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Scattering from a Tenuous Random Medium with Applications in Optics

Jonathan M Blackledge, Fellow, IET, Fellow, IMA, Fellow, IoP, Fellow, RSS

Abstract—Mathematical models for the scattering of light (and other radiation) tend to fall into two categories based on a weak field condition (single scattering processes) or a strong field condition (multiple scattering processes). In the latter case, the complexity of deterministic models coupled with available solutions often fail to provide results that are of value to engineering systems (e.g. imaging systems). For this reason, multiple scattering problems are often approached using stochastic modelling methods whose foundations lie in random walk theory where the amplitude and phase of the scattered field are taken to conform to an appropriate statistical process and distribution. In the case of intermediate scattering processes (scattering by a tenuous random medium), where the scattered field is neither ‘weak’ or ‘strong’, the problem is reduced to finding a suitable approach for constructing and solving a mathematical model that is, ideally, of value to an engineering system.

In this paper, we consider the basis for describing strong scattering in terms of diffusive processes based on the diffusion equation. Intermediate strength scattering is then considered in terms of an (inhomogeneous) fractional diffusion equation which is studied using results from fractional calculus. The diffusion equation for modelling intermediate strength scattering is based on a generalization of the diffusion equation to fractional form. This equation can be justified in terms of the generalization of a random walk model with no statistical bias in the phase to a random walk that has a phase bias and is thus, only ‘partially’ or ‘fractionally’ diffusive. Green’s function solutions to the fractional diffusion equation are studied and results derived that provide a model for an incoherent image obtained from light scattered by a tenuous medium. Applications include image enhancement of star fields and other cosmological bodies imaged through interstellar dust clouds, an example of which is provided in this paper.

Index Terms—Random Media, Multiple Scattering, Diffusion, Fractional Diffusion, Fractional Calculus, Astronomical Imaging

I. INTRODUCTION

IT is well known that the transformation from the object to the image plane is given by a convolution of the Object Function with the Point Spread Function (PSF) of the imaging system [1], [2], [3]. For images generated by the scattering of radiation in the object plane, this transform is based on the weak scattering approximation under the assumption that multiple scattering processes do not contribute to the characteristics of the Object Function. However, in the case

of radiation scattering from a dense random medium, multiple scattering processes tend to contribute, often significantly, to the recorded field. Multiple scattering typically leads to random outcomes, particularly with regard to the scattering of coherent radiation [6]. The random fluctuations in the multiply-scattered intensity leads to noise in an image, which, together with other background noise sources is incorporated into an ‘additive noise function’. The mathematical model for an image is then given by the convolution of the PSF with the Object Function plus noise. This is the standard ‘imaging model’ used in ‘Fourier Optics’ and other ‘scatter imaging’ systems.

A conventional approach to modelling light scattering in random media is to consider the scattering function to be a stochastic function with a characteristic Probability Density Function (PDF) under the weak scattering approximation. In the far field, the scattering amplitude is then given by the Fourier transform of the scattering function. The intensity of the scattered field (i.e. the measurable quantity, at optical frequencies and above) is determined by the Fourier transform of the autocorrelation of the scattering function. Various autocorrelation models can then be considered with regard to computing the intensity function. The inverse scattering problem is reduced to estimating the correlation function by Fourier inversion and then solving the phase reconstruction problem [4], [5] to recover the scattering function from its autocorrelation function.

Multiple scattering processes are often modelled using a statistical approach [7]. The aim is to develop a model of the PDF for the scattered field itself rather than for the scattering function. This involves concepts traditionally associated with the kinetic theory of gases in which the random motion of particles undergoing elastic collisions and following ‘random walks’ are ‘replaced’ with the random scattering of an electric field, for example, from multiple scattering sites. The total contribution of the multiple scattering process after N scattering interactions is given by [6]

$$E = \sum_{j=1}^N r_j \exp(i\phi_j)$$

where the amplitude r , the phase ϕ and N are independent random variables. While this approach provides physically informative models for the PDF that can be used for statistical image segmentation to locate statistically significant features, it does not help in the development of algorithms for the extraction of ‘information from noise’. On the other hand, random walk models provide the basis for diffusion processes

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in general. This is the essential ‘link’ to modelling multiple scattering processes in terms of solutions to the diffusion equation [8] for the intensity of light.

In certain circumstances, multiple scattering may only involve a small number of interactions such that the randomness is not completely averaged out. This occurs when light interacts with tenuous media, for example, and is considered to be one of the most difficult scenarios to model accurately. Diffusion processes are not applicable in such cases. In this paper, we study an approach to solving this problem using the fractional diffusion equation. Section II discusses the limitations associated with using the scalar wave equation in optics while Section III addresses formal direct and inverse scattering solutions to the inhomogeneous wave equation. Section IV introduces the conventional approach to modelling scattering in random media where the scatterer is considered to be a stochastic function under the weak field condition (single scattering). Section V examines a statistical approach for modelling the PDF of a scattered field under the strong field condition (multiple scattering) using a random walk approach. The basis for models relating to optical diffusion and fractional diffusion are given in Sections VI and VII, respectively. These models provide results where the inverse scattering problems can be cast in terms the deconvolution of an image with the PSFs defined for diffusion (multiple scattering from a dense medium) and fractional diffusion (intermediate scattering from a tenuous medium) processes, as presented in Section VIII. A deconvolution algorithm is derived in this same section using Bayesian estimation which is applied to the enhancement of astronomical images as presented in Section IX.

II. THE INHOMOGENEOUS WAVE EQUATION IN OPTICS

The inhomogeneous wave equation for applications in optics is based on certain limiting conditions which need to be assessed in terms of the governing equations of electromagnetism. For a charge-free non-conductive dielectric that is linear, isotropic and inhomogeneous, Maxwell’s macroscopic equations are, e.g. [8], [9], [10], [11] (using the International Systems of Units and where \mathbf{r} and t denote the three-dimensional space vector $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ and time respectively)

$$\nabla \cdot \epsilon \mathbf{E} = 0 \quad (1)$$

$$\nabla \cdot \mu \mathbf{H} = 0 \quad (2)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}. \quad (4)$$

where $\mathbf{E}(\mathbf{r}, t)$ is the electric field, $\mathbf{H}(\mathbf{r}, t)$ is the magnetic field, $\epsilon(\mathbf{r})$ is the permittivity and $\mu(\mathbf{r})$ is the permeability. The isotropy condition used here implies that there is no directional bias to the inhomogeneous characteristics of the medium, i.e. the permittivity and permeability are scalar functions of space only. The linearity condition implies that the medium is not affected by the propagation of electromagnetic waves - the dielectric is invariant of electric field strength [12].

Equations (1) - (4) can be decoupled to provide a wave equation for the electric and magnetic fields. In the former case, starting with equation (3), we divide through by μ and take the curl of the resulting equation giving

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) = -\frac{\partial}{\partial t} \nabla \times \mathbf{H}.$$

By taking the derivative with respect to time t of equation (4), we obtain

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

so that, from the previous equation, we can then write

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) = -\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Expanding the first term, multiplying through by μ and noting that

$$\mu \nabla \left(\frac{1}{\mu} \right) = -\nabla \ln \mu$$

we get

$$\nabla \times \nabla \times \mathbf{E} + \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \mu \frac{\partial \mathbf{E}}{\partial t} = (\nabla \ln \mu) \times \nabla \times \mathbf{E}. \quad (5)$$

Form equation (1), we have

$$\epsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \epsilon = 0 \quad \text{or} \quad \nabla \cdot \mathbf{E} = -\mathbf{E} \cdot \nabla \ln \epsilon$$

and hence, using the vector identity

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} + \nabla (\nabla \cdot \mathbf{E}),$$

from equation (5), we obtain the following inhomogeneous wave equation for the electric field:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla (\mathbf{E} \cdot \nabla \ln \epsilon) - (\nabla \ln \mu) \times \nabla \times \mathbf{E}.$$

where $c = 1/\sqrt{\epsilon\mu}$. This equation is inhomogeneous in ϵ , μ (and thus, the wave speed c), and it models the behaviour of an electric wavefield \mathbf{E} in an inhomogeneous dielectric. In optics, interest often focuses on the behaviour of the scattered wavefield generated by variations in the material parameters ϵ and μ which, for a random medium, are randomly distributed in space. The model used is based on the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) U(\mathbf{r}, t) = 0 \quad (6)$$

for the scalar electric field U where $I = |U|^2$ is the intensity of the scattered field. In this case, the term $(\nabla \ln \mu) \times \nabla \times \mathbf{E}$ is usually set to zero under the condition that variations in the permeability are negligible compared to variations in the permittivity (i.e. $\mu = \mu_0$ where μ_0 is the permeability of free space). However, in order to set $\nabla (\mathbf{E} \cdot \nabla \ln \epsilon) \sim 0$ without imposing any conditions on \mathbf{E} , it is required to assume that $\nabla \epsilon / \epsilon \rightarrow 0$, implying that the gradient in permittivity is small compared to its magnitude. This is not always a viable condition particularly for an inhomogeneity that is of compact support with volume V and large values of ϵ . Further, elimination of this term produces a scalar wave equation which does not take into account polarization effects. Thus, application of equation (6) as a governing wave equation must

be understood to be a limiting approach to modelling scattering processes in optics. However, with regard to multiple scattering in a random medium (especially volume scattering), polarization effects tend to get ‘averaged out’ as a result of the many interactions that take place, each generating an arbitrary change in polarization. The measured intensity therefore becomes independent of polarization and a scalar wave equation can be used although it should be noted that, as a medium becomes more tenuous, polarization effects can become more significant.

III. FORMAL METHODS OF SOLUTION

Formal methods of solution are based on the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = -k^2\gamma(\mathbf{r})u(\mathbf{r}, k)$$

which can be derived quite generally from the (inhomogeneous) time dependent wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right)U(\mathbf{r}, t) = 0$$

by letting

$$\frac{1}{c^2} = \frac{1}{c_0^2}(1 + \gamma)$$

where γ is a dimensionless quantity (the scattering function), $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is a constant (wave speed) and ϵ_0 is the permittivity of free space. With $U(\mathbf{r}, t) = u(\mathbf{r}, \omega)\exp(i\omega t)$ where ω is the (angular) frequency we have

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = -k^2\gamma(\mathbf{r})u(\mathbf{r}, k)$$

where the wavenumber k is given by (for wavelength λ)

$$k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}$$

Noting that $\epsilon = \epsilon_0\epsilon_r$ where $\epsilon_r(\mathbf{r}) \geq 1$ is the relative permittivity, the scattering function is given by $\gamma = \epsilon_r - 1$ and is assumed to be of compact support so that

$$\gamma(\mathbf{r}) \exists \forall \mathbf{r} \in V.$$

The general solution for u at a point \mathbf{r}' is [13], [14]

$$u(\mathbf{r}', k) = k^2 \int_V g\gamma u d^3\mathbf{r} + \oint_S (g\nabla u - u\nabla g) \cdot \hat{\mathbf{n}} d^2\mathbf{r}$$

where g is the ‘outgoing’ free space Green’s function given by

$$g(\mathbf{r} | \mathbf{r}', k) = \frac{\exp(ik |\mathbf{r} - \mathbf{r}'|)}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

which is the solution to the equation

$$(\nabla^2 + k^2)g(\mathbf{r} | \mathbf{r}', k) = -\delta^3(\mathbf{r} - \mathbf{r}')$$

and $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface element $d^2\mathbf{r}$ of a closed surface S . To compute the surface integral, a condition for the behaviour of u on the surface S of γ must be chosen. Consider the case where the incident wavefield u_i is a simple plane wave of unit amplitude

$$\exp(ik \cdot \mathbf{r})$$

satisfying the homogeneous wave equation

$$(\nabla^2 + k^2)u_i(\mathbf{r}, k) = 0.$$

By choosing the condition $u(\mathbf{r}, k) = u_i(\mathbf{r}, k)$ on the surface of γ , we obtain the result

$$u(\mathbf{r}', k) = k^2 \int_V g\gamma u d^3\mathbf{r} + \oint_S (g\nabla u_i - u_i\nabla g) \cdot \hat{\mathbf{n}} d^2\mathbf{r}.$$

Using Green’s theorem to convert the surface integral back into a volume integral, we have

$$\oint_S (g\nabla u_i - u_i\nabla g) \cdot \hat{\mathbf{n}} d^2\mathbf{r} = \int_V (g\nabla^2 u_i - u_i\nabla^2 g) d^3\mathbf{r}.$$

Noting that

$$\nabla^2 u_i = -k^2 u_i$$

and that

$$\nabla^2 g = -\delta^3 - k^2 g$$

we obtain

$$\int_V (g\nabla^2 u_i - u_i\nabla^2 g) d^3\mathbf{r} = \int_V \delta^3 u_i d^3\mathbf{r} = u_i.$$

Hence, by choosing the field u to be equal to the incident wavefield u_i on the surface of γ , we obtain a solution of the form

$$u = u_i + u_s$$

where, with $r \equiv |\mathbf{r}|$,

$$\begin{aligned} u_s(\mathbf{r}, k) &= k^2 g(r, k) \otimes_3 \gamma(\mathbf{r}) u(\mathbf{r}, k) \\ &\equiv k^2 \int_V g(\mathbf{r} | \mathbf{r}', k) \gamma(\mathbf{r}') u(\mathbf{r}', k) d^3\mathbf{r}'. \end{aligned}$$

Here, \otimes_3 is taken to denote the three-dimensional convolution integral as defined above. The function u_s is the scattered wavefield.

