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MODELS OF INTERNAL WAVES IN THE PRESENCE OF CURRENTS

Alan Compelli, Rossen Ivanov

A fluid system consisting of two domains is examined. The system is considered as being bounded at the bottom and top by a flatbed and wave-free surface respectively. An internal wave propagating in one direction, driven by gravity, acts as a free common interface between the fluids. Various current profiles are considered. The Hamiltonian of the system is determined and expressed in terms of canonical wave-related variables. Limiting behaviour is examined and compared to that of other known models. The linearised equations as well as long-wave approximations are formulated. The presented models provide potential applications to modelling of internal geophysical waves.

1. Introduction

In the context of surface waves wave heights of the order of tens of metres are considered *large*. For instance a 20.4 metre surface wave measured off the Northwest coast of Ireland in December 2011 is the largest ever measured by Met Éireann (the Irish meteorological service).

However, internal waves of the order of hundreds of metres often propagate unnoticed beneath the surface. Satellite imaging now allows us to observe and measure these waves. Waves of more than 170 metres have been observed in the Luzon Strait between Taiwan and Luzon Island in the South China Sea whilst

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traffic passed through the busy shipping route oblivious to the massive internal waves passing underneath.

The addition of currents to internal wave systems introduces various degrees of complexity depending on the current profile under study. Waves-current interaction is of interest to various groups including marine biologists, oceanographers and climate scientists.

The presented study draws from previous studies using Euler's equations beginning with Zakharov's irrotational studies on deep water [1] and rotational finite depth studies such as [2, 3]. The aim is to develop nonlinear models that capture the main features of the dynamics and lend themselves to approximate models.

Several other irrotational [4, 5, 6, 7] and rotational models [8, 9, 10, 11] and in particular the model of 2-media systems with internal waves such as [12, 13, 14, 15, 16, 17, 18] form the framework for the presented findings. The consideration of wave-current interactions has been explored in several recent publications [19, 20, 21, 22, 23, 24].

This paper aims to generalise the study of internal waves by considering adjacent media which have different vorticities and current profiles. Recovery of already known special cases will be demonstrated and approximate models will also be formulated.

2. Preliminaries

A two-dimensional bounded system consisting of two fluids is examined as per Figure 1. An internal wave, driven by gravity, acts as a free common interface between the fluids. The bottom of the system is bounded by an impermeable flat-bed. The top of the system is also considered to be a flat surface. It is important to note that, by comparison to [19], this is not equivalent to assuming the amplitude of the surface waves as being small. One has to keep in mind that no matter how small the surface waves are there is a coupling between surface and internal waves.

The function $\eta(x,t)$ describes the elevation of the internal wave with the mean of η assumed to be zero, that is

$$\int_{\mathbb{R}} \eta(x,t)dx = 0$$

and hence the domains $\Omega = \{(x,y) \in \mathbb{R}^2 : -h < y < \eta(x,t)\}$ and $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : \eta(x,t) < y < h_1\}$ are defined.

