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Describing Ideals of Endomorphism Rings

Brendan Goldsmith and Simone Pabst

§1 Introduction

It is well known that the ring of linear transformations of a finite dimensional vector space is simple, i.e. it has no non-trivial proper two-sided ideals. It is, perhaps, not so well known that the (two-sided) ideals in the ring of linear transformations of an infinite dimensional vector space can be characterized by a single cardinal invariant (R. Baer, [1]). It is therefore reasonably natural to ask if there is a generalization of Baer's result to ideals in the endomorphism ring of a wider class of modules. The purpose of this present work is to explore this possibility.

A first generalization is to replace the underlying field of the vector space by a ring. A natural extension of the concept of a field is a discrete valuation ring since a discrete valuation ring modulo its Jacobson radical is a field. Recall (see e.g. [5]) that R is a discrete valuation ring if R is a principal ideal ring with exactly one maximal ideal or, alternatively, with one prime element p . In particular the ring of p -adic rationals, \mathbb{Z}_p , is a discrete valuation ring and the reader may replace all references to discrete valuation rings with the p -adic rationals without any serious loss in generality. Also, recall that if E is a ring then the Jacobson radical of E is defined to be the intersection of all maximal ideals. It is well known that this is equivalent to the set of all elements $x \in \mathbf{R}$ such that rxs is quasi-regular for all $r, s \in R$, i.e. those elements x for which $1 - rxs$ is a unit.

Since a vector space over a field F is a free F -module, an obvious question to ask is whether a corresponding characterization of ideals in the endomorphism ring of a free module over a discrete valuation ring exists. We address this problem in §3 and obtain Baer's Theorem for vector spaces as a corollary to our main result Theorem 3.3.

Whilst free modules over a discrete valuation ring are an obvious generalization of vector spaces it is, perhaps, not so obvious that complete modules over a complete

discrete valuation ring R have many similar properties to vector spaces (cf. [4]). Recall that a complete discrete valuation ring is a discrete valuation ring which is complete in its p -adic topology (i.e. the linear topology with basis of neighbourhoods of 0 given by $p^n R$, $n \geq 0$) where p is the only prime element of R . In §4 we discuss the ideal structure of the endomorphism ring of such modules and achieve a complete characterization modulo the Jacobson radical.

We conclude this introduction by reviewing a number of standard concepts in module / abelian group theory. If R is discrete valuation ring with prime element p then we say that an R -submodule H of the R -module G is *pure* in G if $H \cap p^n G = p^n H$ for all $n \geq 0$. If we consider the module G as a topological module furnished with the p -adic topology (i.e. a basis of neighbourhoods of 0 is given by $p^n G$, $n \geq 0$), then H is pure in G precisely if the p -adic topology on H coincides with the induced subspace topology. Also an R -module D is *divisible* if given any $d \in D$ we can solve the equation $p^n x = d$ in D ; a module is *reduced* if it has no non-trivial divisible submodules. (It is well known that a torsion-free divisible R -module is a direct sum of copies of the quotient field Q of R .) Notice that if $H \subseteq G$ then G/H divisible is equivalent to H being dense in the p -adic topology on G .

Finally note that maps are written on the right and the word ideal will always mean a two-sided ideal.

§2 Preliminaries

In this section we state a few well-known results on modules over complete discrete valuation rings. Throughout the section R shall be a complete discrete valuation ring with prime element p . In considering torsion-free R -modules the concept of a basic submodule is useful; a submodule B of a torsion-free R -module A is called a *basic submodule* of A if B is a free R -module such that A/B is divisible and B is pure in A .

The first lemma is due to R.B. Warfield [8]; it tells us how to obtain a basic submodule from the R/pR -vector space A/pA .

Lemma 2.1 *Let A be a torsion-free R -module and $\pi : A \rightarrow A/pA$ the natural epimorphism. Let $\{x_i \mid i \in I\}$ be an R/pR -basis of A/pA and choose $y_i \in A$ ($i \in I$) such that $y_i\pi = x_i$. Then the submodule B generated by $\{y_i \mid i \in I\}$ is a basic submodule of A . Moreover every basic submodule of A arises in this way. \square*

The lemma ensures the existence of a basic submodule B and the uniqueness of its rank $rk(B)$ where $rk(B)$ is the usual rank of a free R -module. Hence we may define the *rank of a torsion-free R -module A* as the rank of its basic submodule, i.e. $rk(A) = rk(B)$. Since the divisibility of the quotient module A/B is equivalent to the density of B in A in the p -adic topology we obtain the following well-known characterization of complete reduced torsion-free R -modules; (2) is essentially due to I. Kaplansky [5] and (3) is similar to a results proved for p -groups by H. Leptin [6].