A formal solution for u is obtained through the iteration [17]

$$u_{n+1}(\mathbf{r}, k) = u_i(\mathbf{r}, k) + k^2 g(r, k) \otimes_3 \gamma(\mathbf{r}) u_n(\mathbf{r}, k)$$

where $n = 0, 1, 2, 3, \dots$ with $u_0 = u_i$. If this series converges, then it must converge to the solution. To investigate the convergence, it is convenient to use operator notation and write

$$u_{n+1} = u_i + \hat{I}u_n$$

where \hat{I} is the convolution integral operator

$$\hat{I} = k^2 g \otimes_3 \gamma.$$

At each iteration n , we consider the solution to be given by

$$u_n = u + \epsilon_n$$

where ϵ_n is the error associated with the solution at iteration n and u is the exact solution. A necessary condition for convergence is then: $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Since

$$u + \epsilon_{n+1} = u_i + \hat{I}(u + \epsilon_n) = u_i + \hat{I}u + \hat{I}\epsilon_n$$

we can write

$$\epsilon_{n+1} = \hat{I}\epsilon_n$$

because $u = u_i + \hat{I}u$. Thus

$$\epsilon_1 = \hat{I}\epsilon_0; \quad \epsilon_2 = \hat{I}\epsilon_1 = \hat{I}(\hat{I}\epsilon_0); \quad \epsilon_3 = \hat{I}\epsilon_2 = \hat{I}[\hat{I}(\hat{I}\epsilon_0)]; \quad \dots$$

or

$$\epsilon_n = \hat{I}^n \epsilon_0$$

from which it follows that

$$\|\epsilon_n\| = \|\hat{I}^n \epsilon_0\| \leq \|\hat{I}^n\| \times \|\epsilon_0\| \leq \|\hat{I}\|^n \|\epsilon_0\|.$$

The condition for convergence therefore becomes

$$\lim_{n \rightarrow \infty} \|\hat{I}\|^n = 0$$

which requires that

$$\|\hat{I}\| < 1$$

or

$$k^2 \|g(r, k) \otimes_3 \gamma(\mathbf{r})\| < 1.$$

Since

$$\|g(r, k) \otimes_3 \gamma(\mathbf{r})\| \leq \|g(r, k)\| \|\gamma(\mathbf{r})\|$$

by considering γ to be a sphere of radius R with volume V , using an ℓ_2 norm and considering an upper bound, we can write this condition in the form

$$\bar{\gamma} < \frac{1}{k^2 R^2}$$

where $\bar{\gamma}$ is the root-mean-square value

$$\bar{\gamma} = \sqrt{\frac{\int_V \gamma^2 d^3\mathbf{r}}{\int_V d^3\mathbf{r}}}$$

which must be satisfied for the series to converge.

A. Weak and Strong Field Conditions

The scattered field can be written in the form

$$\begin{aligned} u_s(\mathbf{r}, k) &= k^2 g(r, k) \otimes_3 \gamma(\mathbf{r}) u_i(\mathbf{r}, k) \\ &+ k^4 g(r, k) \otimes_3 \gamma(\mathbf{r}) [g(r) \otimes_3 \gamma(\mathbf{r}) u_i(\mathbf{r}, k)] \\ &+ \dots \end{aligned}$$

which can be interpreted as follows:

$$\begin{aligned} &\text{Scattered wavefield} \\ &= \\ &\text{Wavefield generated by **single scattering** events} \\ &+ \\ &\text{Wavefield generated by **double scattering** events} \\ &+ \\ &\text{Wavefield generated by **triple scattering** events} \\ &+ \\ &\vdots \end{aligned}$$

Each term in this ‘Born series’ expresses the effects due to single, double and triple etc. scattering, i.e. the wavefields generated by an increasing number of interactions. A ‘strong

field’ is one in which the scattered field is generated by many multiple scattering events (i.e. many terms in the series solution given above). A ‘weak field’ is one in which the scattered field is generated from single scattering events alone. This requires that $\bar{\gamma}(kR)^2 \ll 1$ and provides a solution for the scattered field given by

$$u_s(\mathbf{r}, k) = k^2 g(r, k) \otimes_3 \gamma(\mathbf{r}) u_i(\mathbf{r}, k)$$

which is known as the Born approximation. The condition required for this approximation to apply shows that, in principle, large values of γ can occur so long as its root mean square value over the volume V is small compared to $(kR)^{-2}$ when γ is said to be a ‘weak scatterer’. Note that when k or R approach zero, this condition is easy to satisfy and that Born scattering is more likely to occur in situations when $\lambda/R \gg 1$ where λ is the wavelength. If $\lambda/R \sim 1$ then the value of $\bar{\gamma}$ must be small for Born scattering to occur.

For an incident plane wave, each term in the Born series includes scaling by r^{-1} , r^{-2} , r^{-3} etc. so that multiple-scattering gets ‘weaker by the term’. This is due to the form of the Green’s function in three-dimensions which scales as r^{-1} , the intensity of the field being r^{-2} - the inverse square law. Thus, if the scattering function is characterized by a number of scattering ‘sites’ (i.e. isolated positions in space where γ is non-zero and of compact support) then provided the distance between these sites is large, the effect of multiple scattering will be insignificant. However, if these sites are close together where the effect of a multiple scattered wavefield falling off as $1/r^2$, $1/r^3$ etc. is not appreciable, then multiple scattering events will contribute significantly to the scattered field. In this sense, we can interpret the meaning of a ‘weak field’ and a ‘strong field’ in terms of the ‘density’ of scattering sites over the volume V being low or high respectively. For $\lambda \sim R$ where R is the characteristic size of the scatterer, the Born approximation holds if $\bar{\gamma} \ll 1$. This is a quantification of the principle that the density of isolated scattering sites, from which a scattering function is composed, is low.

B. Weak Scattering in the Far Field

In the far field, the Green’s function has the form

$$g(\mathbf{r} \mid \mathbf{r}', k) = \frac{1}{4\pi r'} \exp(ikr') \exp(-ik\hat{\mathbf{n}}_s \cdot \mathbf{r}), \quad \frac{r}{r'} \ll 1$$

where $r = |\mathbf{r}|$, $r' = |\mathbf{r}'|$ and $\hat{\mathbf{n}}_s = \mathbf{r}'/r'$. Thus, when the incident field is a (unit) plane wave

$$u_i = \exp(ik\hat{\mathbf{n}}_i \cdot \mathbf{r})$$

where $\hat{\mathbf{n}}_i$ points in the direction of the incident field, the Born scattered field is given by (for some fixed value of $r' \gg r$)

$$u_s(\hat{\mathbf{n}}_s, \hat{\mathbf{n}}_i, k) = \frac{\exp(ikr')}{4\pi r'} A(\hat{\mathbf{n}}_s - \hat{\mathbf{n}}_i, k)$$

where A is the ‘scattering amplitude’ defined as

$$A(\hat{\mathbf{n}}_s - \hat{\mathbf{n}}_i, k) = k^2 \int_V \exp[-ik(\hat{\mathbf{n}}_s - \hat{\mathbf{n}}_i) \cdot \mathbf{r}] \gamma(\mathbf{r}) d^3\mathbf{r}.$$

From this result, it is clear that the scattered field is determined by the Fourier transform of the scattering function γ and that,

in principle, the scattering function can therefore be recovered from the scattered field by Fourier inversion. Observe that when $\hat{\mathbf{n}}_s = \hat{\mathbf{n}}_i$

$$A(k) = k^2 \int_V \gamma(\mathbf{r}) d^3 \mathbf{r}.$$

This is called the forward-scattered field. In terms of Fourier analysis, it represents the zero frequency or ‘DC (Direct Current) level’ of the spectrum of γ . Another special case arises when $\hat{\mathbf{n}}_s = -\hat{\mathbf{n}}_i$. The scattered field that is produced in this case is called the back-scattered field and is given by

$$A(\hat{\mathbf{n}}_i, k) = k^2 \int_V \exp(-2ik\hat{\mathbf{n}}_i \cdot \mathbf{r}) \gamma(\mathbf{r}) d^3 \mathbf{r}.$$

C. Inverse Scattering

Using operator notation, the Born series can be written as

$$u = u_i + \hat{I}_i \gamma + \hat{I}_i (\gamma \hat{I}_i \gamma) + I_i [\gamma \hat{I}_i (\gamma \hat{I}_i \gamma)] + \dots$$

where γ is either the scattering potential (for Schrödinger scattering) or $k^2 \gamma$ (for Helmholtz scattering) and

$$\hat{I}_i = \int d^3 \mathbf{r} u_i g, \quad \hat{I} = \int d^3 \mathbf{r} g.$$

If we let $\epsilon U = u - u_i$ and [18]

$$\gamma = \sum_{j=1}^{\infty} \epsilon^j \gamma_j.$$

Then

$$\begin{aligned} \epsilon U &= \hat{I}_i [\epsilon \gamma_1 + \epsilon^2 \gamma_2 + \epsilon^3 \gamma_3 + \dots] \\ &+ \hat{I}_i [(\epsilon \gamma_1 + \epsilon^2 \gamma_2 + \epsilon^3 \gamma_3 + \dots) \hat{I}_i (\epsilon \gamma_1 + \epsilon^2 \gamma_2 + \epsilon^3 \gamma_3 + \dots)] \\ &+ \hat{I}_i \{(\epsilon \gamma_1 + \epsilon^2 \gamma_2 + \epsilon^3 \gamma_3 + \dots) \hat{I}_i [(\epsilon \gamma_1 + \epsilon^2 \gamma_2 + \epsilon^3 \gamma_3 + \dots) \\ &\quad \hat{I}_i (\epsilon \gamma_1 + \epsilon^2 \gamma_2 + \epsilon^3 \gamma_3 + \dots)]\} + \dots \end{aligned}$$

Equating terms with common coefficients ϵ, ϵ^2 etc. we have for $j = 1$:

$$U = \hat{I}_i \gamma_1; \quad \gamma_1 = \hat{I}_i^{-1} U.$$

For $j = 2$:

$$0 = \hat{I}_i \gamma_2 + \hat{I}_i (\gamma_1 \hat{I}_i \gamma_1); \quad \gamma_2 = -\hat{I}_i^{-1} [\hat{I}_i (\gamma_1 \hat{I}_i \gamma_1)]$$

and so on. By computing the functions γ_j using this iterative method, the scattering function γ is obtained by summing γ_j for $\epsilon = 1$. This approach provides a formal exact inverse scattering solution but it is not unconditional, i.e. the inverse solution is only applicable when the Born series converges to the exact scattering solution.

For $j = 1$, the solution γ_1 is that obtained under the Born approximation, i.e. given that

$$u = u_i + \hat{I}_i \gamma = u_i(\mathbf{r}, k) + k^2 g(r, k) \otimes_3 \gamma(\mathbf{r}) u_i(\mathbf{r}, k)$$

then

$$\gamma = \hat{I}_i^{-1} (u - u_i).$$

IV. SCATTERING FROM RANDOM MEDIA

Analysis of scattering from a random medium ideally requires a model for the physical behaviour of the random variable(s) that is derived from basic principles. This involves modelling the scattered field in terms of its interaction with an ensemble of ‘scattering sites’ based on an assumed stochastic process. If the density of these scattering sites is low enough so that multiple scattering is minimal, then we can apply Born scattering to develop a model for the intensity of a wavefield interacting with a ‘random Born scatterer’.

