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On Modules which are Self-Slender.

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Abstract

This paper is an examination of the dual of the fundamental isomorphism relating homomorphism groups involving direct sums and direct products over arbitrary index sets. We prove that for every cardinal μ, with $\mu^{\aleph_0} = \mu$, there exists a non-slender self-slender self-small group of cardinality μ^+.

1 Introduction

There are two fundamental isomorphisms in homological algebra relating homomorphism groups involving direct sums and direct products over an arbitrary index set I – see for example [5, Theorems 43.1, 43.2]:

$$\mathrm{Hom}(G, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Hom}(G, A_i) \tag{1.1}$$

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$$\mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \cong \prod_{i\in I} \mathrm{Hom}(A_i, G) \tag{1.2}$$

In this work we look at some of the consequences of trying to "dualize"(1.2) by interchanging direct sums and direct products, and especially in the situation where each of the groups A_i is isomorphic to G itself. Recall that interchanging direct sums and direct products in (1.1) leads to the notion of *small* groups and *self–small* groups - see [1].

Thus we shall say that a group G is *strongly slender* if $\mathrm{Hom}(\prod_{i\in I} A_i, G) = \bigoplus_{i\in I} \mathrm{Hom}(A_i, G)$ for all indexed families $\{A_i : i \in I\}$, where we are using "=" to mean that the canonical mapping from $\bigoplus_{i\in I} \mathrm{Hom}(A_i, G) \to \mathrm{Hom}(\prod_{i\in I} A_i, G)$ is an isomorphism; in a similar fashion we say that G is *strongly self–slender* if the above is true when each $A_i \cong G$. We shall say that a group G is *self–slender*, if for all homomorphisms $\sigma : \prod_{n<\omega} G_n \to G($ where $G_n \cong G$ for each $n,)$ $\sigma \restriction G_n = 0$ for all but a finite number of n. In fact the notion of self–slenderness is easily seen to be equivalent to the statement that if $\sigma : \prod_{\alpha<\kappa} G_\alpha \to G$, (where $G_\alpha \cong G$ for each α), $\sigma \restriction G_\alpha = 0$ for all but a finite number of $\alpha :$ in one direction this is immediate, so suppose that G is self–slender and ϕ is a map from $\prod_{\alpha<\kappa} G_\alpha \to G$ which does not vanish on an infinite set I of α's. Then ϕ restricted to $\prod_{\alpha\in I} G_\alpha \to G$ would contradict the self–slenderness of G. At the outset it is important to note that strong slenderness and strong self-slenderness have possibly weaker generalized concepts, where we require the existence of some, not necessarily the canonical, isomorphism; there is a clear analogy with the notions of E-rings and generalized E-rings – see [7, Chapter 13]. Clearly a strongly self–slender group is generalized strongly self–slender. The concept of self-slenderness, in an equivalent formulation, has been studied by Faticoni in [3, 4] but working only in models of (ZFC + V = L).

These notions, which have their origin in Abelian group theory, can easily be generalized to a module setting. Let R be a commutative domain which is slender as an R-module (otherwise the discussion may be vacuous) and let $\mathbb{S}$ be a countable multiplicatively closed subset of regular elements of R with $1 \in \mathbb{S}$, such that R is $\mathbb{S}$-reduced and $\mathbb{S}$-torsion-free, i.e. R is an $\mathbb{S}$-ring in the terminology of [7]; further details of this basic situation may be found at [7, p.13]. Throughout we shall restrict attention to R-modules which are $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced. The notions of strong slenderness and self–slenderness are now identical to the group situation if one replaces the word "group"by "R-module" and interprets $\mathrm{Hom}(-,-)$ as $\mathrm{Hom}_R(-,-)$.