Proposition 2.2 *The following properties of a reduced torsion-free R -module A are equivalent;*

- (1) *A is complete.*
- (2) *If B is a basic submodule of A , then A is the completion of B .*
- (3) *If B is a basic submodule of A , then every R -homomorphism $B \rightarrow A$ extends uniquely to an R -endomorphism of A . \square*

Next we state a few facts on complete reduced torsion-free R -modules; the proofs can be found in [2], [5], and [7].

Lemma 2.3 *Let A, A' be complete reduced torsion-free R -modules.*

- (1) *If θ is an endomorphism of A , then both $\ker\theta$ and $\text{Im}\theta$ are complete.*
- (2) *If S is a pure submodule of A and is complete, then S is a direct summand of A .*
- (3) *A is isomorphic to A' if and only if $rk(A) = rk(A')$. \square*

We finish this section with a standard result on the Jacobson radical of the endomorphism ring of a complete reduced torsion-free R -module; a proof may be found in [7].

Proposition 2.4 *Let E denote the endomorphism ring $\text{End}(A)$ of the complete reduced torsion-free R -module A . Then*

- (i) $J(E) = pE = Ep$,
- (ii) $E/J(E) \cong \text{End}_{R/pR}(A/pA)$,
- (iii) $J(E) = \{\phi \in E \mid A\phi \subseteq pA\}$. □

§3 Free modules over discrete valuation rings

Here we shall discuss ideals of endomorphism rings of free R -modules over discrete valuation rings R ; we will deduce Baer's Theorem on vector spaces as a corollary to our main result. Before we restrict our attention to modules over discrete valuation rings we prove a result (Proposition 3.2) which is true for modules in general. The definition of a direct endomorphism will be useful; an endomorphism μ of A is called *k-direct* [*im-direct*] if $\ker(\mu)$ [$\text{Im}(\mu)$] is a direct summand of A , and μ is said to be a *direct endomorphism* if it is both *k-direct* and *im-direct*. First we need:

Lemma 3.1 *If σ is a direct endomorphism of A , then there exists an endomorphism η of A such that*

- (a) $\sigma\eta$ and $\eta\sigma$ are both idempotent, and
- (b) $\text{Im}(\sigma) = \text{Im}(\eta\sigma)$, $\text{Im}(\sigma\eta) = \text{Im}(\eta)$, $\ker(\eta) = \ker(\eta\sigma)$, and $\ker(\sigma\eta) = \ker(\sigma)$.

Proof: By our assumption, we may write $A = \text{Im}(\sigma) \oplus S$ and $A = \ker(\sigma) \oplus T$. Then the restriction $\sigma \upharpoonright T : T \rightarrow \text{Im}(\sigma)$ is an isomorphism. Hence there exists $\tau : \text{Im}(\sigma) \rightarrow T$ such that $\tau(\sigma \upharpoonright T) = \text{id}_{\text{Im}(\sigma)}$, and $(\sigma \upharpoonright T)\tau = \text{id}_T$. Now let $\eta : A \rightarrow A$ be defined by $\eta \upharpoonright \text{Im}(\sigma) = \tau$ and $\ker(\eta) = S$. We shall see that η is the required endomorphism.

First we show that $\sigma\eta$ and $\eta\sigma$ are idempotent. Let x be an arbitrary element of A , then $x(\sigma\eta)^2 = x\sigma\eta\sigma\eta = x\sigma\tau\sigma = x\sigma\eta$; so $(\sigma\eta)^2 = \sigma\eta$. Now let $x = a\sigma + s \in A$ ($a \in A$, $s \in S$). Then $x(\eta\sigma)^2 = x\eta\sigma\eta\sigma = a\underbrace{\sigma\eta\sigma\eta}_{=\sigma\eta}\sigma = (a\sigma)\eta\sigma = x\eta\sigma$ i.e. $(\eta\sigma)^2 = \eta\sigma$.