In the far field, the Born scattered field (i.e. the scattering amplitude) is given by the Fourier transform of the scattering function. If this function is known *a priori*, then the scattering amplitude can be determined. This is an example of a deterministic model. If the scattering function is stochastic (i.e. a randomly distributed scatterer) such that it can only be quantified in terms of a statistical distribution (i.e. the probability density function (PDF) - denoted by Pr) then we can simulate the (Born) scattered field by designing a random number generator that outputs deviates that conform to this distribution. The Fourier transform of this stochastic field then provides the Born scattering amplitude. Thus, given a three dimensional Helmholtz scattering function $\gamma(\mathbf{r})$, $\mathbf{r} \in V$ with $\text{Pr}[\gamma(\mathbf{r})]$ known *a priori*, the scattering amplitude A is given by

$$A(\hat{\mathbf{N}}, k) = k^2 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r}) \gamma(\mathbf{r}) d^3 \mathbf{r}$$

where $\hat{\mathbf{N}} = \hat{\mathbf{n}}_s - \hat{\mathbf{n}}_i$ and $\gamma(\mathbf{r})$ is a stochastic function whose deviates conform to the PDF $\text{Pr}[\gamma(\mathbf{r})]$.

The intensity of the scattering amplitude is given by

$$\begin{aligned} I(\hat{\mathbf{N}}, k) &= |A(\hat{\mathbf{N}}, k)|^2 = A(\hat{\mathbf{N}}, k) A^*(\hat{\mathbf{N}}, k) \\ &= k^4 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r}) \gamma(\mathbf{r}) d^3 \mathbf{r} \int_V \exp(ik\hat{\mathbf{N}} \cdot \mathbf{r}') \gamma^*(\mathbf{r}') d^3 \mathbf{r}'. \end{aligned}$$

Using the autocorrelation theorem, we have

$$I(\hat{\mathbf{N}}, k) = k^4 \int_V \exp(-ik\hat{\mathbf{N}} \cdot \mathbf{r}) \Gamma(\mathbf{r}) d^3 \mathbf{r}$$

where Γ is the autocorrelation function given by

$$\Gamma(\mathbf{r}) = \int_V \gamma(\mathbf{r}') \gamma^*(\mathbf{r}' + \mathbf{r}) d^3 \mathbf{r}' = \int_V \gamma(\mathbf{r}') \gamma(\mathbf{r}' + \mathbf{r}) d^3 \mathbf{r}'$$

since $\gamma = \epsilon_r - 1$ is real.

This result allows us to evaluate the intensity of the Born scattered amplitude by computing the Fourier transform of the autocorrelation of the scattering function which is taken to be composed of a number of scatterers distributed at random throughout V . This requires the autocorrelation function to be defined for a particular type of random scatterer. Thus, a random medium can be characterized via its autocorrelation function by measuring the scattered intensity and inverse Fourier transforming the result.

From the autocorrelation theorem, the characteristics of the autocorrelation function can be formulated by considering its expected spectral properties since

$$\Gamma(\mathbf{r}) \leftrightarrow |\tilde{\gamma}(\mathbf{k})|^2$$

where $\tilde{\gamma}$ is the Fourier transform of γ , \mathbf{k} is the spatial frequency vector and \leftrightarrow denotes the transformation from real space \mathbf{r} to Fourier space \mathbf{k} . Hence, in order to evaluate the most likely form of the autocorrelation function we can consider the properties of the power spectrum of the scattering function. If this function is ‘white’ noise, for example (i.e. its Power Spectral Density Function or PSDF is a constant), then the autocorrelation function is a delta function whose Fourier transform is a constant. However, in practice, we can expect that few scattering functions have a PSDF characterized by white noise, rather, the PSDF will tend to decay as the frequency increases. For example, we can consider a model for the PSDF based on the Gaussian function

$$|\tilde{\gamma}(\mathbf{k})|^2 = \tilde{\gamma}_0^2 \exp\left(-\frac{k^2}{k_0^2}\right)$$

where $\tilde{\gamma}_0 = \tilde{\gamma}(0)$, $k = |\mathbf{k}|$ and k_0 is the standard deviation which is a measure of the correlation length. This PSDF yields an autocorrelation function which is of the same type, i.e. a Gaussian function. If the ‘geometry’ of the scattering function is statistically self-affine, then we can model the scattering function as a random fractal whose PSDF is characterized by (for a Topological Dimension of 3 and Fractal Dimension denoted by D_F) [15]

$$|\tilde{\gamma}(\mathbf{k})|^2 \sim \frac{1}{k^{11-2D_F}}$$

where $3 < D_F < 4$, the autocorrelation function being characterized by [15]

$$\Gamma(\mathbf{r}) \sim \frac{1}{r^{D_F-2.5}}.$$

Other issues in determining the nature of the autocorrelation function are related to the physical conditions imposed on the stochastic characteristics of the scatterer.

The method discussed above can be used to model the (Born) scattered intensity from a random medium which requires an estimate of the autocorrelation of the scattering function to be known. However, this approach assumes that the density of scattering sites from which the scatterer is composed is low so that the Born approximation is valid. When the density of scattering sites increases and multiple scattering is present, the problem become progressively more complex. One approach to overcoming this problem is to resort to a stochastic theory which involves developing a statistical model, not for the scattering function, but for the scattered field itself. This is discussed in the following section.

V. STATISTICAL MODELS

Random walk methods are used as the basis for generating stochastic scattering models where the scattering of a wavefield from one scattering site to another is taken to be a random walk in the (complex) plane with arbitrary amplitude

and phase variations. We consider the wavefield E (e.g. the electric field) to be given by

$$E = \sum_{j=1}^N r_j \exp(i\phi_j) = R \exp(i\Phi)$$

where r, ϕ and N are independent random variables. Both r and ϕ are assumed to be continuous random variables and N is discrete. We can write E as a vector, whose components are the real and imaginary parts of E , i.e.

$$\mathbf{E} = (E_{\text{real}}, E_{\text{imag}}).$$

It is useful to work in terms of the characteristic function of a complex random variable

$$\mathbf{U} = (U_{\text{real}}, U_{\text{imag}})$$

defined as (2D inverse Fourier transform)

$$C(\mathbf{U}) = \langle \exp(i\mathbf{E} \cdot \mathbf{U}) \rangle = \int \exp(i\mathbf{E} \cdot \mathbf{U}) P(\mathbf{E}) d\mathbf{E}$$

where the integral is taken to over all \mathbf{E} and where $P(\mathbf{E})$ is the Probability Density Function (PDF) of \mathbf{E} . Thus, P can be computed from C via the 2D Fourier transform, i.e.

$$P(\mathbf{E}) = \frac{1}{(2\pi)^2} \int \exp(-i\mathbf{E} \cdot \mathbf{U}) C(\mathbf{U}) d\mathbf{U}$$

where the integral is taken over all \mathbf{U} .

The aim of this calculation is to find an expression for P . This is done by first computing $C(\mathbf{U}) = \langle \exp(i\mathbf{E} \cdot \mathbf{U}) \rangle$ and then taking the inverse Fourier transform to evaluate P . The calculation of the characteristic function will be based on the following assumptions: (i) The phase is uniformly distributed which represents strong scattering; (ii) the scattering events at each site are independent; (iii) N conforms to a negative binomial distribution of the form

$$P_N = \binom{N + \alpha - 1}{N} \frac{(\bar{N}/\alpha)^N}{(1 + \bar{N}/\alpha)^{N+\alpha}}$$

where \bar{N} is the mean of the distribution and α is a ‘bunching’ parameter. Clearly $\alpha > \bar{N}$ for P_N to be a proper PDF. Assumption (iii) above is based on a birth-death-migration processes which is representative of the distribution of scatterers.

To find $\langle \exp(i\mathbf{E} \cdot \mathbf{U}) \rangle$ we write \mathbf{E} and \mathbf{U} in terms of their real and imaginary components, i.e.

$$\mathbf{E} = (R \cos \Phi, R \sin \Phi), \quad \mathbf{U} = (U \cos \chi, U \sin \chi)$$

where $U \equiv |\mathbf{U}|$. Here R is the resultant amplitude and Φ is the resultant phase that is detected:

$$\begin{aligned} \mathbf{E} \cdot \mathbf{U} &= R \cos \Phi U \cos \chi - R \sin \Phi U \sin \chi \\ &= U \sum_{j=1}^N r_j (\cos \phi_j \cos \chi - \sin \phi_j \sin \chi) = U \sum_{j=1}^N r_j \cos(\phi_j + \chi). \end{aligned}$$

Hence, the characteristic function for a random walk with N steps is

$$C_N(\mathbf{U}) = \langle \exp[iU \sum_{j=1}^N r_j \cos(\phi_j + \chi)] \rangle.$$

Since

$$\exp(x_1 + x_2 + \dots + x_N) = \exp(x_1) \exp(x_2) \exp(x_3) \dots \exp(x_N),$$

$$C_N(\mathbf{U}) = \left\langle \prod_{j=1}^N \exp[iUr_j \cos(\phi_j + \chi)] \right\rangle.$$

The variables r, ϕ and N are independent. Assumption (ii) given above means that r_j is independent of r_k , i.e. a scattering event at site j is independent of a scattering event at site k . The net effect of this assumption is to eliminate conditional probabilities from the scattering process. In this case, the product can be taken outside the average, giving

$$C_N(\mathbf{U}) = \prod_{j=1}^N \langle \exp[iUr_j \cos(\phi_j + \chi)] \rangle.$$

The term $\langle \exp[iUr_j \cos(\phi_j + \chi)] \rangle$ is an average over both the amplitude distribution and the phase distribution. Assuming that the phases are uniformly distributed (strong scattering), the integral for the phase can be written as

$$\langle \exp[iUr_j \cos(\phi_j + \chi)] \rangle_\phi = \int_{-\pi}^{\pi} \exp(iUr_j \cos(\phi + \chi)) P_j(\phi) d\phi$$

where P_j is the uniform phase distribution defined as

$$P_j(\phi) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \phi < \pi; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the integral

$$I = \int_{-\pi}^{\pi} \exp[iUr_j \cos(\phi + \chi)] d\phi.$$

To evaluate this integral we use the following identity

$$\exp(i\alpha \cos \theta) = J_0(\alpha) + 2 \sum_{k=1}^{\infty} i^k J_k(\alpha) \cos k\theta$$

where J_k is the Bessel function of order k . Then

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \left[J_0(\alpha) + 2 \left(\sum_{k=1}^{\infty} i^k J_k(\alpha) \cos k\theta \right) \right] d\theta \\ &= [J_0(\alpha)\theta]_{-\pi}^{\pi} + \left[2 \sum_{k=1}^{\infty} \frac{i^k}{k} J_k(\alpha) \sin k\theta \right]_{-\pi}^{\pi} = 2\pi J_0(\alpha). \end{aligned}$$

Hence,

$$\langle \exp(i\mathbf{E} \cdot \mathbf{U}) \rangle_\phi = \langle \exp[iUr_j \cos(\phi_j + \chi)] \rangle_\phi = J_0(Ur_j)$$

where

$$U = \sqrt{U_{\text{real}}^2 + U_{\text{imag}}^2}$$

and

$$C_N(\mathbf{U}) = \prod_{j=1}^N \langle J_0(Ur_j) \rangle_r$$

where

$$\langle J_0(Ur_j) \rangle_r = \int_0^{\infty} J_0(Ur) P_j(r) dr.$$

Here $P_j(r)$ is the PDF for r . Now, if all the scattering processes are similar, then they will all have the same PDF and therefore

$$\prod_{j=1}^N \langle J_0(Ur_j) \rangle_r = \langle J_0(Ur) \rangle_r^N = \left(\int_0^{\infty} J_0(Ur) P(r) dr \right)^N.$$

This result depends on the number of steps N which is itself a random variable, and, in order to proceed further, we must consider a PDF for N . For this purpose we consider the negative binomial distribution - assumption (iii) - and develop an expression for the characteristic function for the mean \bar{N} of N . This is given by

$$\begin{aligned} C_{\bar{N}}(\mathbf{U}) &= \sum_{N=0}^{\infty} P_N C_N(\mathbf{U}) \\ &= \sum_{N=0}^{\infty} \binom{N + \alpha - 1}{N} \frac{(\bar{N}/\alpha)^N}{(1 + \bar{N}/\alpha)^{N+\alpha}} \langle J_0(Ur) \rangle_r^N \\ &= \sum_{N=0}^{\infty} \frac{(N + \alpha - 1)!}{N!(\alpha - 1)!} \left(\frac{(\bar{N}/\alpha) \langle J_0(Ur) \rangle_r}{1 + \bar{N}/\alpha} \right)^N \frac{1}{(1 + \bar{N}/\alpha)^\alpha} \\ &= \frac{1}{(\alpha - 1)!(1 + \bar{N}/\alpha)^\alpha} \sum_{N=0}^{\infty} \frac{(N + \alpha - 1)!}{N!} \mu^N \end{aligned}$$

where

$$\mu = \frac{(\bar{N}/\alpha) \langle J_0(Ur) \rangle_r}{1 + \bar{N}/\alpha}.$$

Now,

$$\begin{aligned} &\sum_{N=0}^{\infty} \frac{(N + \alpha - 1)!}{N!} \mu^N \\ &= (\alpha - 1)! \left(1 + \alpha\mu + \frac{\alpha(1 + \alpha)}{2!} \mu^2 + \dots \right) = (\alpha - 1)! (1 - \mu)^{-\alpha} \end{aligned}$$

and therefore we can write

$$\begin{aligned} C_{\bar{N}}(\mathbf{U}) &= \frac{(\alpha - 1)!}{(\alpha - 1)!(1 + \bar{N}/\alpha)^\alpha} \frac{1}{(1 - \mu)^\alpha} \\ &= \frac{(1 + \bar{N}/\alpha)^\alpha}{(1 + \bar{N}/\alpha)^\alpha (1 + \bar{N}/\alpha - (\bar{N}/\alpha) \langle J_0(Ur) \rangle_r)^\alpha} \\ &= \left(1 + \frac{\bar{N}}{\alpha} (1 - \langle J_0(Ur) \rangle_r) \right)^{-\alpha}. \end{aligned}$$

The calculation of $\langle J_0(Ur) \rangle_r$ is based on a small but important modification whereby we scale r according to $r \rightarrow r/\sqrt{\bar{N}}$. Thus, we consider

$$\langle J_0(Ur) \rangle_r = \int_0^{\infty} P(r) J_0(Ur/\sqrt{\bar{N}}) dr.$$

As $\bar{N} \rightarrow \infty$, this modification of the definition of $\langle J_0(Ur) \rangle_r$ allows us to employ the Frobenius series for J_0 , i.e.