We write $G^I$and $G^{(I)}$ for the direct product $\prod_{i\in I} G$ and direct sum $\bigoplus_{i\in I} G$ except where the latter notation is easier to read. $\mathrm{End}_R(G)$ stands for $\mathrm{Hom}_R(G,G)$, the module of R-endomorphisms of G; the notation $\widehat{G}$ is used to denote the $\mathbb{S}$-completion of the $\mathbb{S}$-reduced $\mathbb{S}$-torsion-free R-module G. Infinite cardinals are usually denoted by κ, λ, μ; all other notation is standard and may be found in [2], [5, 6], [7]; in particular all groups shall be additively written Abelian groups. Recall that a cardinal κ is *ω-measurable*, if there exists a countably complete non-principal ultrafilter over κ. The family of countably complete ultrafilters over κ is denoted $\mathcal{D}(\kappa)$; it contains all the principal ultrafilters over κ. An uncountable cardinal κ is *measurable* if there exists a κ-complete non-principal ultrafilter over κ. The least ω-measurable cardinal is measurable and measurable cardinals are strongly inaccessible. If κ is measurable, then there are at least 2^κ κ-complete non-principal ultrafilters over κ and so $\mathcal{D}(\kappa)$ contains at least 2^κ elements. It is, however, consistent with ordinary set theory (ZFC) that no measurable cardinals exist. For example, ZFC+V=L implies that there are no measurable cardinals (and hence no ω-measurable cardinals). We shall use $\aleph_m$ to denote the first measurable cardinal, if there exist measurable cardinals, otherwise the condition $\kappa < \aleph_m$ is vacuously true for every cardinal κ. To avoid confusion, we point out that in Fuchs [6] the term measurable refers to a non-trivial countably additive two-valued measure; this is what we call ω-measurable, as in [2]. Finally recall the beth function: for a cardinal κ and an ordinal α, define $\beth_\alpha(\kappa)$ by: $\beth_0(\kappa) = \kappa, \beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$, and for a limit ordinal α, $\beth_\alpha(\kappa) = \sup_{\beta<\alpha} \beth_\beta(\kappa)$.

2 Strongly Slender and Self-Slender Modules

A simple observation, which is certainly well known, will be of fundamental use; a proof is included for completeness.

Lemma 2.1 *Let K be an R-module of cardinality less than κ and suppose that $K \le H$. Then $|\operatorname{Hom}_R(K^\lambda, H)|$ is at least as large as the number of κ-complete ultrafilters on λ.*

Proof. Suppose U is a κ-complete ultrafilter on λ. For $a \in K^\lambda$, let $\phi_U(a) = k$ iff for $k \in K$, $a^{-1}(k) = \{\alpha < \lambda : a(\alpha) = k\} \in U$. The κ-completeness of U ensures the map ϕ_U is well defined since the sets $a^{-1}(k)$, $(k \in K)$ are a partition of λ into fewer than κ subsets – see, for example, [2, Lemma II 2.6]. It is straightforward to verify that the map $\phi_U : K^\lambda \to K$ is a homomorphism and if U and V are different κ-complete ultrafilters, then $\phi_U \ne \phi_V$. Since $K \le H$, the result follows immediately. ∎

It is well known from the earliest observations of Łoś that measurable cardinals play a key role in determining homomorphism groups when the domain is a direct product. To avoid constant repetition we single out two well-known facts that we shall use constantly in the remainder of this work; proofs of these facts may be found in [2, II Corollary 3.3, III Corollary 3.6].

Fact 1. For any cardinal λ and slender ring T, if $\{A_i : i < \lambda\}$ is a family of T-modules, then $\operatorname{Hom}_T(\prod_{i<\lambda} A_i, T) \cong \bigoplus_{D\in\mathcal{D}(\lambda)} \operatorname{Hom}_T(\prod_{i<\lambda} A_i/D, T)$.

Fact 2. If T is a ring of non-ω-measurable cardinality, then for any cardinal λ, if D is a countably complete ultrafilter over λ, then $T^\lambda/D \cong T$.

The following result gives an algebraic characterization of the non-existence of measurable cardinals which illustrates clearly their fundamental role.

Proposition 2.2 *For a slender ring R, the following conditions are equivalent:*

(i) Every countable subring T of R which is T-reduced, is strongly slender.

(ii) Some countable subring T of R is strongly self-slender.