To prove part (b) we make the following calculations;

$$\underline{\text{Im}(\eta\sigma)} = A\eta\sigma = A\sigma\eta\sigma = A\sigma\tau\sigma = A\sigma = \underline{\text{Im}(\sigma)};$$

$$\begin{aligned}
\underline{\text{Im}(\sigma\eta)} &= A\sigma\eta = A\sigma\tau = T = \underline{\text{Im}(\eta)}; \\
\underline{\text{ker}(\eta\sigma)} &= \{x \in A \mid x\eta\sigma = 0\} = \{x \in A \mid x\eta \in \text{ker}(\sigma)\} \\
&= \{x \in A \mid x\eta \in \text{ker}(\sigma) \cap T = 0\} = \underline{\text{ker}(\sigma)} \\
\underline{\text{ker}(\sigma\eta)} &= \{x \in A \mid x\sigma\eta = 0\} = \{x \in A \mid x\sigma\tau = 0\} \\
&= \{x \in A \mid x\sigma = 0\} = \underline{\text{ker}(\sigma)}
\end{aligned}$$

This completes the proof. \square

Proposition 3.2 *Let I be an ideal of the endomorphism ring $\text{End}(A)$ of A such that all $\mu \in I$ are k -direct. Moreover assume that I contains a direct endomorphism σ . If α is an im -direct endomorphism of A such that $\text{Im}(\alpha)$ is isomorphic to a direct summand of $\text{Im}(\sigma)$, then α belongs to the ideal I .*

Proof: Let I , σ , α be as above. Then we can write $A = \text{Im}(\sigma) \oplus S = \text{Im}(\alpha) \oplus T$ and $\text{Im}(\sigma) = R \oplus C$ where $R \cong_{\phi'} \text{Im}(\alpha)$. We extend ϕ' to an endomorphism ϕ of A by $\phi \upharpoonright R = \phi'$ and $\phi \upharpoonright C \oplus S = 0$.

Now consider the endomorphism $\sigma\phi \in I$. Since $\sigma\phi$ belongs to I it is k -direct. Moreover $A\sigma\phi = (R \oplus C)\phi = R\phi = R\phi' = \text{Im}\alpha$, which is a direct summand of A . Hence $\sigma\phi$ is direct and we may apply Lemma 3.1 to $\sigma\phi$. Thus there exists $\eta \in \text{End}(A)$ such that $\eta\sigma\phi$ is an idempotent and $\text{Im}(\eta\sigma\phi) = \text{Im}(\sigma\phi) = \text{Im}(\alpha)$. Therefore, for any $x \in A$, there is $y \in A$ with $x\alpha = y\eta\sigma\phi$; so

$$(x\alpha)\eta\sigma\phi = y(\eta\sigma\phi)^2 = y\eta\sigma\phi = x\alpha, \text{ i.e. } \alpha = \alpha\eta\sigma\phi \in I \text{ since } I \text{ is an ideal. } \square$$

Now let R be a discrete valuation ring and A a free R -module. The next theorem tells us something about ideals I containing a direct endomorphism of a given rank; recall that the *rank of an endomorphism* σ is defined to be the rank of the free R -module $\text{Im}\sigma$.

Theorem 3.3 *Let I be an ideal of $\text{End}(A)$. If I contains a direct endomorphism of rank κ , then I contains all endomorphisms of rank less than or equal to κ .*

Proof: First we show that all endomorphisms of A are k -direct. Let μ be any endomorphism of A , then $A/\text{ker}(\mu) \cong \text{Im}(\mu)$ where $\text{Im}(\mu)$ is free and hence projective. Thus there exists a homomorphism $\phi: \text{Im}(\mu) \rightarrow A$ with $\phi\mu = \text{id}_A$. We

show that $A = \ker(\mu) \oplus (\text{Im}(\mu))\phi$. Let $x \in \ker(\mu) \cap (\text{Im}(\mu))\phi$, then there is $y \in A$ such that $x = y\mu\phi$ and $0 = x\mu = y\mu \underbrace{\phi\mu}_{=\text{id}_A} = y\mu$, so $x = y\mu\phi = 0\phi = 0$. Also, $a = a\mu\phi + (a - a\mu\phi)$ with $(a - a\mu\phi)\mu = a\mu - a\mu \underbrace{\phi\mu}_{=\text{id}_A} = 0$ for any $a \in A$. Thus $\ker(\mu)$ is a direct summand of A , i.e. μ is k -direct.