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{2^6} - \dots$$

then

$$\langle J_0(Ur) \rangle_r$$

$$\begin{aligned}
&= \int_0^{\infty} P(r)dr - \frac{1}{4} \int_0^{\infty} \frac{U^2 r^2}{\bar{N}} P(r)dr + \frac{1}{2^6} \int_0^{\infty} \frac{U^4 r^4}{\bar{N}^2} P(r)dr - \dots \\
&= 1 - \frac{1}{4} \frac{U^2}{\bar{N}} \langle r^2 \rangle + \frac{1}{2^6} \frac{U^4}{\bar{N}^2} \langle r^4 \rangle - \dots
\end{aligned}$$

where

$$\langle r^n \rangle = \int_0^{\infty} r^n P(r)dr.$$

Hence, we can write

$$\begin{aligned}
&C_{\bar{N}}(\mathbf{U}) \\
&= \left[1 + \frac{\bar{N}}{\alpha} \left(1 - \left(1 - \frac{1}{4} \frac{U^2}{\bar{N}} \langle r^2 \rangle + \frac{1}{2^6} \frac{U^4}{\bar{N}^2} \langle r^4 \rangle - \dots \right) \right) \right] \\
&= \left(1 + \frac{1}{4} \frac{U^2}{\alpha} \langle r^2 \rangle - \frac{1}{2^6} \frac{U^4}{\bar{N}\alpha} \langle r^4 \rangle + \dots \right)^{-\alpha}
\end{aligned}$$

and

$$C(\mathbf{U}) = \lim_{\bar{N} \rightarrow \infty} C_{\bar{N}}(\mathbf{U}) = \left(1 + \frac{1}{4} \frac{U^2}{\alpha} \langle r^2 \rangle \right)^{-\alpha}.$$

This result allows us to compute the PDF of $\mathbf{E} = R \exp(i\Phi)$ which can be obtained by evaluating the Fourier integral of $C(\mathbf{U})$, i.e.

$$\begin{aligned}
P(\mathbf{E}) &= \frac{1}{(2\pi)^2} \int_{\forall \mathbf{U}} \exp(-i\mathbf{E} \cdot \mathbf{U}) C(\mathbf{U}) d\mathbf{U} \\
&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{\exp(-i\mathbf{E} \cdot \mathbf{U})}{\left(1 + \frac{1}{4} \frac{U^2}{\alpha} \langle r^2 \rangle \right)^{\alpha}} U dU d\chi.
\end{aligned}$$

Integrating over χ generates a Bessel function as before

$$P(\mathbf{E}) = \frac{1}{2\pi} \int_0^{\infty} \frac{U J_0(UR)}{\left(1 + \frac{1}{4} \frac{U^2}{\alpha} \langle r^2 \rangle \right)^{\alpha}} dU.$$

Evaluating the final integral gives

$$P(\mathbf{E}) = \frac{1}{2\pi 2^{\alpha-1} \Gamma(\alpha)} \left(\frac{4\alpha}{\langle r^2 \rangle} \right)^{\frac{1+\alpha}{2}} K_{\alpha-1} \left[R \left(\frac{4\alpha}{\langle r^2 \rangle} \right)^{\frac{1}{2}} \right]$$

where $K_{\alpha-1}$ is a modified Bessel function. The PDF of the amplitude follows by integrating $P(\mathbf{E})$ over all values of the phase Φ . However, $P(\mathbf{E})$ is independent of Φ and so this integral yield 2π , i.e.

$$P(R) = \int_{-\pi}^{\pi} P(\mathbf{E}) R d\Phi = 2\pi R P(\mathbf{E})$$

$P(R)$ can therefore be written as

$$P(R) = \frac{\beta^{1+\alpha}}{2^{\alpha-1} \Gamma(\alpha)} R^{\alpha} K_{\alpha-1}(\beta R)$$

where

$$\beta = \left(\frac{4\alpha}{\langle r^2 \rangle} \right)^{\frac{1}{2}}.$$

This is the so called ‘ K -distribution’ whose calculation illustrates the way in which the PDF of an image can be derived

subject to a model for the distribution of the phase (in this case, a uniform phase distribution representing strong scattering) and a statement of the characteristics of the random walk (in this case, a negative binomial distribution for the number of steps N). The PDF derived can then be used to characterize a signal or image (that has been generated by strong and coherent scattering processes) statistically by computing the parameters α , β and $\langle r^2 \rangle$. Although this approach may be of value to the statistical analysis of a signal/image, it does not provide a solution to the inverse scattering problem. For this purpose, optical diffusion models for strong scattering are required as discussed below.

VI. OPTICAL DIFFUSION

When light is scattered by one localized centre, the single scattering approximation can be used, but when these centres are grouped together, multiple light scattering occurs. The randomness of multiple interactions tends to be averaged out by the large number of scattering events that occur leading to a deterministic distribution of intensity. This is exemplified by a light passing through thick fog, for example. In this sense, multiple scattering is highly analogous to diffusion, and the terms multiple scattering and diffusion are interchangeable in many contexts. Optical elements designed to produce multiple scattering are thus known as diffusers. The diffusion equation can then be used to model such systems in the same way as it can be used to model temperature distributions and particle concentrations, for example, and any system that is the result of a large ensemble of particles/waves undergoing random elastic collisions/interactions (at least when there is no directional bias associated with the scattering processes).

Suppose we consider the three-dimensional diffusion of light to be based on a three-dimensional random walk. Each scattering event is taken to be a point of the random walk in which a ray of light changes its direction randomly (any direction between 0 and 4π radians). The light field is taken to be composed of a complex of rays, each of which propagates through the diffuser in a way that is incoherent and uncorrelated in time. If this is the case, then the propagation of light can be considered to be analogous to the process of (classical) diffusion. Instead of modelling the process in terms of the (inhomogeneous) wave equation (as discussed in Section III)

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})} \frac{\partial^2}{\partial t^2} \right) u(\mathbf{r}, t) = 0$$

with intensity given by $I(\mathbf{r}, t) = |u(\mathbf{r}, t)|^2$ we consider the intensity to be given by the solution of the homogeneous diffusion equation

$$\left(\nabla^2 - \frac{1}{D} \frac{\partial}{\partial t} \right) I(\mathbf{r}, t) = 0$$

with initial condition $I(\mathbf{r}, t) = I_0(\mathbf{r})$ at $t = 0$. This assumes that the diffusivity D is constant throughout the diffuser which is taken to be an isotropic medium. This is analogous to assuming that $\text{Pr}[c(\mathbf{r})]$ for a random scattering model (based on a solution to the wave equation) is the same throughout the diffuser and thus, that the autocorrelation function $\Gamma(\mathbf{r})$

required to compute the intensity (as discussed in Section IV) is stationary.

In multiple wave scattering theory, we consider a wavefront travelling through space and scattering from a site that changes the direction of propagation. The mean free path is taken to be the average number of wavelengths taken by the wavefront to propagate from one interaction to another as described by the free space Green's function. After scattering from many sites, the wavefront can be considered to have diffused through the 'diffuser'. Here, the mean free path is a measure of the density of scattering sites, which in turn, is a measure of the diffusivity of the medium D . As D becomes larger, the medium is taken to be increasingly tenuous allowing for a greater 'flux' of light (i.e. Ficks law for the diffusion flux \mathbf{J} given by $\mathbf{J} = -D\nabla C$ where C is the 'concentration') with diffusion length \sqrt{Dt} which is a measure of how far the field has 'propagated' by diffusion in time t . In the following section, we derived a diffusion equation for light intensity $|u(\mathbf{r}, k)|^2$ given the homogeneous wave equation for the light field $u(\mathbf{r}, k)$.

A. Derivation of the Diffusion Equation

Consider the three-dimensional homogeneous time dependent wave equation

$$\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} u = 0$$

where c_0 is taken to be a constant (light speed) and $I = |u|^2$ defines the intensity of light Let

$$u(x, y, z, t) = \phi(x, y, z, t) \exp(i\omega t)$$

where it is assumed that field ϕ varies significantly slowly in time compared with $\exp(i\omega t)$ and note that

$$u^*(x, y, z, t) = \phi^*(x, y, z, t) \exp(-i\omega t)$$

is also a solution to the wave equation. Differentiating

$$\nabla^2 u = \exp(i\omega t) \nabla^2 \phi,$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u &= \exp(i\omega t) \left(\frac{\partial^2}{\partial t^2} \phi + 2i\omega \frac{\partial \phi}{\partial t} - \omega^2 \phi \right) \\ &\simeq \exp(i\omega t) \left(2i\omega \frac{\partial \phi}{\partial t} - \omega^2 \phi \right) \end{aligned}$$

when

$$\left| \frac{\partial^2 \phi}{\partial t^2} \right| \ll 2\omega \left| \frac{\partial \phi}{\partial t} \right|.$$

Under this condition, the wave equation reduces to

$$(\nabla^2 + k^2)\phi = \frac{2ik}{c_0} \frac{\partial \phi}{\partial t}$$

where $k = \omega/c_0$. However, since u^* is also a solution,

$$(\nabla^2 + k^2)\phi^* = -\frac{2ik}{c_0} \frac{\partial \phi^*}{\partial t}$$

and thus,

$$\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^* = \frac{2ik}{c_0} \left(\phi^* \frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi^*}{\partial t} \right)$$

which can be written in the form

$$\nabla^2 I - 2\nabla \cdot (\phi \nabla \phi^*) = \frac{2ik}{c_0} \frac{\partial I}{\partial t}$$

where $I = \phi \phi^* = |\phi|^2$. Let ϕ be given by

$$\phi(\mathbf{r}, t) = A(\mathbf{r}, t) \exp(ik\hat{\mathbf{n}} \cdot \mathbf{r})$$

where $\hat{\mathbf{n}}$ is a unit vector and A is the amplitude function. Differentiating, and noting that $I = A^2$, we obtain

$$\hat{\mathbf{n}} \cdot \nabla A = \frac{2}{c_0} \frac{\partial A}{\partial t}$$

or

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) A(x, y, z, t) = \frac{2}{c_0} \frac{\partial}{\partial t} A(x, y, z, t)$$

which is the unconditional continuity equation for the amplitude A of a wavefield

$$u(\mathbf{r}, t) = A(\mathbf{r}, t) \exp[i(k\hat{\mathbf{n}} \cdot \mathbf{r} + \omega t)]$$

where A varies slowly with time.