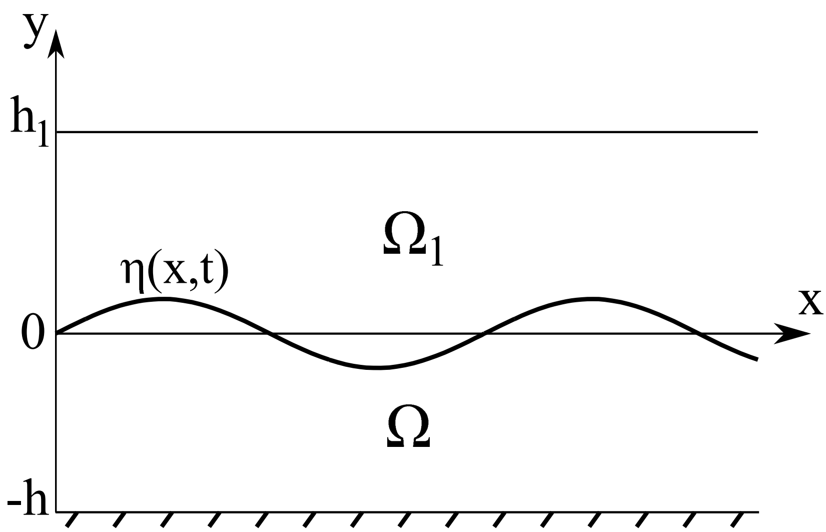

Figure 1: System setup. The function $\eta(x,t)$ describes the elevation of the internal wave which propagates in x-direction.

The system is considered incompressible with ρ and ρ_1 being the respective constant densities of the lower and upper media and stability is given by the immiscibility condition $\rho > \rho_1$.

The velocity fields $\mathbf{V}(x,y,t) = (u,v)$ and $\mathbf{V}_1(x,y,t) = (u_1,v_1)$ of the lower and upper media respectively are defined in terms of the respective velocity potentials

$$\left\{ \begin{array}{l} \varphi \equiv \tilde{\varphi} + \kappa x \text{ for } \Omega \\ \varphi_1 \equiv \tilde{\varphi}_1 + \kappa_1 x \text{ for } \Omega_1. \end{array} \right. \tag{1}$$

The motivation for the decomposition (1) is the separation of the velocity potential to wave-motion components, given by $\tilde{\varphi}$ and $\tilde{\varphi}_1$, and the κx and $\kappa_1 x$ terms, generating constant horizontal velocity components of the flows in the corresponding domains.

Stream functions ψ and ψ_1 are introduced as

$$\left\{ \begin{array}{l} \left. \begin{array}{l} u = \tilde{\varphi}_x + \gamma y + \kappa = \psi_y \\ v = \tilde{\varphi}_y = -\psi_x \end{array} \right\} \text{for } \Omega \\ \left. \begin{array}{l} u_1 = \tilde{\varphi}_{1,x} + \gamma_1 y + \kappa_1 = \psi_{1,y} \\ v_1 = \tilde{\varphi}_{1,y} = -\psi_{1,y} \end{array} \right\} \text{for } \Omega_1 \end{array} \right. \tag{2}$$

where $\gamma = u_y - v_x$ and $\gamma_1 = u_{1,y} - v_{1,x}$ are the constant non-zero vorticities (see [19]–[24] for the reasoning behind the choices in (2)).

The following assumption is made: $\eta(x,t)$, $\tilde{\varphi}(x,y,t)$ and $\tilde{\varphi}_1(x,y,t)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ with respect to the x variable (for any y and t). The assumption of course implies that for large absolute values of x the internal wave attenuates and so

(3) $$\lim_{|x|\to\infty} \eta(x,t) = 0, \quad \lim_{|x|\to\infty} \tilde{\varphi}(x,y,t) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \tilde{\varphi}_1(x,y,t) = 0.$$

Effectively the current profiles in Ω and Ω_1 are

$$U(y) = \gamma y + \kappa, \qquad U_1(y) = \gamma_1 y + \kappa_1.$$

As the system under study is a geophysical system and we are considering equatorial motion the following Coriolis forces per unit mass have to be taken into account:

(4) $$\begin{cases} \mathbf{F} = 2\omega \nabla \psi \text{ for } \Omega \\ \mathbf{F}_1 = 2\omega \nabla \psi_1 \text{ for } \Omega_1 \end{cases}$$

with ω being the rotational speed of Earth.

Interface velocity potentials $\phi(x,t)$ and $\phi_1(x,t)$ are introduced defined as

(5) $$\begin{cases} \phi := \tilde{\varphi}(x,\eta(x,t),t) \text{ for } \Omega \\ \phi_1 := \tilde{\varphi}_1(x,\eta(x,t),t) \text{ for } \Omega_1. \end{cases}$$

The variable $\xi(x,t)$, defined as [14]

(6) $$\xi := \rho\phi - \rho_1\phi_1$$

plays the important role of momentum. It belongs to $\mathcal{S}(\mathbb{R})$.