Now let $\sigma \in I$ be a direct endomorphism of A of rank κ , i.e. $A = \text{Im}(\sigma) \oplus S$ and $rk(\text{Im}\sigma) = \kappa$. Next we prove that all direct (im-direct) endomorphisms α with $rk\alpha \leq \kappa$ are elements of I . Let α be such an endomorphism, then

$A = \text{Im}(\alpha) \oplus T$ and $rk\alpha = rk(\text{Im}(\alpha)) \leq \kappa$. Since $\text{Im}(\alpha)$, $\text{Im}(\sigma)$ are free R -modules with $rk(\text{Im}(\alpha)) \leq rk(\text{Im}(\sigma))$, there exists a direct summand of $\text{Im}(\sigma)$ which is isomorphic to $\text{Im}(\alpha)$. Thus we may apply Proposition 3.2 which implies $\alpha \in I$.

Finally let ϕ be any endomorphism of A with $rk\phi \leq \kappa$. Since ϕ is k -direct we may express A as $A = \ker(\phi) \oplus C$ where $C \cong A/\ker(\phi) \cong \text{Im}(\phi)$. Thus

$rk(C) = rk\phi \leq \kappa$. Let π be the projection of A onto C with $\ker(\pi) = \ker(\phi)$. Obviously π is a direct endomorphism with $rk\pi = rk(C) \leq \kappa$; thus $\pi \in I$. Hence $\phi = \pi\phi$ is an element of I and this completes proof. \square

Note that the previous theorem holds for free modules over any ring R having the property that submodules of free modules are free; e.g. all principal ideal domains have this property. So, in particular Theorem 3.3 holds for a field; in this case we get even more, namely we can characterize the ideals of the endomorphism ring $\text{End}(A)$ of a vector space A .

Corollary 3.4 *Let A be a vector space over a field R . Then the only ideals of $\text{End}(A)$ are the ideals E_κ ($\kappa \geq \aleph_0$) defined by $E_\kappa = \{\alpha \in \text{End}(A) \mid rk\alpha < \kappa\}$.*

Proof: Note first, that all endomorphisms of a vector space A are direct. Hence, in this case Theorem 3.3 reads as:

If σ is an element of an ideal I , then I contains every endomorphism α with $rk\alpha \leq rk\sigma$.

It is easy to check that, for each $\kappa \geq \aleph_0$, E_κ is an ideal of $\text{End}(A)$; write E_0 for E_{\aleph_0} , the ideal of all finite rank endomorphisms.

Now, let $I \neq 0$ be an arbitrary ideal which is properly contained in $\text{End}(A)$. Since I is non-trivial there exists a non-zero endomorphism $\sigma \in I$. If σ is of infinite rank then, obviously, $E_0 \subseteq I$. So suppose σ is of finite rank $n \geq 1$, then I contains all endomorphisms of rank less than or equal to n . Thus if e is an element of a given basis B then I contains the projection π_e onto the one-dimensional subspace generated by e along the subspace generated by the remaining basis elements. Therefore all finite sums $\sum_{i=1}^k \pi_{e_i}$ ($e_i \in B$) of such projections belong to I and hence, for any $k \in \mathbb{N}$, there is an endomorphism of rank k belonging to I . This implies that all endomorphisms of finite rank are contained in the ideal I and so in either case we deduce $E_0 \subseteq I$.

Moreover $E_{\kappa+1} \subseteq I$ whenever $\eta \in I$ for some η of rank κ . Let $\tau + 1$ be the smallest cardinal with $E_{\tau+1} \not\subseteq I$ (we may consider a successor cardinal since the ideals E_κ form a smooth increasing chain). Then all $\eta \in I$ have rank less than τ and hence $I \subseteq E_\tau$. Also $E_\tau \subseteq I$ by the minimality of $\tau + 1$, thus $I = E_\tau$. \square

§4 Complete modules over complete discrete valuation rings

In the last section we turn our attention to complete reduced torsion-free R -modules A over complete discrete valuation rings R . Recall that the rank of a reduced torsion-free R -module over a complete discrete valuation ring is the rank of a basic submodule B of A (see §2). Again we define the rank of an endomorphism as the rank of its image. Moreover we call an endomorphism α of A a *pure endomorphism* if $\text{Im}(\alpha)$ is a pure submodule of A .