The equation

$$\nabla^2 I - 2\nabla \cdot (\phi \nabla \phi^*) = \frac{2ik}{c_0} \frac{\partial I}{\partial t}$$

is valid for $k = k_0 - i\kappa$ (i.e. $\omega = \omega_0 - i\kappa c_0$) and so, by equating the real and imaginary parts, we have

$$D\nabla^2 I + 2\text{Re}[\nabla \cdot (\phi \nabla \phi^*)] = \frac{\partial I}{\partial t}$$

and

$$\text{Im}[\nabla \cdot (\phi \nabla \phi^*)] = -\frac{k_0}{c_0} \frac{\partial I}{\partial t}$$

respectively where $D = c_0/2\kappa$, so that under the condition

$$\text{Re}[\nabla \cdot (\phi \nabla \phi^*)] = 0$$

we obtain

$$D\nabla^2 I = \frac{\partial I}{\partial t}.$$

This is the diffusion equation for the intensity of light I . The condition required to obtain this result can be justified by applying a boundary condition on the surface S of a volume V over which the equation is taken to conform. Using the divergence theorem

$$\begin{aligned} \text{Re} \int_V \nabla \cdot (\phi \nabla \phi^*) d^3\mathbf{r} &= \text{Re} \oint_S \phi \nabla \phi^* \cdot \hat{\mathbf{n}} d^2\mathbf{r} \\ &= \oint_S (\phi_r \nabla \phi_r + \phi_i \nabla \phi_i) \cdot \hat{\mathbf{n}} d^2\mathbf{r}. \end{aligned}$$

Now, if

$$\phi_r(\mathbf{r}, t) \nabla \phi_r(\mathbf{r}, t) = -\phi_i(\mathbf{r}, t) \nabla \phi_i(\mathbf{r}, t), \quad \mathbf{r} \in S$$

then the surface integral is zero and

$$D\nabla^2 I(\mathbf{r}, t) = \frac{\partial}{\partial t} I(\mathbf{r}, t), \quad \mathbf{r} \in V.$$

This boundary condition can be written as

$$\frac{\nabla \phi_r}{\nabla \phi_i} = -\tan\theta$$

where θ is the phase of the field ϕ which implies that the amplitude A of ϕ is constant on the boundary (i.e. $A(\mathbf{r}, t) = A_0$, $\mathbf{r} \in S$, $\forall t$), since

$$\begin{aligned} \frac{\nabla A_0 \cos \theta(\mathbf{r}, t)}{\nabla A_0 \sin \theta(\mathbf{r}, t)} &= -\frac{A_0 \sin \theta(\mathbf{r}, t) \nabla \theta(\mathbf{r}, t)}{A_0 \cos \theta(\mathbf{r}, t) \nabla \theta(\mathbf{r}, t)} \\ &= -\tan \theta(\mathbf{r}, t), \quad \mathbf{r} \in S. \end{aligned}$$

Suppose we record the intensity I of a light field in the xy -plane for a fixed value of z . Then for $z = z_0$ say,

$$I(x, y, t) \equiv I(x, y, z_0, t)$$

so that

$$\frac{\partial}{\partial t} I(x, y, t) = D \nabla^2 I(x, y, t).$$

Let this two-dimensional diffusion equation be subject to the initial condition

$$I(x, y, 0) = I_0(x, y).$$

Then, at any time $t > 0$, it can be assumed that light diffusion is responsible for blurring the image I_0 and that as time increases, the image becomes progressively more diffused, the solution being given by (see Appendix I), for the infinite domain [8], [14]

$$I(x, y, t) = \frac{1}{4\pi Dt} \exp \left[-\left(\frac{x^2 + y^2}{4Dt} \right) \right] \otimes_2 I_0(x, y)$$

where \otimes_2 denotes the two-dimensional convolution integral.

B. Inverse Solution

If we record an image at a time $t = T$ then by Taylor expanding I at $t = 0$ we can write

$$I(x, y, 0) = I(x, y, T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n \left[\frac{\partial^n}{\partial t^n} I(x, y, t) \right]_{t=T}$$

From the diffusion equation

$$\begin{aligned} \frac{\partial^2 I}{\partial t^2} &= D \nabla^2 \frac{\partial I}{\partial t} = D^2 \nabla^4 I \\ \frac{\partial^3 I}{\partial t^3} &= D \nabla^2 \frac{\partial^2 I}{\partial t^2} = D^3 \nabla^6 I \end{aligned}$$

and so on. Thus, by induction, we can write

$$\left[\frac{\partial^n}{\partial t^n} I(x, y, t) \right]_{t=T} = D^n \nabla^{2n} I(x, y, T).$$

Substituting this result into the series for I_0 given above, we get

$$\begin{aligned} I_0(x, y) &= I(x, y, T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (DT)^n \nabla^{2n} I(x, y, T) \\ &\sim I(x, y, T) - DT \nabla^2 I(x, y, T), \quad DT \ll 1. \end{aligned}$$

VII. HURST PROCESSING AND FRACTIONAL DIFFUSION

The diffusion equation models a macroscopic field which is the result of an ensemble of incoherent random walks characterised by a \sqrt{t} scaling law. Hurst processes, describe random walks that have a directional bias and are characterised by the scaling law t^H , $H \in (0.5, 1]$ [21]. As the value of H approaches 1, the random processes become increasingly persistent. In terms of the multiple scattering of light from a random medium, increasing persistence relates to multiple scattering from fewer sites so that the light path has a greater directional bias. We consider the characterisation of this by generalizing the diffusion operator

$$\nabla^2 - \sigma \frac{\partial}{\partial t}$$

to the fractional form [22], [23]

$$\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q}$$

where $q \in [1, 2]$ and $D^q = 1/\sigma^q$ is the fractional diffusivity. Fractional diffusive processes can therefore be interpreted as intermediate between diffusive processes proper (random phase walks with $H = 0.5$; diffusive processes with $q = 1$) and ‘propagative processes’ (coherent phase walks for $H = 1$; propagative processes with $q = 2$). It should be noted that the fractional diffusion operator given above is the result of a phenomenology. It is a generalisation of a well known differential operator to fractional form which follows from a physical analysis of a fully incoherent random process and its generalisation to fractional form, just as the Hurst exponent H is a generalisation of the \sqrt{t} scaling law. The solution to fractional partial differential equations of this type requires application of the fractional calculus [24] - [28] which is developed in the following section. For readers that are not familiar with fractional calculus, Appendix II provides a brief overview of the subject area and introduces results that are used throughout the rest of this paper.

A. Solution to the Fractional Diffusion Equation

Consider the fractional diffusion equation for the intensity $I(x, y, t)$ of light in the image plane z given by

$$\nabla^2 I(\mathbf{r}, t) - \sigma^q \frac{\partial^q}{\partial t^q} I(\mathbf{r}, t) = I_0(\mathbf{r}) \delta(t)$$

where $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ and $I_0(\mathbf{r})$ is a source function with an impulse at $t = 0$. For $q = 1$, the solution to this equation in the infinite domain is (with $r = |\mathbf{r}|$ and $I(\mathbf{r}, t = 0) = 0$ as shown in Appendix I)

$$I(\mathbf{r}, t) = I_0(\mathbf{r}) \otimes_2 G(r, t)$$

where, for $t > 0$,

$$G(r, t) = \frac{1}{4\pi t} \exp \left[-\left(\frac{\sigma r^2}{4t} \right) \right]$$

which is the solution of

$$\left(\nabla^2 - \sigma \frac{\partial}{\partial t} \right) G(r, t) = -\delta^2(r) \delta(t).$$

For the fractional diffusion equation, we consider a similar (Green's function) solution but where the Green's function is given by the solution of

$$\left(\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q}\right) G(\mathbf{r}, t) = -\delta^2(\mathbf{r})\delta(t).$$

Using the Fourier based operator for a fractional derivative, we can transform this equation into the form

$$(\nabla^2 + \Omega_q^2)g(\mathbf{r} | \mathbf{r}', \omega) = -\delta^2(\mathbf{r} - \mathbf{r}')$$

where

$$g(\mathbf{r} | \mathbf{r}', \omega) = \int_{-\infty}^{\infty} G(\mathbf{r} | \mathbf{r}', t) \exp(-i\omega t) dt,$$

$$\Omega_q^2 = -i\omega\sigma, \quad \Omega_q = \pm i(i\omega\sigma)^{q/2}.$$

Note that for $q = 2$, this equation becomes

$$(\nabla^2 + k^2)g(\mathbf{r} | \mathbf{r}', \omega) = \delta^2(\mathbf{r} - \mathbf{r}')$$

where $k = \pm\omega\sigma$. This equation defines the Green's function for the time independent wave operator in two-dimensions, the 'out going' Green's functions being given by [8], [16], [14]

$$g(\mathbf{r} | \mathbf{r}', k) = \frac{i}{4} H_0(k | \mathbf{r} - \mathbf{r}' |)$$

$$\simeq \frac{1}{\sqrt{8\pi}} \exp(i\pi/4) \frac{\exp(ik | \mathbf{r} - \mathbf{r}' |)}{\sqrt{k | \mathbf{r} - \mathbf{r}' |}}, \quad k | \mathbf{r} - \mathbf{r}' | \gg 1$$

where H_0 is the Hankel function. Generalizing this result, for $q \in (1, 2)$, by writing the exponential function in its series form, with $R = | \mathbf{r} - \mathbf{r}' |$ we have, for $\Omega_q = i(i\omega\sigma)^{q/2}$,

$$\begin{aligned} G(R, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \frac{\exp(i\pi/4)}{\sqrt{8\pi}} \frac{\exp[-(i\omega\sigma)^{q/2} R]}{\sqrt{iR}(i\omega\sigma)^{q/4}} \\ &= \frac{1}{\sqrt{8\pi R}} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \dots \\ &\dots \left(\frac{1}{(i\omega\sigma)^{q/4}} - (i\omega\sigma)^{q/4} R + \frac{1}{2!} (i\omega\sigma)^{3q/4} R^2 - \dots \right) \\ &= \frac{1}{\sqrt{8\pi R}} \frac{1}{\sigma^{q/4} t^{1-q/4}} - \sqrt{\frac{R}{8\pi}} \sigma^{q/4} \delta^{q/4}(t) \\ &+ \frac{1}{\sqrt{8\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} R^{(2n+1)/2} \sigma^{3nq/4} \delta^{3nq/4}(t). \end{aligned}$$

Simplification of this infinite sum can be addressed by considering suitable asymptotics, the most significant of which (for arbitrary values of R) is the case when the (fractional) diffusivity D is large. In particular, we note that as $\sigma \rightarrow 0$,

$$G(R, t) = \frac{1}{\sqrt{8\pi R} \sigma^{q/4} t^{1-(q/4)}}.$$

Thus, we can consider a solution to the two-dimensional fractional diffusion equation (for a tenuous medium when $\sigma \rightarrow 0$)

$$\left(\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q}\right) I(\mathbf{r}, t) = I_0(\mathbf{r})\delta(t)$$

of the form

$$I(x, y) = \frac{1}{2\sqrt{2\pi}} \frac{1}{(DT)^{1-q/4}} \frac{1}{(x^2 + y^2)^{1/4}} \otimes_2 I_0(x, y).$$

Comparing this solution with the solution to the two-dimensional diffusion equation, i.e.

$$I(x, y) = \frac{1}{4\pi DT} \exp\left[-\left(\frac{x^2 + y^2}{4DT}\right)\right] \otimes_2 I_0(x, y),$$

we observe that when the diffusivity is large and the diffusion time $t = T$ is small such that $DT = 1$, the difference between an image obtained by a full two-dimensional diffuser and a fractional diffuser is compounded in the difference between the convolution of the initial image with (ignoring scaling) the functions $\exp(-R^2/4)$ and $1/\sqrt{R}$, respectively. Compared with the Gaussian (at least for $DT \geq 1$), the function $R^{-1/2}$ decays more rapidly and hence will have broader spectral characteristics leading to an output that is less 'diffused' than that produced by the convolution of the input with a Gaussian. In terms of the fractional diffusion equation being used to model scattering in a tenuous medium, this is to be expected.