3. Hamiltonian Formulation

The Hamiltonian $H = H(\eta,\xi)$ is the total energy of the system. The dynamics can be represented in a Hamiltonian form [2, 3, 17, 18]

(7) $$\begin{cases} \eta_t = \delta_\xi H \\ \xi_t = -\delta_\eta H - \Gamma \displaystyle\int_{-\infty}^{x} \frac{\delta H}{\delta\xi(x')} dx' \end{cases}$$

where

(8) $$\Gamma := \rho\gamma - \rho_1\gamma_1 + 2\omega(\rho - \rho_1).$$

We introduce the Dirichlet-Neumann operators [14, 15]

$$\begin{cases} G(\eta)\phi = (\tilde{\varphi}_{\mathbf{n}})\sqrt{1+(\eta_x)^2} \text{ for } \Omega \\ G_1(\eta)\phi_1 = (\tilde{\varphi}_{1,\mathbf{n}_1})\sqrt{1+(\eta_x)^2} \text{ for } \Omega_1 \end{cases} \tag{9}$$

where $\varphi_{\mathbf{n}}$ and $\varphi_{1,\mathbf{n}_1}$ are the normal derivatives of the velocity potentials, at the interface, for outward normals $\mathbf{n}$ and $\mathbf{n}_1$ (noting that $\mathbf{n} = -\mathbf{n}_1$).

We define

$$\mu := \big((\gamma-\gamma_1)\eta + (\kappa-\kappa_1)\big)\eta_x. \tag{10}$$

Recalling from (6) that $\xi = \rho\phi - \rho_1\phi_1$ and defining $B := \rho G_1(\eta) + \rho_1 G(\eta)$ one can express the velocity potentials as

$$\begin{cases} \phi = B^{-1}\big(\rho_1\mu + G_1(\eta)\xi\big) \\ \phi_1 = B^{-1}\big(\rho\mu - G(\eta)\xi\big). \end{cases} \tag{11}$$

The Hamiltonian is [23]

$$\begin{aligned} H(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\,dx - \frac{1}{2}\rho\rho_1\int_{\mathbb{R}} \mu B^{-1}\mu\,dx \\ - \int_{\mathbb{R}} (\gamma\eta+\kappa)\xi\eta_x\,dx + \rho_1\int_{\mathbb{R}} \mu B^{-1}G(\eta)\xi\,dx \\ + \frac{\rho}{6\gamma}\int_{\mathbb{R}} (\gamma\eta+\kappa)^3 dx - \frac{\rho_1}{6\gamma_1}\int_{\mathbb{R}} (\gamma_1\eta+\kappa_1)^3 dx + \frac{1}{2}g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2 dx + \int_{\mathbb{R}} \mathfrak{h}_1 dx \end{aligned} \tag{12}$$

where

$$\mathfrak{h}_1 = \frac{\rho_1\kappa_1^3}{6\gamma_1} - \frac{\rho\kappa^3}{6\gamma}.$$

Considering the following case: $\gamma_1 = \gamma$ and $\kappa_1 = \kappa$ then $\mu = 0$ then the Hamiltonian in [21] is recovered as

$$\begin{aligned} H(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}} (G(\eta)\xi B^{-1}G_1(\eta))\xi\,dx - \int_{\mathbb{R}} (\gamma\eta+\kappa)\xi\eta_x\,dx \\ + \frac{\rho-\rho_1}{6\gamma}\int_{\mathbb{R}} [(\gamma\eta+\kappa)^3 - \kappa^3]dx + \frac{1}{2}g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2 dx. \end{aligned} \tag{13}$$