First we present a result which is similar to Theorem 3.3; for a complete reduced torsion-free R -module A over a complete discrete valuation ring R we can prove

Theorem 4.1 *Let I be an ideal of $\text{End}(A)$. If I contains a pure endomorphism σ of rank κ then I contains all endomorphisms α with $rk(\alpha) \leq \kappa$.*

Proof: Firstly we show that any endomorphism μ of A is k -direct. By Lemma 2.3 it suffices to show that $\ker(\mu)$ is pure in A for any $\mu \in \text{End}(A)$. If $x = p^n a$ with $x \in \ker \mu$ and $a \in A$, we have $(p^n a)\mu = x\mu = 0$. Hence $p^n(a\mu) = 0$ which implies

$a\mu = 0$ since A is torsion-free. So, $p^n A \cap \ker(\mu) = p^n \ker(\mu)$, i.e. $\ker(\mu)$ is pure in A and thus μ is k -direct for any $\mu \in \text{End}(A)$.

Let σ be a pure endomorphism in I and assume first that α is a pure endomorphism of A . Then, by Lemma 2.3, both $\text{Im}(\sigma)$ and $\text{Im}(\alpha)$ are direct summands of A , i.e. we may write A as $A = \text{Im}(\sigma) \oplus S = \text{Im}(\alpha) \oplus T$. If B_α, B_σ are basic submodules of $\text{Im}(\alpha)$ and $\text{Im}(\sigma)$ respectively, then there exists a direct summand D of B_σ of rank $rk\alpha = rk(B_\alpha) \leq rk(B_\sigma) = rk\sigma$ which is isomorphic to B_α . We may extend this isomorphism to an isomorphism of the completions $\text{Im}(\alpha)$ and \widehat{D} of B_α and D respectively. Moreover, since D is pure in B_σ , the completion \widehat{D} is pure in $\text{Im}(\sigma)$, hence \widehat{D} is a direct summand of $\text{Im}(\sigma)$ which is isomorphic to $\text{Im}(\alpha)$. Thus we may apply Proposition 3.2 which implies $\alpha \in I$. We have shown that all pure endomorphisms α with $rk\alpha \leq rk\sigma$ are contained in I ; so in particular all idempotents π with $rk\pi \leq rk\sigma$ belong to I .

Finally, let ϕ be any endomorphism of A with $rk\phi \leq rk\sigma$. Then $A = \ker(\phi) \oplus C$ where $C \cong A/\ker(\phi) \cong \text{Im}(\phi)$, so $rk(C) = rk\alpha$. If π denotes the projection onto C with $\ker(\pi) = \ker(\phi)$ then $\pi \in I$ since $rk\pi = rk(C) = rk\phi \leq rk\sigma$. Therefore $\phi = \pi\phi$ is an element of I . \square

The previous theorem, however, does not characterize the ideals of $\text{End}(A)$ since there are ideals which don't contain a pure endomorphism, e.g. $p\text{End}(A)$. Instead of using similar arguments as in the case of free R -modules we shall now use Corollary 3.4 on vector spaces to determine the ideals I of $\text{End}(A)$ modulo their Jacobson radicals. First we consider the Jacobson radicals $J(E_\kappa)$ of the ideals E_κ where $E_\kappa = \{\eta \in \text{End}(A) \mid rk\eta < \kappa\}$ for $\kappa \geq \aleph_0$.

Lemma 4.2 *Let E_κ be an ideal as defined above. Then the Jacobson radical $J(E_\kappa)$ coincides with the ideal pE_κ .*

Proof: First we show that $pE_\kappa = E_\kappa \cap p\text{End}(A)$. Let $p\alpha \in E_\kappa$ with $\alpha \in \text{End}(A)$, then $A = A_1 \oplus \ker(p\alpha)$ and $rk(A_1) < \kappa$. Since A is torsion-free $\ker(p\alpha) = \ker(\alpha)$. Hence $A\alpha = A_1\alpha$ and $rk\alpha < \kappa$, i.e. $\alpha \in E_\kappa$. By Proposition 2.4

$p\text{End}(A) = J(\text{End}(A))$, so it follows $E_\kappa \cap p\text{End}(A) = E_\kappa \cap J(\text{End}(A))$. But

$E_\kappa \cap J(\text{End}(A)) = J(E_\kappa)$ since E_κ is an ideal of $\text{End}(A)$. Therefore

$pE_\kappa = E_\kappa \cap p\text{End}(A) = E_\kappa \cap J(\text{End}(A)) = J(E_\kappa)$; this completes the proof. \square

Next we will show that $E_\kappa/J(E_\kappa)$ is isomorphic to a corresponding ideal of the vector space A/pA over the field R/pR .