B. Inverse Solution

Let I_0 be represented as a Taylor series at some time $T > 0$, i.e.

$$I(\mathbf{r}, 0) = I(\mathbf{r}, T) + T \left[\frac{\partial}{\partial t} I(\mathbf{r}, t) \right]_{t=T} - \frac{T^2}{2!} \left[\frac{\partial^2}{\partial t^2} I(\mathbf{r}, t) \right]_{t=T} + \dots$$

Now, since

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-q}}{\partial t^{1-q}} \frac{\partial^q}{\partial t^q} u$$

then from the fractional diffusion equation

$$\frac{\partial u}{\partial t} = D^q \frac{\partial^{1-q}}{\partial t^{1-q}} \nabla^2 u$$

and

$$\frac{\partial^2}{\partial t^2} u$$

$$\begin{aligned} &= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} \left(D^q \frac{\partial^{1-q}}{\partial t^{1-q}} \nabla^2 u \right) = D^q \frac{\partial^{1-q}}{\partial t^{1-q}} \nabla^2 \frac{\partial u}{\partial t} \\ &= D^q \frac{\partial^{1-q}}{\partial t^{1-q}} \nabla^2 \left(D^q \frac{\partial^{1-q}}{\partial t^{1-q}} \nabla^2 u \right) = D^{2q} \frac{\partial^{1-q}}{\partial t^{1-q}} \left(\frac{\partial^{1-q}}{\partial t^{1-q}} \nabla^4 u \right) \end{aligned}$$

so that in general,

$$\frac{\partial^n u}{\partial t^n} = D^{nq} \frac{\partial^{n(1-q)}}{\partial t^{n(1-q)}} \nabla^{2n} u.$$

Because (see Appendix II)

$$\frac{\partial^{-q}}{\partial t^{-q}} I(\mathbf{r}, t) = \frac{1}{\Gamma(q) t^{1-q}} \otimes I(\mathbf{r}, t)$$

we can write the Taylor series for the field at $t = 0$ in terms of the field at $t = T$ as

$$\begin{aligned} I(\mathbf{r}, 0) &= I(\mathbf{r}, T) + \frac{TD^q}{\Gamma(q)} \left[\frac{\partial}{\partial t} \left(\frac{1}{t^{1-q}} \otimes \nabla^2 I(\mathbf{r}, t) \right) \right]_{t=T} \\ &- \frac{T^2 D^{2q}}{2! \Gamma(2q)} \left[\frac{\partial^2}{\partial t^2} \left(\frac{1}{t^{1-2q}} \otimes \nabla^4 I(\mathbf{r}, t) \right) \right]_{t=T} \end{aligned}$$

$$+ \frac{T^3 D^{3q}}{3! \Gamma(3q)} \left[\frac{\partial^3}{\partial t^3} \left(\frac{1}{t^{1-3q}} \otimes \nabla^6 I(\mathbf{r}, t) \right) \right]_{t=T} - \dots$$

Note that for $T \ll 1$,

$$I(\mathbf{r}, 0) = I(\mathbf{r}, T) + \frac{TD^q}{\Gamma(q)} \left[\frac{\partial}{\partial t} \left(\frac{1}{t^{1-q}} \otimes \nabla^2 I(\mathbf{r}, t) \right) \right]_{t=T}$$

and under the condition that

$$\left[\frac{\partial}{\partial t} \left(\frac{1}{t^{1-q}} \otimes I(\mathbf{r}, t) \right) \right]_{t=T} = I(\mathbf{r}, T)$$

we can write

$$I(\mathbf{r}, 0) = I(\mathbf{r}, T) + \frac{TD^q}{\Gamma(q)} \nabla^2 I(\mathbf{r}, T).$$

VIII. DECONVOLUTION

In the presence of additive noise $n(x, y)$, the deconvolution problem is as follows: Given that

$$I(x, y) = p(x, y) \otimes_2 I_0(x, y) + n(x, y)$$

where $\Pr[n(x, y)]$ is known (ideally), find an estimate for I_0 . This is a common problem in optics (digital image processing) known as the deconvolution problem whose solution is fundamental to image restoration and reconstruction [2], [3]. In terms of the material presented in this paper, there are two Point Spread Functions (PSF) $p(x, y)$ that have been considered: For full diffusion (strong scattering)

$$p(x, y) = \frac{1}{4\pi DT} \exp \left[- \left(\frac{x^2 + y^2}{4DT} \right) \right]$$

and for fractional diffusion (intermediate scattering in a tenuous medium with large diffusivity)

$$p(x, y) = \frac{1}{2\sqrt{2\pi}} \frac{1}{(DT)^{1-q/4}} \frac{1}{(x^2 + y^2)^{1/4}}.$$

We note that

$$\frac{1}{4\pi DT} \exp \left[- \left(\frac{x^2 + y^2}{4DT} \right) \right] \leftrightarrow \exp[-4DT(k_x^2 + k_y^2)]$$

and [15]

$$\frac{1}{2\sqrt{2\pi}} \frac{1}{(DT)^{1-q/4}} \frac{1}{(x^2 + y^2)^{1/4}} \\ \uparrow \\ \frac{\sqrt{\pi}\Gamma(0.75)}{\Gamma(0.25)(DT)^{1-q/4}} \frac{1}{(k_x^2 + k_y^2)^{3/4}}$$

where Γ denotes the Gamma function. In the latter case, the filter is a ‘fractal filter’ and thus, if I_0 is characterised by white noise, then the output I is a Mandelbrot surface with a fractal dimension of 2.5 [19], [20], [15]. In the absence of noise, the inverse solution for I_0 can be written in the form (evaluating the Gamma functions)

$$I_0(x, y) = 1.67(DT)^{1-q/4} \nabla^{\frac{3}{2}} I(x, y),$$

a result that is based on the application of the fractional Laplacian or Riesz operator [15]

$$\nabla^q \leftrightarrow |\mathbf{k}|^q.$$

Figure 1 shows the effect of filtering an image using full diffusion and fractional diffusion for $DT = 1$. Comparison of the results shows that fractional diffusion does not blur the image to the same extent which is to be expected given the physical characteristics under which fractional diffusion processes are taken to occur, i.e. in terms of intermediate multiple scattering events in a tenuous rarefied medium.

$$\frac{1}{4\pi DT} \exp \left[- \left(\frac{x^2 + y^2}{4DT} \right) \right]$$

$$\frac{1}{2\sqrt{2\pi}} \frac{1}{(DT)^{1-q/4}} \frac{1}{(x^2 + y^2)^{1/4}}$$

Fig. 1. Comparison between the effect of diffusion (centre) and fractional diffusion (bottom) on a binary image (top) for $DT = 1$.

There are a range of approaches to solving the one-dimensional and two-dimensional deconvolution problem in practice (i.e. with additive noise) leading to the classification of different ‘inverse filters’. If *a priori* information on the statistics of the noise function and the object function is available, then Bayesian estimation methods are preferable in the design of filters whose performance will then depend on statistical parameters such as the standard deviation. In some cases, an estimate of $\Pr[n(x, y)]$ can be obtained by taking an image (and a number of images to obtain a statistically significant result) with zero input, i.e. with $I_0 = 0$. This provides a method of validating an idealised PDF through data fitting and, thus, determination of the statistical parameters

from which a theoretical PDF is composed. In cases when experimental determinism is not practically possible, statistical models are used directly. This includes models such as the K-distribution discussed and derived in Section V. However, with regard to incoherent imaging systems, the noise function tends to be Gaussian distributed - a result of the noise being a linear combination of many different independent noise source which combine to produce Gaussian noise (a consequence of the Central Limit Theorem).

A. Bayesian Estimation

Using Bayes rule, the aim is to find an estimate for I_0 such that

$$\frac{\partial}{\partial I_0} \ln \Pr[n(x, y)] + \frac{\partial}{\partial I_0} \ln \Pr[I_0(x, y)] = 0.$$

Consider the following models for the PDFs: (i) Gaussian statistics for the noise when (ignoring scaling and where σ_n^2 is the standard deviation of n)

$$\Pr[n(x, y)] =$$

$$\exp\left(-\frac{1}{\sigma_n^2} \int \int [(I(x, y) - p(x, y) \otimes_2 I_0(x, y))^2 dx dy]\right).$$

(ii) Gaussian statistics for the object function where (ignoring scaling and where $\sigma_{I_0}^2$ is the standard deviation of I_0)

$$\Pr[I_0(x, y)] = \exp\left(-\frac{1}{\sigma_{I_0}^2} \int \int I_0^2(x, y) dx dy\right).$$

Differentiating, these statistical models yield the equation

$$I(x, y) \odot_2 p(x, y) = \frac{\sigma_n^2}{\sigma_{I_0}^2} f(x, y) + [p(x, y) \otimes_2 f(x, y)] \odot_2 p(x, y)$$

where \odot_2 denotes the two-dimensional correlation integral. In Fourier space, this equation becomes

$$\tilde{I}(k_x, k_y) P^*(k_x, k_y) = \frac{1}{\Gamma^2} \tilde{I}_0(x, y) + |P(k_x, k_y)|^2 I_0(k_x, k_y)$$

The filter $F(k_x, k_y)$ for Gaussian statistics is therefore given by

$$F(k_x, k_y) = \frac{P^*(k_x, k_y)}{|P(k_x, k_y)|^2 + \sigma_n^2/\sigma_{I_0}^2}$$

where σ_n/σ_{I_0} defines the signal-to-noise ratio of $I(x, y)$. and $\tilde{I}_0(k_x, k_y) = F(k_x, k_y) \tilde{I}(k_x, k_y)$. The reconstruction for I_0 is then given by

$$I_0(x, y) =$$

$$\frac{1}{(2\pi)^2} \int \int F(k_x, k_y) \tilde{I}(k_x, k_y) \exp(ik_x x) \exp(ik_y y) dk_x dk_y \quad (7)$$

B. Adaptive Filtering

Given $P(k_x, k_y)$, the performance of this filter depends on the value of $\Sigma = \sigma_n^2/\sigma_{I_0}^2$. In general, as $\Sigma \rightarrow 0$ the reconstruction sharpens but at the expense of 'ringing'. Thus, an optimum value of Σ is obtained by computing I_0 over a range of values of Σ and, for each reconstruction, computing the ratio of the number of zero crossings Z_c to the sum of the magnitude of a digital gradient $\sum |DI_0[i, j]|$, i.e.

$$R = \frac{Z_c}{\sum |DI_0[i, j]|}$$

This ratio is based on the principle that an optimum reconstruction is one which provides a sharp image with minimal ringing, i.e. a reconstruction for which R is a minimum. This principle has been applied in the example results given in the following section. Note that the Fourier based approach to image restoration relies on the ability to implement the convolution and correlation theorems. This requires that the data has been recorded by an (optical) imaging system that is isoplanatic (i.e. the Point Spread Function is stationary).

IX. EXAMPLE APPLICATIONS: IMAGE ENHANCEMENT IN ASTRONOMY

We consider examples of image reconstruction based on equation (7) for fully diffusive and fractional diffusive models using the optimization procedure discussed in the previous section for the following 'digital Laplacian'

$$DI_0[i, j] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

A. Deconvolution for Full Diffusion

Figure 2 shows the application of equation (7) where (ignoring scaling and with $\sigma = 4DT$)

$$P(k_x, k_y) = \exp[-\sigma(k_x^2 + k_y^2)].$$

In this example, the diffusion of the object has been generated by turbulence of the earths atmosphere through which light from the object has been fully diffused. In this case, the reconstruction depends on the value of both σ and Σ and an optimization scheme based on computing $I_0[i, j; \sigma, \Sigma]$ for $\min R$.

B. Deconvolution for Fractional Diffusion

Fractional diffusion models apply to scattering processes that occur in a tenuous and extremely rarefied medium. In applied optics, one of the most common examples of this phenomena occurs in astronomy and the processes associated with light scattering from cosmic dust which is composed of particles which are a few molecules to the order of 10^{-4} metres in size. Cosmic dust is defined in terms of its astronomical location including intergalactic dust, interstellar dust, interplanetary dust and circumplanetary dust (such as in a planetary ring). In our own Solar System, interplanetary dust is generated from sources such as comet dust, asteroidal

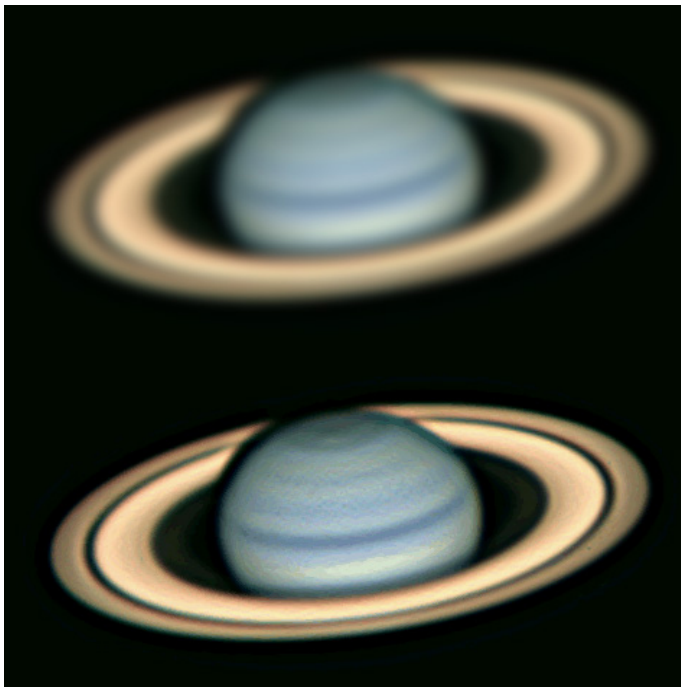


Fig. 2. Diffusion based deconvolution (below) of an image of Saturn observed by a ground based telescope with light diffused by the atmosphere (above).

dust, dust from the Kuiper belt and interstellar dust passing through our solar system. This dust is responsible for zodiacal light which is produced by sunlight reflecting off dust particles. Cosmic dust can be categorised in terms of different types of nebulae associated with different physical causes and processes. These include: diffuse nebula, infrared reflection nebula, supernova remnants and molecular clouds, for example. However, in a more general sense, cosmic dust often characterises the interstellar medium which is the gas and dust that pervade interstellar space. This medium consists of an extremely dilute (by terrestrial standards) mixture of ions, atoms, molecules, and larger dust grains, consisting of about 99% gas and 1% dust by mass. Densities range from a few thousand to a few hundred million particles per cubic meter with an average value in the Milky Way Galaxy, for example, of a million particles per cubic meter. In comparison with the scattering of light from earth-based random media, for example, the interstellar medium is highly diffuse and therefore ideal for applying light scattering models based on fractional diffusion when $D \rightarrow \infty$.