(iii) There are no measurable cardinals

Proof. *(iii)* implies *(ii)*. Since $\mathbb{Z}$ is the prime subring of R, it suffices to show that $\mathbb{Z}$ is strongly self-slender. This follows immediately from Facts 1 & 2 since $|\mathcal{D}(\lambda)| = \lambda$ as λ is non-measurable.

(iii) implies *(i)*. It suffices to note that if T is countable and T-reduced, then T is slender; now apply Facts 1 & 2.

(i) implies *(ii)*. The prime subring of R is $\mathbb{Z}$, which is reduced, slender and so, by *(i)*, is strongly self-slender.

(ii) implies *(iii)*. Since T is countable and strongly self-slender, it follows from Facts 1 & 2 that $\mathrm{Hom}(T^\kappa, T) \cong \bigoplus_\kappa \mathrm{End}\, T \cong \bigoplus\limits_{D \in \mathcal{D}(\kappa)} \mathrm{Hom}_T(T^\kappa/D, T) \cong \bigoplus\limits_{D \in \mathcal{D}(\kappa)} \mathrm{End}\, T$. However if κ is measurable, then $|\mathcal{D}(\kappa)| \geq 2^\kappa$ and we are finished by comparing cardinalities. ∎

If G is strongly slender, then clearly any R-homomorphism from R^ω into G vanishes on almost all components and hence G is slender. We begin with a well-known observation:

Proposition 2.3 *Assuming that no measurable cardinals exist, then the following holds*

(i) An R-module G is strongly slender if, and only if G is slender.

(ii) If G is slender, then G is generalized strongly self–slender.

Proof. By assumption the cardinalities of G and the index set I are not ω-measurable and hence the results follow from [2, III Corollary 3.7]. ∎

The situation changes dramatically if measurable cardinals exist. Note that in the next result we are not restricted to the first measurable cardinal $\aleph_m$.

Proposition 2.4 *If there exists a measurable cardinal κ and G is an R-module of cardinality less than κ, then G is not generalized strongly self–slender.*

Proof. Apply Lemma 2.1 with $K = H = G$ and $\lambda = \kappa$. However if κ is measurable then there are 2^κ κ-complete ultrafilters on κ – see [2, II Exercise 12 p.50]. Thus $|\mathrm{Hom}_R(G^\kappa, G)| \geq 2^\kappa > \kappa = |\mathrm{Hom}_R(G, G)^{(\kappa)}|$ and so G is not self–slender. ∎

Corollary 2.5 *If there exists a proper class of measurable cardinals, then no R-module is generalized strongly self–slender.*

Slender modules are self-slender, and this suggests the obvious question whether self–slender modules are necessarily slender. We shall address this question in the next section.

In the context of modules over an $\mathbb{S}$-ring, it is difficult to give necessary conditions for a module to be a generalized strongly self–slender module. However, in the context of Abelian group theory, where cotorsion-freeness may be characterized by exclusion of certain groups, there is an immediate necessary condition:

Proposition 2.6 *If G is a generalized strongly self–slender Abelian group, then G is cotorsion-free.*

Proof. It suffices to show that G cannot have a direct summand isomorphic to either $\mathbb{Z}(p^n)$, $\mathbb{Q}$ or J_p for any prime p. Note that if G decomposes as a direct sum $K \oplus H$, then $\mathrm{Hom}(G^\lambda, G)$ has cardinality at least as large as $|\mathrm{Hom}(K^\lambda, K)|$. Now if $\lambda = |G|$ and $\kappa = 2^\lambda$, it is immediate that $\mathrm{Hom}(G,G)^{(\lambda)}$ has cardinality at most κ. However, $\mathbb{Z}(p^n)^\lambda \cong \mathbb{Z}(p^n)^{(\kappa)}$, $\mathbb{Q}^\lambda \cong \mathbb{Q}^{(\kappa)}$ and so $|\mathrm{Hom}(\mathbb{Z}(p^n)^\lambda, \mathbb{Z}(p^n))| = |\mathrm{Hom}(\mathbb{Q}^\lambda, \mathbb{Q})| = 2^\kappa$. Moreover $J_p^\lambda \cong \widehat{\bigoplus_\kappa J_p}$ and, since J_p is algebraically compact, $\mathrm{Hom}(\widehat{\bigoplus_\kappa J_p}, J_p) \cong \mathrm{Hom}(\bigoplus_\kappa J_p, J_p)$ so that $|\mathrm{Hom}(J_p^\lambda, J_p)| = |J_p^\kappa| = 2^\kappa$. Thus neither $Z(p^n)$, $\mathbb{Q}$ nor J_p is a summand of G, as required. ∎

It is now rather easy to show that a generalized self–slender Abelian group with non-trivial dual is slender, provided that there are no measurable cardinals.