Considering the following case: $\gamma_1 = \gamma = 0$ and $\kappa_1 = \kappa$ then $\mu = 0$ then the Hamiltonian in [22] is recovered as

$$H(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}} (G(\eta)\xi B^{-1}G_1(\eta))\xi\, dx - \kappa \int_{\mathbb{R}} \xi\eta_x\, dx + \frac{1}{2}g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2 dx. \tag{14}$$

Finally, considering the case: $\gamma_1 \neq \gamma$ and $\kappa_1 = \kappa = 0$ then $\mu = (\gamma-\gamma_1)\eta\eta_x$ then the Hamiltonians in [17, 18] are recovered as

$$\begin{aligned} H(\eta,\xi) = {} & \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\, dx + \rho_1(\gamma-\gamma_1)\int_{\mathbb{R}} \eta\eta_x B^{-1}G(\eta)\xi\, dx \\ & - \frac{1}{2}\rho\rho_1(\gamma-\gamma_1)^2 \int_{\mathbb{R}} \eta\eta_x B^{-1}\eta\eta_x\, dx - \gamma\int_{\mathbb{R}} \xi\eta\eta_x\, dx \\ & + \frac{1}{6}(\rho\gamma^2 - \rho_1\gamma_1^2)\int_{\mathbb{R}} \eta^3 dx + \frac{1}{2}g(\rho-\rho_1)\int_{\mathbb{R}} \eta^2 dx. \end{aligned} \tag{15}$$

The additional complexity of surface waves is considered in [19, 20].

4. Scales and Expansions

Let a represent the average amplitude of the waves $\eta(x,t)$ under consideration and $\varepsilon = a/h_1$ a small parameter.

The Dirichlet-Neumann operators have the following structure

$$\bar{G} = \bar{G}^{(0)} + \bar{G}^{(1)} + \bar{G}^{(2)} + \dots \tag{16}$$

where $\bar{G}^{(n)} \sim \bar{\eta}^n \partial_{\bar{x}}^{n+1}$, i.e. $\bar{G}^{(n)} = \dfrac{\varepsilon^n}{h_1} G^{(n)}$, and similarly for G_1, and so

$$\begin{aligned} \bar{G}(\bar{\eta}) &= \bar{D}\tanh(h\bar{D}) + \bar{D}\bar{\eta}\bar{D} - \bar{D}\tanh(h\bar{D})\bar{\eta}\bar{D}\tanh(h\bar{D}) + \mathcal{O}(\varepsilon^2) \\ \bar{G}_1(\bar{\eta}) &= \bar{D}\tanh(h_1\bar{D}) - \bar{D}\bar{\eta}\bar{D} + \bar{D}\tanh(h_1\bar{D})\bar{\eta}\bar{D}\tanh(h_1\bar{D}) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

where $\bar{D} = -i\partial/\partial\bar{x}$.

The Hamiltonian, $H^{(2)}$, is given by:

$$(17)\quad H^{(2)} = \frac{1}{2}\int\limits_{\mathbb{R}} \xi \frac{D\tanh(hD)\tanh(h_1 D)}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\xi\,dx + \frac{1}{2}\rho\rho_1(\kappa-\kappa_1)^2\int\limits_{\mathbb{R}}\eta\frac{D}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\eta\,dx - \kappa\int\limits_{\mathbb{R}}\xi\eta_x\,dx + \rho_1(\kappa-\kappa_1)\int\limits_{\mathbb{R}}\eta_x\frac{\tanh(hD)}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\xi\,dx + \frac{1}{2}A_1\int\limits_{\mathbb{R}}\eta^2\,dx$$

where

$$A_1 = \rho\gamma\kappa - \rho_1\gamma_1\kappa_1 + g(\rho-\rho_1).$$

This gives the linear equations

$$(18)\quad \eta_t + \kappa\eta_x = \frac{D\tanh(hD)\tanh(h_1 D)}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\xi + \frac{\rho_1(\kappa-\kappa_1)\tanh(hD)}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\eta_x$$

and

$$(19)\quad \xi_t + \kappa\xi_x + \Gamma\int_{-\infty}^{x}\eta_t\,dx' = \frac{\rho_1(\kappa-\kappa_1)\tanh(hD)}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\xi_x - \frac{\rho\rho_1(\kappa-\kappa_1)^2 D}{\rho\tanh(h_1 D)+\rho_1\tanh(hD)}\eta - A_1\eta.$$