Figure 2 shows the application of equation (7) where (ignoring scaling)

$$P(k_x, k_y) = \frac{1}{(k_x^2 + k_y^2)^{3/4}}$$

from an optical image obtained with the Hubble Space Telescope. This image is part of the constellation of Perseus as observed through an interstellar dust cloud that covers nearly 4 degrees on the sky observed 1,000 light-years away.

X. CONCLUSIONS

We have considered different approaches to modelling light scattering through random media including: formal scattering



Fig. 3. Fractional diffusion based deconvolution (right) for $\sigma_n/\sigma_{I_0} = 1$ of a dust clouded star field (left) in the constellation of Pegasus.

methods, cross-correlation models for a scattering function under the weak field condition, statistical modelling of the wavefield and application of the diffusion equation for modelling multiple scattering processes. The formal scattering approach provides inverse solutions that are, in general, not of any practical value to signal and image processing. Cross-correlation methods are of value in modelling the intensity distribution but are not generally applicable to image processing problems. While statistical modelling methods are useful for developing theoretical PDFs of images and their statistical evaluation, they are not directly applicable for image enhancement.

The use of a fully diffusive process for modelling strong (multiple) scattering provides a result that is applicable in terms of solving the inverse scattering problem which is compounded in terms of developing a suitable deconvolution algorithm. We have extended this approach to model intermediate scattering by generalizing the diffusion equation to the fractional form

$$\left(\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q} \right) I(\mathbf{r}, t) = I_0(\mathbf{r})\delta(t)$$

where $I(x, y, t)$ is a light intensity image, D^{-1} is the fractional diffusivity and $q \in (1, 2)$. A solution has been considered based on a Fourier transform representation of a fractional derivative for which the initial condition $I(\mathbf{r}, t = 0)$ (used to solve the diffusion equation) is not required. An asymptotic result has then been derived for the case when $\sigma \rightarrow 0$ that is compounded in an Optical Transfer Function given by $(k_x^2 + k_y^2)^{-0.75}$.

APPENDIX I

THE DIFFUSION EQUATION

The homogeneous diffusion equation [8]

$$\nabla^2 u(\mathbf{r}, t) = \sigma \frac{\partial}{\partial t} u(\mathbf{r}, t), \quad \sigma = \frac{1}{D}$$

where D is the ‘Diffusivity’, differs in many aspects from the scalar wave equation. The most important single feature is the asymmetry of the diffusion equation with respect to time. For the wave equation, if $u(\mathbf{r}, t)$ is a solution, so is $u(\mathbf{r}, -t)$. However, if $u(\mathbf{r}, t)$ is a solution of

$$\nabla^2 u = \sigma \frac{\partial u}{\partial t}$$

the function $u(\mathbf{r}, -t)$ is not; it is a solution of the quite different equation,

$$\nabla^2 u(\mathbf{r}, -t) = -\sigma \frac{\partial}{\partial t} u(\mathbf{r}, -t).$$

Thus, unlike the wave equation, the diffusion equation differentiates between past and future. This is because the diffusing field u represents the behaviour of some average property of an ensemble (e.g. of particles) which cannot in general go back to an original state. Causality must therefore be considered in the solution to the diffusion equation. This in turn leads to the use of the one-sided Laplace transform (i.e. a causal transform) for solving the equation with respect to t (compared to the Fourier transform - a non-causal transform - used to solve the wave equation with respect to t).

A. Green's Function for the Diffusion Equation

To obtain a general solution to the diffusion equation, we need to evaluate the Green's function G for the diffusion equation subject to the causality condition

$$G(\mathbf{r} | \mathbf{r}_0, t | t_0) = 0 \quad \text{if } t < t_0.$$

This can be accomplished for one-, two- and three-dimension simultaneously [14]. With $R = |\mathbf{r} - \mathbf{r}_0|$ and $\tau = t - t_0$ we require the solution of the equation

$$\left(\nabla^2 - \sigma \frac{\partial}{\partial \tau} \right) G(R, \tau) = -\delta^n(R) \delta(\tau), \quad \tau > 0$$

where n is 1, 2 or 3 depending on the number of dimensions. One way of solving this equation is to first take the Laplace transform with respect to τ , then solve for G (in Laplace space) and inverse Laplace transform. This requires an initial condition to be specified (the value of G at $\tau = 0$). Another way to solve this equation is to take its Fourier transform with respect to R , solve for G (in Fourier space) and then inverse Fourier transform. Here, we adopt the latter approach. Let

$$G(R, \tau) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{k}, \tau) \exp(i\mathbf{k} \cdot \mathbf{R}) d^n \mathbf{k}$$

and

$$\delta^n(R) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \exp(i\mathbf{k} \cdot \mathbf{R}) d^n \mathbf{k}.$$

Then the equation for G reduces to

$$\sigma \frac{\partial \tilde{G}}{\partial \tau} + k^2 \tilde{G} = \delta(\tau)$$

which has the solution

$$\tilde{G} = \frac{1}{\sigma} \exp(-k^2 \tau / \sigma) H(\tau)$$

where $H(\tau)$ is the step function

$$H(\tau) = \begin{cases} 1, & \tau > 0; \\ 0, & \tau < 0. \end{cases}$$

Hence, the Green's functions are given by

$$\begin{aligned} G(R, \tau) &= \frac{1}{\sigma (2\pi)^n} H(\tau) \int_{-\infty}^{\infty} \exp(i\mathbf{k} \cdot \mathbf{R}) \exp(-k^2 \tau / \sigma) d^n \mathbf{k} \\ &= \frac{1}{\sigma (2\pi)^n} H(\tau) \left(\int_{-\infty}^{\infty} \exp(ik_x R_x) \exp(-k_x^2 \tau / \sigma) dk_x \right) \dots \end{aligned}$$

By rearranging the exponent in the integral, it becomes possible to evaluate each integral exactly. Thus, with

$$\begin{aligned} ik_x R_x - k_x^2 \frac{\tau}{\sigma} &= - \left(k_x \sqrt{\frac{\tau}{\sigma}} - i \frac{R_x}{2} \sqrt{\frac{\sigma}{\tau}} \right)^2 - \left(\frac{\sigma R_x^2}{4\tau} \right) \\ &= - \frac{\tau}{\sigma} \xi^2 - \left(\frac{\sigma R_x^2}{4\tau} \right) \end{aligned}$$

where

$$\xi = k_x - i \frac{\sigma R_x}{2\tau},$$

the integral over k_x becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp \left[- \left(\frac{\tau}{\sigma} \xi^2 \right) - \left(\frac{\sigma R_x^2}{4\tau} \right) \right] d\xi \\ &= e^{-\left(\frac{\sigma R_x^2}{4\tau} \right)} \int_{-\infty}^{\infty} e^{-\left(\tau \xi^2 / \sigma \right)} d\xi = \sqrt{\frac{\pi \sigma}{\tau}} \exp \left[- \left(\frac{\sigma R_x^2}{4\tau} \right) \right] \end{aligned}$$

with similar results for the integrals over k_y and k_z giving the result

$$G(R, \tau) = \frac{1}{\sigma} \left(\frac{\sigma}{4\pi\tau} \right)^{\frac{n}{2}} \exp \left[- \left(\frac{\sigma R^2}{4\tau} \right) \right] H(\tau).$$

The function G satisfies an important property which is valid for all n :

$$\int_{-\infty}^{\infty} G(R, \tau) d^n \mathbf{r} = \frac{1}{\sigma}, \quad \tau > 0.$$

This is the expression for the conservation of the Green's function associated with the diffusion equation. For example, if we consider the diffusion of heat, then if at a time t_0 and at a point in space \mathbf{r}_0 a source of heat is introduced instantaneously (i.e. a heat impulse), then the heat diffuses out through the medium characterized by σ in such a way that the total heat energy is unchanged.

B. Green's Function Solution to the Diffusion Equation

Working in three dimensions, let us consider the general solution to the equation

$$\left(\nabla^2 - \sigma \frac{\partial}{\partial t} \right) u(\mathbf{r}, t) = -f(\mathbf{r}, t)$$

where f is a source function of compact support ($\mathbf{r} \in V$) and define the Green's function as the solution to the equation

$$\left(\nabla^2 - \sigma \frac{\partial}{\partial t} \right) G(\mathbf{r} | \mathbf{r}_0, t | t_0) = -\delta^3(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0)$$

It is convenient to first take the Laplace transform of these equations with respect to $\tau = t - t_0$ to obtain

$$\nabla^2 \bar{u} - \sigma[-u_0 + p\bar{u}] = -\bar{f}$$

and

$$\nabla^2 \bar{G} + \sigma[-G_0 + p\bar{G}] = -\delta^3$$

where

$$\bar{u}(\mathbf{r}, p) = \int_0^\infty u(\mathbf{r}, \tau) \exp(-p\tau) d\tau,$$

$$\bar{G}(\mathbf{r} | \mathbf{r}_0, p) = \int_0^\infty G(\mathbf{r} | \mathbf{r}_0, \tau) \exp(-p\tau) d\tau,$$

$$\bar{f}(\mathbf{r}, p) = \int_0^\infty f(\mathbf{r}, \tau) \exp(-p\tau) d\tau.$$

$$u_0 \equiv u(\mathbf{r}, \tau = 0) \quad \text{and} \quad G_0 \equiv G(\mathbf{r} | \mathbf{r}_0, \tau = 0) = 0.$$

Pre-multiplying the equation for \bar{u} by \bar{G} and the equation for \bar{G} by \bar{u} , subtracting the two results and integrating over V we obtain

$$\int_V (\bar{G} \nabla^2 \bar{u} - \bar{u} \nabla^2 \bar{G}) d^3 \mathbf{r} + \sigma \int_V u_0 \bar{G} d^3 \mathbf{r} = - \int_V \bar{f} \bar{G} d^3 \mathbf{r} + \bar{u}(\mathbf{r}_0, p).$$

Using Green's theorem and rearranging the result gives

$$\begin{aligned} \bar{u}(\mathbf{r}_0, p) &= \int_V \bar{f}(\mathbf{r}, p) \bar{G}(\mathbf{r} | \mathbf{r}_0, p) d^3 \mathbf{r} + \sigma \int_V u_0(\mathbf{r}) \bar{G}(\mathbf{r} | \mathbf{r}_0, p) d^3 \mathbf{r} \\ &\quad + \oint_S (\bar{g} \nabla \bar{u} - \bar{u} \nabla \bar{g}) \cdot \mathbf{n} d^2 \mathbf{r}. \end{aligned}$$

Finally, taking the inverse Laplace transform and using the convolution theorem for Laplace transforms, we can write

$$\begin{aligned} u(\mathbf{r}_0, \tau) &= \int_0^\tau \int_V f(\mathbf{r}, \tau') G(\mathbf{r} | \mathbf{r}_0, \tau - \tau') d^3 \mathbf{r} d\tau' \\ &\quad + \sigma \int_V u_0(\mathbf{r}) G(\mathbf{r} | \mathbf{r}_0, \tau) d^3 \mathbf{r} \\ &\quad + \int_0^\tau \oint_S [G(\mathbf{r} | \mathbf{r}_0, \tau') \nabla u(\mathbf{r}, \tau - \tau') \\ &\quad - u(\mathbf{r}, \tau') \nabla G(\mathbf{r} | \mathbf{r}_0, \tau - \tau')] \cdot \hat{\mathbf{n}} d^2 \mathbf{r} d\tau'. \end{aligned}$$

The first two terms are convolutions of the Green's function with the source function and the initial field $u(\mathbf{r}, \tau = 0)$ respectively.