Theorem 2.7 *If there are no measurable cardinals, an Abelian group G with $\mathrm{Hom}(G,\mathbb{Z}) \neq 0$ is slender if, and only if, G is generalized strongly self–slender.*

Proof. We have already seen in Proposition 2.3 that slender Abelian groups are generalized strongly self–slender provided there are no measurable cardinals. Conversely suppose for a contradiction that G is generalized strongly self–slender but not slender. It follows from Proposition 2.6 that G is a cotorsion-free Abelian group which is *not* slender. Thus we have $P \leq G$, where $P \cong \mathbb{Z}^\omega$. Hence $\mathrm{Hom}(G^\kappa, P) \leq \mathrm{Hom}(G^\kappa, G)$ so that one has the inequality $|\mathrm{Hom}(G^\kappa, G)| \geq |\mathrm{Hom}(G^\kappa, P)|$. However $\mathrm{Hom}(G^\kappa, P) \cong \prod_\omega \mathrm{Hom}(G^\kappa, \mathbb{Z}) \cong \prod_\omega \bigoplus_\kappa \mathrm{Hom}(G, \mathbb{Z})$, the last equality coming from the fact that $\mathbb{Z}$ is slender and $|G|$ is not ω-measurable i.e. $\mathrm{Hom}(G^\kappa, P) \cong \prod_\omega Y$, where $Y = \bigoplus_\kappa \mathrm{Hom}(G,\mathbb{Z})$; note that our assumption on the dual of G means that Y is non-trivial. Now choose the cardinal κ so that $\kappa > |\mathrm{Hom}(G,G)|$ *and* κ has cofinality ω; e.g. take $\kappa = \beth_\omega(|\mathrm{End}(G)|)$. Thus $|\mathrm{Hom}(G^\kappa, P)| = |Y^\omega| = \kappa^\omega > \kappa$ since $Y \neq 0$. However if G is self–slender, then $\mathrm{Hom}(G^\kappa, G) \cong \bigoplus_\kappa \mathrm{Hom}(G,G)$, so that $|\mathrm{Hom}(G^\kappa, G)| = \kappa$ which contradicts the fact that $|\mathrm{Hom}(G^\kappa, G)| \geq |\mathrm{Hom}(G^\kappa, P)|$. Thus G cannot be generalized self–slender as required. ∎

Our final example in this section shows that neither the class of generalized strongly self–slender R-modules, nor the class of strongly self–slender R-modules, is closed under direct products.

Example 2.8 *Let K be any slender R-module which is not of ω-measurable cardinality. Now consider the R-module $G = \prod_\omega K$. Let $\mu = |\mathrm{Hom}_R(K,K)|$ and $\lambda = \beth_\omega(\mu^{\aleph_0})$. Then on one hand*

$$\bigoplus_\lambda \mathrm{Hom}_R(G,G) \cong \bigoplus_\lambda \prod_\omega \mathrm{Hom}_R(\prod_\omega K, K) \cong \bigoplus_\lambda \prod_\omega \bigoplus_\omega \mathrm{Hom}_R(K,K)$$

and this has cardinality λ. On the other hand

$$\mathrm{Hom}_R(G^\lambda, G) \cong \prod_\omega \mathrm{Hom}_R(G^\lambda, K) \cong \prod_\omega \bigoplus_{D \in \mathcal{D}(\lambda)} \mathrm{Hom}_R(G^\lambda/D, K).$$