Looking for a solution which is a superposition of sine and cosine waves, we represent η and ξ as

$$(20)\quad \begin{cases} \eta(x,t) = \eta_0 e^{-i(\Omega(k)t-kx)} \\ \xi(x,t) = \xi_0 e^{-i(\Omega(k)t-kx)} \end{cases}$$

where k is the wave number and $\Omega(k)$ is the angular frequency. The wave speed c is given by

$$(21)\quad c(k) = \frac{\Omega(k)}{k}.$$

Introducing the functions

$$(22)\quad \begin{cases} T(k) = \tanh(kh) \\ T_1(k) = \tanh(kh_1) \end{cases}$$

one can obtain the following quadratic equation for the wave speed

$$(c-\kappa)^2+\frac{2(\kappa-\kappa_1)\rho_1 kT+\Gamma TT_1}{k(\rho_1 T+\rho T_1)}(c-\kappa)-\frac{(A_1-\kappa\Gamma)TT_1}{k(\rho_1 T+\rho T_1)} + \frac{(\kappa-\kappa_1)^2\rho_1 T(\rho_1 T-\rho T_1)}{(\rho_1 T+\rho T_1)^2}=0. \tag{23}$$

For $\kappa=\kappa_1$ we have:

$$A_1-\kappa\Gamma=g(\rho-\rho_1)-2\kappa\omega(\rho-\rho_1),$$

which is a quantity usually close to $g(\rho-\rho_1)$ and thus positive (see the discussion below). Then the quadratic equation becomes

$$(c-\kappa)^2+\frac{\Gamma TT_1}{k(\rho_1 T+\rho T_1)}(c-\kappa)-\frac{(A_1-\kappa\Gamma)TT_1}{k(\rho_1 T+\rho T_1)}=0. \tag{24}$$

This has solutions:

$$c_\pm(k)=\kappa+\frac{1}{2}\left[-f_1\pm\sqrt{f_1^2+4f_2}\right] \tag{25}$$

where

$$f_1(k)=\frac{\Gamma\tanh(kh)\tanh(kh_1)}{k(\rho_1\tanh(kh)+\rho\tanh(kh_1))} \tag{26}$$

and

$$f_2(k)=\frac{(A_1-\kappa\Gamma)\tanh(kh)\tanh(kh_1)}{k(\rho_1\tanh(kh)+\rho\tanh(kh_1))} \tag{27}$$

corresponding to right moving ($c_+>\kappa$) and left moving ($c_-<\kappa$) waves with respect to a *moving* observer, moving with the a velocity κ.

The reference frame for the *moving* observer is obtained by a Galilean transformation $X\to x-\kappa t$, $T\to t$,

$$\partial_T\to\partial_t+\kappa\partial_x,\qquad \partial_X\to\partial_x$$

giving

$$\eta_T=\frac{DT(D)T_1(D)}{\rho T_1(D)+\rho_1 T(D)}\xi+\frac{\rho_1(\kappa-\kappa_1)T(D)}{\rho T_1(D)+\rho_1 T(D)}\eta_x \tag{28}$$

and

$$\xi_T + \Gamma\partial_x^{-1}\eta_T = (\Gamma\kappa - A_1)\eta + \frac{\rho_1(\kappa-\kappa_1)T(D)}{\rho T_1(D)+\rho_1 T(D)}\xi_x + \frac{\rho\rho_1(\kappa-\kappa_1)^2 D}{\rho T_1+\rho_1 T}\eta. \tag{29}$$