By way of a simple example, suppose we consider the source term to be zero and the volume of interest is the infinite domain, so that the surface integral is zero. Then we have

$$u(\mathbf{r}_0, \tau) = \sigma \int_V u_0(\mathbf{r}) G(\mathbf{r} | \mathbf{r}_0, \tau) d^3 \mathbf{r}.$$

In one dimension, this reduces to

$$u(x_0, \tau) = \sqrt{\frac{\sigma}{4\pi\tau}} \int_{-\infty}^\infty \exp\left[-\frac{\sigma(x_0 - x)^2}{4\tau}\right] u_0(x) dx, \quad \tau > 0.$$

Observe that the field u at a time $t > 0$ is given by the convolution of the field at time $t = 0$ with the (Gaussian) function

$$\sqrt{\frac{\sigma}{4\pi t}} \exp\left(-\frac{\sigma x^2}{4t}\right).$$

In two-dimensions, the equivalent result is

$$u(x, y, t) = \frac{\sigma}{4\pi t} \exp\left[-\left(\frac{\sigma(x^2 + y^2)}{4t}\right)\right] \otimes_2 u_0(x, y). \quad (5)$$

APPENDIX II

OVERVIEW OF FRACTIONAL CALCULUS

Fractional calculus (e.g. [24], [25], [26], [27], [28]) is the study of the calculus associated with fractional differentials and a fractional integrals which, in the main, are based on generalizations of results obtained using integer calculus. For example, the classical fractional integral operators are the Riemann-Liouville transform [24]

$$\hat{I}^q f(t) = \frac{1}{\Gamma(q)} \int_{-\infty}^t \frac{f(\tau)}{(t - \tau)^{1-q}} d\tau, \quad q > 0$$

and the Weyl transform

$$\hat{I}^q f(t) = \frac{1}{\Gamma(q)} \int_t^\infty \frac{f(\tau)}{(t - \tau)^{1-q}} d\tau, \quad q > 0$$

where

$$\Gamma(q) = \int_0^\infty t^{q-1} \exp(-t) dt.$$

For integer values of q (i.e. when $q = n$ where n is a non-negative integer), the Riemann-Liouville transform reduces to the standard Riemann integral. This transform is just a (causal) convolution of the function $f(t)$ with $t^{q-1}/\Gamma(q)$. For fractional differentiation, we can perform a fractional integration of appropriate order and then differentiate to an appropriate integer order. The reason for this is that direct fractional differentiation can lead to divergent integrals. Thus, the fractional differential operator \hat{D}^q for $q > 0$ is given by

$$\hat{D}^q f(t) \equiv \frac{d^q}{dt^q} f(t) = \frac{d^n}{dt^n} [\hat{I}^{n-q} f(t)].$$

where

$$\hat{I}^{q-n} f(t) = \frac{1}{\Gamma(n-q)} \int_{-\infty}^t \frac{f(\tau)}{(t - \tau)^{1+q-n}} d\tau, \quad n - q > 0$$

in which the value of $\hat{I}^{q-n} f(t)$ at a point t depends on the behaviour of $f(t)$ from $-\infty$ to t via a convolution with the kernel $t^{n-q}/\Gamma(q)$. The convolution process is dependent on the history of the function $f(t)$ for a given kernel and thus, in this context, we can consider a fractional derivative defined via the result above to have memory.

A. The Laplace Transform and the Half Integrator

It is informative at this point to consider the application of the Laplace transform to identify an ideal integrator and then a half integrator. The Laplace transform is given by

$$\hat{L}[f(t)] \equiv F(p) = \int_0^{\infty} f(t) \exp(-pt) dt$$

and from this result we can derive the transform of a derivative given by

$$\hat{L}[f'(t)] = pF(p) - f(0)$$

and the transform of an integral given by

$$\hat{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{p} F(p).$$

Now, suppose we have a standard time invariant linear system whose input is $f(t)$ and whose output is given by

$$s(t) = f(t) \otimes g(t)$$

where the convolution is causal, i.e.

$$s(t) = \int_0^t f(\tau) g(t - \tau) d\tau.$$

Suppose we let

$$g(t) = H(t) = \begin{cases} 1, & t > 0; \\ 0, & t < 0. \end{cases}$$

Then, $G(p) = 1/p$ and the system becomes an ideal integrator:

$$s(t) = f(t) \otimes H(t) = \int_0^t f(t - \tau) d\tau = \int_0^t f(\tau) d\tau.$$

Now, consider the case when we have a time invariant linear system with an impulse response function given by

$$g(t) = \frac{H(t)}{\sqrt{t}} = \begin{cases} |t|^{-1/2}, & t > 0; \\ 0, & t < 0. \end{cases}$$

The output of this system is $f \otimes g$ and the output of such a system with input $f \otimes g$ is $f \otimes g \otimes g$. Now

$$\begin{aligned} g(t) \otimes g(t) &= \int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \int_0^{\sqrt{t}} \frac{2x dx}{x\sqrt{t-x^2}} \\ &= 2 \left[\sin^{-1} \left(\frac{x}{\sqrt{t}} \right) \right]_0^{\sqrt{t}} = \pi. \end{aligned}$$

Hence,

$$\frac{H(t)}{\sqrt{\pi t}} \otimes \frac{H(t)}{\sqrt{\pi t}} = H(t)$$

and the system defined by the impulse response function $H(t)/\sqrt{\pi t}$ represents a ‘half-integrator’ with a Laplace transform given by

$$\hat{L}\left[\frac{H(t)}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{p}}.$$

This result provides an approach to working with fractional integrators and/or differentiators using the Laplace transform. Fractional differential and integral operators can be defined and used in a similar manner to those associated with conventional or integer order calculus and we now provide an overview of such operators.

B. Operators of Integer Order

The following operators are all well-defined, at least with respect to all test functions $u(t)$ say which are (i) infinitely differentiable and (ii) of compact support (i.e. vanish outside some finite interval).

Integral Operator:

$$\hat{I}u(t) \equiv \hat{I}^1 u(t) = \int_{-\infty}^t u(\tau) d\tau.$$

Differential Operator:

$$\hat{D}u(t) \equiv \hat{D}^1 u(t) = u'(t).$$

Identify Operator:

$$\hat{I}^0 u(t) = u(t) = \hat{D}^0 u(t).$$

Now,

$$\hat{I}[\hat{D}u](t) = \int_{-\infty}^t u'(\tau) d\tau = u(t)$$

and

$$\hat{D}[\hat{I}u](t) = \frac{d}{dt} \int_{-\infty}^t u(\tau) d\tau = u(t)$$

so that

$$\hat{I}^1 \hat{D}^1 = \hat{D}^1 \hat{I}^1 = \hat{I}^0.$$

For n (integer) order:

$$\begin{aligned} \hat{I}^n u(t) &= \int_{-\infty}^t d\tau_{n-1} \dots \int_{-\infty}^{\tau_2} d\tau_1 \int_{-\infty}^{\tau_1} u(\tau) d\tau, \\ \hat{D}^n u(t) &= u^{(n)}(t) \end{aligned}$$

and

$$\hat{I}^n [\hat{D}^n u](t) = u(t) = \hat{D}^n [\hat{I}^n u](t).$$

C. Convolution Representation

Consider the function

$$t_+^{q-1}(t) \equiv |t|^{q-1} H(t) = \begin{cases} |t|^{q-1}, & t > 0; \\ 0, & t < 0. \end{cases}$$

which, for any $q > 0$ defines a function that is locally integrable. We can then define an integral of order n in terms of a convolution as

$$\hat{I}^n u(t) = \left(u \otimes \frac{1}{(n-1)!} t_+^{n-1} \right) (t)$$

$$\begin{aligned}
&= \frac{1}{(n-1)!} \int_{-\infty}^t (t-\tau)^{n-1} u(\tau) d\tau \\
&= \frac{1}{(n-1)!} \int_{-\infty}^t \tau^{n-1} u(t-\tau) d\tau
\end{aligned}$$

In particular,

$$\hat{I}^1 u(t) = (u \otimes H)(t) = \int_{-\infty}^t u(\tau) d\tau.$$

These are classical (absolutely convergent) integrals and the identity operator admits a formal convolution representation, using the delta function, i.e.

$$\hat{I}^0 u(t) = \int_{-\infty}^{\infty} \delta(\tau) u(t-\tau) d\tau$$

where

$$\delta(t) = \hat{D}H(t).$$

Similarly,

$$\hat{D}^n u(t) \equiv \hat{I}^{-n} u(t) = \int_{-\infty}^{\infty} \delta^{(n)}(\tau) u(t-\tau) d\tau = u^{(n)}(t).$$

On the basis of the material discussed above, we can now formally extend the integral operator to fractional order and consider the operator

$$\begin{aligned}
\hat{I}^q u(t) &= \frac{1}{\Gamma(q)} \int_{-\infty}^{\infty} u(\tau) t_+^{q-1} (t-\tau) d\tau \\
&= \frac{1}{\Gamma(q)} \int_{-\infty}^t u(\tau) t_+^{q-1} (t-\tau) d\tau
\end{aligned}$$

where

$$\Gamma(q) = \int_0^{\infty} t^{q-1} \exp(-t) dt, \quad q > 0$$

with the fundamental property that

$$\Gamma(q+1) = q\Gamma(q).$$

Here, I^q is an operator representing a time invariant linear system with impulse response function $t_+^{q-1}(t)$ and transfer function $1/p^q$. For the cascade connection of I^{q_1} and I^{q_2} we have

$$\hat{I}^{q_1} [\hat{I}^{q_2} u(t)] = \hat{I}^{q_1+q_2} u(t).$$

This classical convolution integral representation holds for all real $q > 0$ (and formally for $q = 0$, with the delta function playing the role of an impulse function and with a transfer function equal to the constant 1).

D. Fractional Differentiation

For $0 < q < 1$, if we define the (Riemann-Liouville) derivative of order q as

$$\hat{D}^q u(t) \equiv \frac{d}{dt} [\hat{I}^{1-q} u](t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{-\infty}^t (t-\tau)^{-q} u(\tau) d\tau,$$

then,

$$\hat{D}^q u(t) = \frac{1}{\Gamma(1-q)} \int_{-\infty}^t (t-\tau)^{-q} u'(\tau) d\tau \equiv \hat{I}^{1-q} u'(t).$$

Hence,

$$\hat{I}^q [\hat{D}^q u] = \hat{I}^q [\hat{I}^{1-q} u'] = \hat{I}^1 u' = u$$

and \hat{D}^q is the formal inverse of the operator \hat{I}^q . Given any $q > 0$, we can always write $\lambda = n - 1 + q$ and then define

$$\hat{D}^\lambda u(t) = \frac{1}{\Gamma(1-q)} \frac{d^n}{dt^n} \int_{-\infty}^t u(\tau) (t-\tau)^{-q} d\tau.$$

D^q is an operator representing a time invariant linear system consisting of a cascade combination of an ideal differentiator and a fractional integrator of order $1 - q$. For D^λ we replace the single ideal differentiator by n such that

$$\hat{D}^0 u(t) = \frac{1}{\Gamma(1)} \frac{d}{dt} \int_{-\infty}^t u(\tau) d\tau = u(t) \equiv \int_{-\infty}^{\infty} u(\tau) \delta(t-\tau) d\tau$$

and

$$\begin{aligned}
\hat{D}^n u(t) &= \frac{1}{\Gamma(1)} \frac{d^{n+1}}{dt^{n+1}} \int_{-\infty}^t u(\tau) d\tau \\
&= u^{(n)}(t) \equiv \int_{-\infty}^{\infty} u(\tau) \delta^{(n)}(t-\tau) d\tau.
\end{aligned}$$

In addition to the conventional and classical definitions of fractional derivatives and integrals, more general definitions are available including the Erdélyi-Kober fractional integral [29]

$$\frac{t^{-p-q+1}}{\Gamma(q)} \int_0^t \frac{\tau^{p-1}}{(t-\tau)^{1-q}} f(\tau) d\tau, \quad q > 0, \quad p > 0$$

which is a generalisation of the Riemann-Liouville fractional integral and the integral

$$\frac{t^p}{\Gamma(q)} \int_t^{\infty} \frac{\tau^{-q-p}}{(\tau-t)^{1-q}} f(\tau) d\tau, \quad q > 0, \quad p > 0$$

which is a generalization of the Weyl integral. Further definitions exist based on the application of hypergeometric functions and operators involving other special functions such as the Maier G-function and the Fox H-function [28]. Moreover, all such operators leading to a fractional integral of the Riemann-Liouville type and the Weyl type to have the general forms (through induction)

$$\hat{I}^q f(t) = t^{q-1} \int_{-\infty}^t \Phi\left(\frac{\tau}{t}\right) \tau^{-q} f(\tau) d\tau$$

and

$$\hat{I}^q f(t) = t^{-q} \int_t^{\infty} \Phi\left(\frac{t}{\tau}\right) \tau^{q-1} f(\tau) d\tau$$

respectively, where the kernel Φ is an arbitrary continuous function so that the integrals above make sense in sufficiently large functional spaces. Although there are a number of approaches that can be used to define a fractional differential/integral, there is one particular definition, which has wide ranging applications in signal and image processing and is based on the Fourier transform, i.e.

$$\frac{d^q}{dt^q} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^q F(\omega) \exp(i\omega t) d\omega$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

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