of the 'system' are also entirely random. The principal point here is that the flow of information has a direct relationship on the management decisions that are made on behalf of an organisation.

The non-stationary but statistically self-affine nature of the markets leads directly to the use of the Fourier dimension as a measure for quantifying their 'state of coherence'. Just as this parameter can be used as a market index for managing a financial portfolio, so, it may be of value in quantifying the 'state' of any organisation undergoing change (management). The conceptual basis associated with the Fourier dimension and the system behaviour that it reflects leads directly to an approach to management where the principles of openness and transparency articulate the degree of coherence of information flow through an organisation from one level to another. In effect, the sustained organisational approach to managing continuous change is the basis for a portfolio in which $q(t) > 1$ and increases with time.

Fractal type	Fractal Dimension
Fractal Dust	$0 < D_F < 1$
Fractal Curve	$1 < D_F < 2$
Fractal Surface	$2 < D_F < 3$
Fractal Volume	$3 < D_F < 4$
Fractal Time	$4 < D_F < 5$
Hyper-fractals	$5 < D_F < 6$
$\vdots$	$\vdots$

TABLE II
FRACTAL TYPES AND CORRESPONDING FRACTAL DIMENSIONS

The FMH and the self-affine nature of organisations in general provides a model in which the work-force at any one level (i.e. department/section/group etc.) of an organisation can empathise with all other levels by cultivating an understanding in which each level is a reflection of their own, e.g. problems/solutions at middle management are a reflection of the same type of problems/solutions at executive level. This 'empathy' is a two-way entity which differs only in terms of its scale. Sustained organisational change and the example methods of implementing it is a self-affine process and should thus be introduced with this aspect in mind [50]. In tackling problems at any level within an organisation, one is, in effect, taking consideration of such problems above and below that same level in terms of the dynamic behaviour of the 'system' as a whole, a macroeconomy being the antithesis of such a 'system'.

APPENDIX I
RELATIONSHIP BETWEEN THE HURST EXPONENT AND THE TOPOLOGICAL, FRACTAL AND FOURIER DIMENSIONS

Suppose we cut up some simple one-, two- and three-dimensional Euclidean objects (a line, a square surface and a cube, for example), make exact copies of them and then keep on repeating the copying process. Let N be the number of copies that we make at each stage and let r be the length of each of the copies, i.e. the scaling ratio. Then we have

$$Nr^{D_T} = 1, \quad D_T = 1, 2, 3, ...$$

where D_T is the topological dimension. The similarity or fractal dimension is that value of D_F which is usually (but not always) a non-integer dimension 'greater' that its topological dimension (i.e. 0,1,2,3,... where 0 is the dimension of a point on a line) and is given by

$$D_F = -\frac{\log(N)}{\log(r)}.$$

The fractal dimension is that value that is strictly greater than the topological dimension as given in Table II. In each case, as the value of the fractal dimension increases, the fractal becomes increasingly 'space-filling' in terms of the topological dimension which the fractal dimension is approaching. In each

case, the fractal exhibits structures that are self-similar. A self-similar deterministic fractal is one where a change in the scale of a function $f(x)$ (which may be a multi-dimensional function) by a scaling factor λ produces a smaller version, reduced in size by λ, i.e.

$$f(\lambda x) = \lambda f(x).$$

A self-affine deterministic fractal is one where a change in the scale of a function $f(x)$ by a factor λ produces a smaller version reduced in size by a factor λ^q, $q > 0$, i.e.

$$f(\lambda x) = \lambda^q f(x).$$

For stochastic fields, the expression

$$\Pr[f(\lambda x)] = \lambda^q \Pr[f(x)]$$

describes a statistically self-affine field - a random scaling fractal. As we zoom into the fractal, the shape changes, but the distribution of lengths remains the same.

There is no unique method for computing the fractal dimension. The methods available are broadly categorized into two families: (i) Size-measure relationships, based on recursive length or area measurements of a curve or surface using different measuring scales; (ii) application of relationships based on approximating or fitting a curve or surface to a known fractal function or statistical property, such as the variance.

Consider a simple Euclidean straight line ℓ of length $L(\ell)$ over which we 'walk' a shorter 'ruler' of length δ. The number of steps taken to cover the line $N[L(\ell), \delta]$ is then L/δ which is not always an integer for arbitrary L and δ. Since

$$N[L(\ell), \delta] = \frac{L(\ell)}{\delta} = L(\ell)\delta^{-1},$$

$$\Rightarrow 1 = \frac{\ln L(\ell) - \ln N[L(\ell), \delta]}{\ln \delta} = -\left(\frac{\ln N[L(\ell), \delta] - \ln L(\ell)}{\ln \delta}\right)$$

which expresses the topological dimension $D_T = 1$ of the line. In this case, $L(\ell)$ is the Lebesgue measure of the line and if we normalize by setting $L(\ell) = 1$, the latter equation can then be written as

$$1 = -\lim_{\delta \to 0}\left[\frac{\ln N(\delta)}{\ln \delta}\right]$$

since there is less error in counting $N(\delta)$ as δ becomes smaller. We also then have $N(\delta) = \delta^{-1}$. For extension to a fractal curve f, the essential point is that the fractal dimension should satisfy an equation of the form

$$N[F(f), \delta] = F(f)\delta^{-D_F}$$

where $N[F(f), \delta]$ is 'read' as the number of rulers of size δ needed to cover a fractal set f whose measure is $F(f)$ which can be any valid suitable measure of the curve. Again we may normalize, which amounts to defining a new measure F' as some constant multiplied by the old measure to get