Consider the term $(\Gamma\kappa - A_1)\eta$. This evaluates to

$$\rho_1\gamma_1(\kappa_1-\kappa) - (\rho-\rho_1)g + 2\kappa(\rho-\rho_1)\omega. \tag{30}$$

Note that the term $2\kappa(\rho-\rho_1)\omega$ is the only term in the linearised equation that depends on κ but does not depend on the relative velocity $\kappa-\kappa_1$. All other terms depend only on the difference $\kappa-\kappa_1$. This is a consequence of the presence of the Coriolis term representing non-inertial forces in our frame of reference. When $\omega = 0$ the equations, of course, depend only on the relative difference $\kappa-\kappa_1$.

If $\kappa-\kappa_1 \neq 0$ there is a jump in the velocity component, tangent to the surface of the internal wave $y=\eta(x,t)$, i.e. there is a vortex sheet. While such a situation is compatible with the inviscid Euler's equations, in practical situations, where viscosity is always present, such jumps do not occur (otherwise there will be a vortex sheet between the two media). For this reason in our further considerations we take $\kappa_1 = \kappa$. This corresponds to the case where there is no vortex sheet at the boundary between the two layers at $y=\eta(x,t)$.

Recalling the Hamiltonian in (12) where, in this case, $\mu = (\gamma-\gamma_1)\eta\eta_x$

$$\begin{aligned} H(\eta,\xi) = {} & \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\,dx - \frac{1}{2}\rho\rho_1(\gamma-\gamma_1)^2\int_{\mathbb{R}}\eta\eta_x B^{-1}\eta\eta_x\,dx \\ & - \int_{\mathbb{R}}(\kappa+\gamma\eta)\xi\eta_x\,dx + \rho_1(\gamma-\gamma_1)\int_{\mathbb{R}}\eta\eta_x B^{-1}G(\eta)\xi\,dx \\ & + \frac{1}{2}[(\rho-\rho_1)g + (\rho\gamma-\rho_1\gamma_1)\kappa]\int_{\mathbb{R}}\eta^2 dx + \frac{1}{6}(\rho\gamma^2-\rho_1\gamma_1^2)\int_{\mathbb{R}}\eta^3 dx. \end{aligned} \tag{31}$$

5. Long Waves Approximation

We will study the equations under the additional approximation that the wavelengths L are much bigger than h and h_1. Since

$$\bar{L} = h_1 L \Rightarrow \frac{1}{L} = \frac{h_1}{\bar{L}} = \delta.$$

Thus for the wave number $k = 2\pi/L = 2\pi\delta$ we have $k = \mathcal{O}(\delta)$. We further assume that $\delta^2 = \mathcal{O}(\varepsilon)$. Recall that the operator D has an eigenvalue k, thus we shall keep in mind that $D = \mathcal{O}(\delta)$. Moreover the x-derivative of the velocity potentials do not get an extra factor of δ since $\bar{v}$ of order ε remains unchanged. In other words the "wave" component of u is $\tilde{\varphi}_x$ and is of order $\varepsilon \sim \delta^2$, hence $\tilde{\varphi}_x \sim \delta$ and $\xi \sim \delta$.

The Dirichlet-Neumann operators can hence be represented as

$$G(\eta) = \delta\Big(D\tanh(\delta h D)\Big) + \varepsilon\delta^2\Big(D\eta D - D\tanh(\delta h D)\eta D\tanh(\delta h D)\Big) + \mathcal{O}(\varepsilon^2\delta^4) \tag{32}$$

and

$$G_1(\eta) = \delta\Big(D\tanh(\delta h_1 D)\Big) - \varepsilon\delta^2\Big(D\eta D - D\tanh(\delta h_1 D)\eta D\tanh(\delta h_1 D)\Big) + \mathcal{O}(\varepsilon^2\delta^4). \tag{33}$$