Lemma 4.3 *For any cardinal $\kappa \geq \aleph_0$, $E_\kappa/J(E_\kappa) \cong E_\kappa(A/pA)$.*

Proof: Every $\alpha \in E_\kappa$ induces an R/pR -endomorphism on A/pA since $pA\alpha \subseteq pA$. So we may define a map $\Delta: E_\kappa \rightarrow \text{End}_{R/pR}(A/pA)$ by $\alpha\Delta = \bar{\alpha}: A/pA \rightarrow A/pA$ with $(a + pA)\bar{\alpha} = a\alpha + pA$. It is easy to check that Δ is a ring homomorphism. Moreover, the kernel of Δ is $pE_\kappa = J(E_\kappa)$. We show that $\text{Im}\Delta = E_\kappa(A/pA)$. Certainly, $\text{Im}\Delta \subseteq E_\kappa(A/pA)$ since the vector space rank of an endomorphism $\bar{\alpha}$ of A/pA cannot be greater than $rk\alpha$. Now let $\eta: A \rightarrow A/pA$ and $\rho: R \rightarrow R/pR$ be the endomorphisms defined by $a\eta = a + pA$ and $r\rho = r + pR$.

Let us consider an endomorphism ϕ of A/pA of rank less than κ . We can pick an R/pR -basis $\{x_i \mid i \in I\}$ of A/pA such that $x_i\phi = 0$ for all but less than κ of the x_i . Choose $y_i \in A$ such that $y_i\eta = x_i$ for all $i \in I$. Then the module generated by $\{y_i \mid i \in I\}$ is a basic submodule of A by Lemma 2.1. However $x_i\phi = \sum_{j \in I} r_{ij}x_j$ where $r_{ij} \in R/pR$ with $r_{ij} = 0$ for all but finitely many j ; choose $s_{ij} \in R$ with $s_{ij}\phi = r_{ij}$ and $s_{ij} = 0$ whenever $r_{ij} = 0$. Finally, we define $\beta: B \rightarrow B$ by $y_i\beta = \sum_{j \in I} s_{ij}y_j$, i.e. $y_i\beta = 0$ for all but less than κ of the y_i . Thus the unique extension of β to an endomorphism of A has rank $rk(B\beta) < \kappa$ and satisfies $\beta\Delta = \phi$. Hence $\Delta: E_\kappa \rightarrow E_\kappa(A/pA)$ is surjective with $\ker\Delta = pE_\kappa = J(E_\kappa)$, so $E_\kappa/J(E_\kappa) \cong E_\kappa(A/pA)$. \square

We finish the paper with the characterization of the ideals of the endomorphism ring of a complete reduced torsion-free R -module A over a complete discrete valuation ring R ; modulo their Jacobson radical they are characterized by a single cardinal κ .

Theorem 4.4 *If I is an arbitrary ideal of the endomorphism ring of a complete reduced torsion-free R -module A , then either $I \subseteq J(\text{End}(A))$ or $I/J(I) \cong E_\kappa/J(E_\kappa) \cong E_\kappa(A/pA)$ for some cardinal κ .*

Proof: Let I be any ideal of $\text{End}(A)$. We consider the mapping $\Delta: I \longrightarrow \text{End}(A/pA)$ defined by $\alpha\Delta = \bar{\alpha}$ with $(a + pA)\bar{\alpha} = a\alpha + pA$. This defines a ring homomorphism with $\ker(\Delta) = J(I) = I \cap J(\text{End}(A))$ which is either equal to I (i.e. $I \subseteq J(\text{End}(A))$) or it is proper contained in I . In the latter case $I/J(I) \cong K$ for some non-zero ideal K of $\text{End}(A/pA)$. Thus $K = E_\kappa(A/pA)$ for some κ by Corollary 3.4. Therefore $I/J(I) \cong E_\kappa/J(E_\kappa)$ by Lemma 4.3. \square

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