$$D_F = -\lim_{\delta \to 0}\left[\frac{\ln N(\delta)}{\ln \delta}\right]$$

where $N(\delta)$ is taken to be $N[F'(f), \delta]$ for notational convenience. Thus a piecewise continuous field has precise fractal properties over all scales. However, for the discrete (sampled) field

$$D = -\left\langle \frac{\ln N(\delta)}{\ln \delta} \right\rangle$$

where we choose values δ_1 and δ_2 (i.e. the upper and lower bounds) satisfying $\delta_1 < \delta < \delta_2$ over which we apply an averaging processes denoted by $\langle\ \rangle$. The most common approach is to utilise a bi-logarithmic plot of $\ln N(\delta)$ against $\ln \delta$, choose values δ_1 and δ_2 over which the plot is uniform and apply an appropriate data fitting algorithm (e.g. a least squares estimation method or, as used in this paper, Orthogonal Linear Regression) within these limits.

The relationship between the Fourier dimension q and the fractal dimension D_F can be determined by considering this method for analysing a statistically self-affine field. For a fractional Brownian process (with unit step length)

$$A(t) = t^H, \quad H \in (0, 1]$$

where H is the Hurst dimension. Consider a fractal curve covering a time period $\Delta t = 1$ which is divided up into $N = 1/\Delta t$ equal intervals. The amplitude increments ΔA are then given by

$$\Delta A = \Delta t^H = \frac{1}{N^H} = N^{-H}.$$

The number of lengths $\delta = N^{-1}$ required to cover each interval is

$$\Delta A \Delta t = \frac{N^{-H}}{N^{-1}} = N^{1-H}$$

so that

$$N(\delta) = N N^{1-H} = N^{2-H}.$$

Now, since

$$N(\delta) = \frac{1}{\delta^{D_F}}, \quad \delta \to 0,$$

then, by inspection,

$$D_F = 2 - H.$$

Thus, a Brownian process, where $H = 1/2$, has a fractal dimension of 1.5. For higher topological dimensions D_T

$$D_F = D_T + 1 - H.$$

This algebraic equation provides the relationship between the fractal dimension D_F, the topological dimension D_T and the Hurst dimension H. We can now determine the relationship between the Fourier dimension q and the fractal dimension D_F.

Consider a fractal signal $f(x)$ over an infinite support with a finite sample $f_X(x)$, given by

$$f_X(x) = \begin{cases} f(x), & 0 < x < X; \\ 0, & \text{otherwise.} \end{cases}$$

A finite sample is essential as otherwise the power spectrum diverges. Moreover, if $f(x)$ is a random function then for any experiment or computer simulation we must necessarily take a finite sample. Let $F_X(k)$ be the Fourier transform of $f_X(x)$, $P_X(k)$ be the power spectrum and $P(k)$ be the power spectrum of $f(x)$. Then

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(k)\exp(ikx)dk,$$

$$P_X(k) = \frac{1}{X}\left|F_X(k)\right|^2$$

and

$$P(k) = \lim_{X\to\infty} P_X(k).$$

The power spectrum gives an expression for the power of a signal for particular harmonics. $P(k)dk$ gives the power in the range k to $k+dk$. Consider a function $g(x)$, obtained from $f(x)$ by scaling the x-coordinate by some $a>0$, the f-coordinate by $1/a^H$ and then taking a finite sample as before, i.e.

$$g_X(x) = \begin{cases} g(x) = \frac{1}{a^H}f(ax), & 0<x<X; \\ 0, & \text{otherwise.}\end{cases}$$

Let $G_X(k)$ and $P'_X(k)$ be the Fourier transform and power spectrum of $g_X(x)$, respectively. We then obtain an expression for G_X in terms of F_X,

$$G_X(k) = \int_0^X g_X(x)\exp(-ikx)dx =$$

$$\frac{1}{a^{H+1}}\int_0^X f(s)\exp\left(-\frac{iks}{a}\right)ds$$

where $s=ax$. Hence

$$G_X(k) = \frac{1}{a^{H+1}}F_X\left(\frac{k}{a}\right)$$

and the power spectrum of $g_X(x)$ is

$$P'_X(k) = \frac{1}{a^{2H+1}}\frac{1}{aX}\left|F_X\left(\frac{k}{a}\right)\right|^2$$

and, as $X\to\infty$,

$$P'(k) = \frac{1}{a^{2H+1}}P\left(\frac{k}{a}\right).$$

Since $g(x)$ is a scaled version of $f(x)$, their power spectra are equal, and so

$$P(k) = P'(k) = \frac{1}{a^{2H+1}}P\left(\frac{k}{a}\right).$$

If we now set $k=1$ and then replace $1/a$ by k we get

$$P(k) \propto \frac{1}{k^{2H+1}} = \frac{1}{k^\beta}.$$

Now since $\beta = 2H+1$ and $D_F = 2-H$, we have

$$D_F = 2 - \frac{\beta-1}{2} = \frac{5-\beta}{2}.$$

The fractal dimension of a fractal signal can be calculated directly from β using the above relationship. This method also generalizes to higher topological dimensions giving

$$\beta = 2H + D_T.$$

Thus, since

$$D_F = D_T + 1 - H,$$

then $\beta = 5-2D_F$ for a fractal signal and $\beta = 8-2D_F$ for a fractal surface so that, in general,

$$\beta = 2(D_T+1-D_F) + D_T = 3D_T + 2 - 2D_F$$

and

$$D_F = D_T + 1 - H = D_T + 1 - \frac{\beta - D_T}{2} = \frac{3D_T + 2 - \beta}{2},$$

the Fourier dimension being given by $q = \beta/2$.

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