We will keep track only of the scale variables ε, δ. The Hamiltonian has the following expansion to order δ^6:

$$H(\eta,\xi) = \frac{1}{2}\delta^4 \int_{\mathbb{R}} \xi D\left(\alpha_1 + \delta^2(\alpha_3\eta - \alpha_2 D^2)\right) D\xi dx + \delta^4\alpha_5 \int_{\mathbb{R}} \frac{\eta^2}{2} dx - \delta^4\kappa \int_{\mathbb{R}} \eta_x\xi dx - \delta^6\alpha_4 \int_{\mathbb{R}} \eta\eta_x\xi dx + \delta^6\alpha_6 \int_{\mathbb{R}} \frac{\eta^3}{6} dx. \tag{34}$$

We introduce the following constants:

$$\begin{aligned} &\alpha_1 = \frac{hh_1}{\rho_1 h + \rho h_1}, \quad \alpha_2 = \frac{h^2 h_1^2(\rho h + \rho_1 h_1)}{3(\rho_1 h + \rho h_1)^2}, \quad \alpha_3 = \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2}, \\ &\alpha_4 = \frac{\gamma_1\rho_1 h + \gamma\rho h_1}{\rho_1 h + \rho h_1}, \quad \alpha_5 = g(\rho - \rho_1) + (\rho\gamma - \rho_1\gamma_1)\kappa \quad \text{and} \\ &\alpha_6 = \rho\gamma^2 - \rho_1\gamma_1^2. \end{aligned} \tag{35}$$

The Hamiltonian equations for the Hamiltonian (34) in terms of η and $\tilde{u} = \xi_x$ are

$$\begin{aligned} &\eta_T + \alpha_1\tilde{u}_x + \delta^2\alpha_2\tilde{u}_{xxx} + \delta^2(\alpha_3(\eta\tilde{u})_x + \alpha_4\eta\eta_x) = 0 \\ &\tilde{u}_T + \Gamma\eta_T + (\rho - \rho_1)(g - 2\omega\kappa)\eta_x + \delta^2(\alpha_3\tilde{u}\tilde{u}_x + \alpha_4(\tilde{u}\eta)_x + \alpha_6\eta\eta_x) = 0, \end{aligned} \tag{36}$$

where for convenience $\partial_T = \partial_t + \kappa\partial_x$, that is a Galilean change of coordinates.

Since $\omega = 7.3 \times 10^{-5}$ rad/s, $\kappa \sim 1$ m/s, then $g \gg 2\omega\kappa$ and the $2\omega\kappa$ term will be neglected.

One can also exclude η_T from the second equation, which leads to the system

$$
\begin{aligned}
&\eta_T + \alpha_1 \tilde{u}_x + \delta^2 \alpha_2 \tilde{u}_{xxx} + \delta^2(\alpha_3(\eta\tilde{u})_x + \alpha_4 \eta\eta_x) = 0\\
&\tilde{u}_T - \alpha_1 \Gamma \tilde{u}_x + (\rho - \rho_1) g \eta_x - \delta^2 \Gamma \alpha_2 \tilde{u}_{xxx}\\
&\qquad + \delta^2 \left(\alpha_3 \tilde{u}\tilde{u}_x + (\alpha_4 - \Gamma\alpha_3)(\tilde{u}\eta)_x + (\alpha_6 - \Gamma\alpha_4)\eta\eta_x\right) = 0.
\end{aligned} \tag{37}
$$

The linearised equations produce wave speeds

$$
c = \frac{1}{2}\left(-\alpha_1\Gamma \pm \sqrt{\alpha_1\Gamma^2 + 4\alpha_1(\rho - \rho_1)g}\right) \tag{38}
$$

for an observer, moving with the flow, i.e. there are left- ($-$ sign) and right-running ($+$ sign) waves. For a stationary observer the velocities are $c + \kappa$. Moreover, in the leading approximation,

$$
\eta = \frac{\alpha_1}{c}\tilde{u} \qquad \text{and} \qquad \tilde{u} = \frac{c}{\alpha_1}\eta. \tag{39}
$$

One can look for a relation between η and u at the next order of approximation:

$$
\tilde{u} = \frac{c}{\alpha_1}\eta + \delta^2 \sigma \eta_{xx} + \delta^2 \mu \eta^2 \tag{40}
$$

for some constants σ and μ (Johnson's transformation). Then one can express u via η in both equations (37). The condition for the two equations to coincide with terms to order δ^2 leads to the determination of σ and μ as

$$
\sigma = -\frac{c\alpha_2(c + \Gamma\alpha_1)}{\alpha_1^2(2c + \Gamma\alpha_1)}
$$

and

$$
\mu = \frac{\alpha_1\alpha_4(c - \Gamma\alpha_1) - c\alpha_3(c + 2\Gamma\alpha_1) + \alpha_1^2\alpha_6}{2\alpha_1^2(2c + \Gamma\alpha_1)}.
$$

Thus u can be expressed with η to order δ^2 and η satisfies the KdV equation, that represents the coinciding terms of (37), that is

$$
\eta_T + c\eta_x + \delta^2 \frac{c^2\alpha_2}{\alpha_1(2c + \Gamma\alpha_1)}\eta_{xxx} + \delta^2 \frac{3c^2\alpha_3 + 3c\alpha_1\alpha_4 + \alpha_1^2\alpha_6\alpha_2}{\alpha_1(2c + \Gamma\alpha_1)}\eta\eta_x = 0. \tag{41}
$$

In the case when all vorticities are zero this simplifies to

$$\eta_T + c\eta_x + \delta^2 \frac{c\alpha_2}{2\alpha_1}\eta_{xxx} + \delta^2 \frac{3c\alpha_3}{2\alpha_1}\eta\eta_x = 0. \tag{42}$$

The KdV equation represents a balance between a nonlinearity term $\eta\eta_x$, and dispersion term η_{xxx}. Reintroducing ε and δ, it is clear that these terms are scaled like $\varepsilon\eta\eta_x$ and $\delta^2\eta_{xxx}$, so that when $\varepsilon \sim \delta^2$ the interplay between nonlinearity and dispersion is producing smooth and stable in time soliton solutions.

However, there are various geophysical scales and many other situations are possible, including $\delta \sim \varepsilon^2$. In such case $\delta^2 \sim \varepsilon^4 \ll \varepsilon$ and instead of a KdV equation the relevant model is the dispersionless Burgers equation ($\partial_\tau = \partial_T + c\partial_x$):

$$\eta_\tau + \varepsilon \frac{3c^2\alpha_3 + 3c\alpha_1\alpha_4 + \alpha_1^2\alpha_6\alpha_2}{\alpha_1(2c + \Gamma\alpha_1)}\eta\eta_x = 0. \tag{43}$$

Such an equation does not support globally smooth solutions, i.e. the solutions always form a vertical slope and break. Such wave-breaking phenomenon is well known for internal waves in the ocean. This is a mechanism that causes mixing in the deep ocean, with implications for biological productivity and sediment transport.

6. Conclusions

A closed water-wave system provides a simplified model for internal geophysical waves. The Hamiltonian form of the system and the equations of motion in terms of phase space variables (η, ξ) can be calculated with non-canonical Hamiltonian structure.

In particular, small amplitude and long-wave regimes are studied. There are various geophysical scales, allowing for smooth solitons at the KdV regime as well as breaking waves in the very large wavelengths, when the equations are asymptotically equivalent to the dispersionless Burgers equation.

A possible limitation of the model is the assumption of a flat surface, which apparently changes the nature of the internal waves. In the case of a free surface, even in the case of very small amplitudes, the internal wave (in a linear approximation) is usually coupled to the surface wave. This has an impact on the possible propagation speeds.

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