Moreover since K has non-ω-measurable cardinality, so also has G and thus each ultrapower G^λ/D is canonically isomorphic to G – see e.g. [2, II Corollary 3.3]. Note that $\mathcal{D}(\lambda)$ has cardinality λ since *there are always λ principal ultrafilters over λ and the fact that λ is not ω-measurable means that each countably complete ultrafilter over λ is principal.*

Since K is slender $\mathrm{Hom}_R(G,K) \cong \bigoplus_{\omega} \mathrm{Hom}_R(K,K)$ and so $\mathrm{Hom}_R(G^\lambda, G)$ has cardinality $\lambda^{\aleph_0} > \lambda$. Thus G is not generalized self–slender, and hence, a fortiori, not self–slender. Now if we choose $K = R$, then K is self-slender but K^ω is not; in particular note that while $\mathbb{Z}$ is self-slender, the Baer-Specker group $\mathbb{Z}^{\aleph_0}$ is not. Finally we remark that if no measurable cardinals exist, then it follows from Proposition 2.4 that neither the class of generalized strongly self-slender nor the class of strongly self-slender modules is closed under direct products.

3 Self–slender R-modules which are not Slender

In [3] Faticoni remarked that the only known examples of groups which are self–slender and self–small, are the reduced torsion-free groups of finite rank; note that in his terminology 'self-slender groups' are what we have chosen to call 'strongly self-slender' . In this section we show that there are "many" such R-modules and thus, in particular, many such Abelian groups. Moreover, the R-modules need not be slender. Thus we answer the question raised in Section 2 in the negative.

It is well known in the theory of Abelian groups that a homomorphism from a product into a slender group is determined by its restriction to the corresponding direct sum. Our first Lemma indicates how to extend this to self-slenderness by restricting to *R-algebras*.

Lemma 3.1 *If G is a self-slender R-algebra and $\phi : \prod_{n<\omega} G_n \to G$ (where each $G_n \cong G$) is a homomorphism such that $\phi \restriction \bigoplus_{n<\omega} G_n = 0$, then $\phi = 0$.*

Proof. The argument follows exactly as in [2, III Theorem 1.2]: the critical point is that once the summable family $x^{(n)}$ has been constructed, a homomorphism $\theta : G^\omega \to G$ can be defined by the rule $\theta : (g_n) \mapsto \sum_{n\in\omega} g_n x^{(n)} \phi$; this makes sense because G is now an R-algebra. The remainder of the proof is exactly as in [2]. ∎

Our next result, which may be of independent interest, comes from a careful examination of Łoś's original argument on slenderness; we follow the argument as given in [7, Theorem 1.4.13, Corollary 1.4.14] pointing out the necessary changes resulting from the fact that our module is only self–slender and not slender; again a key point is that we work with an R-algebra.

Theorem 3.2 *If $\lambda < \aleph_m$ and G is a self–slender R-algebra, then*

$$\mathrm{Hom}_R(\prod_{\lambda} G, G) = \bigoplus_{\lambda} \mathrm{Hom}_R(G,G).$$

Proof. We begin by showing that a map from $G^\lambda \to G$, which vanishes on $G^{(\lambda)}$, is identically zero. Let $x = \sum_{i<\lambda} x_i e_i \in G^\lambda$ and $\varphi : G^\lambda \to G$ be such that $G^{(\lambda)}\varphi = 0$. We must show that $x\varphi = 0$. For $X \subseteq \lambda$ we consider $x_X = \sum_{i\in X} x_i e_i$ as an element in G^λ in the obvious way. The homomorphism φ and the element x induce a map

$$\varphi^* : \mathfrak{P}(\lambda) \to M \quad (X \mapsto x_X\varphi).$$

If $X, Y \subseteq \lambda$ are disjoint, then $x_X + x_Y = x_{X\cup Y}$, thus $X\varphi^* + Y\varphi^* = (X \cup Y)\varphi^*$ and so φ^* is additive. First we claim that φ^* is also σ–additive; we show, in particular,
that $X_i\varphi^* = 0$ for almost all $i < \omega$ and for all pairwise disjoint subsets $X_i \subseteq \lambda$.

If $\lambda = \bigcup_{i<\omega} X_i$ is an arbitrary partition of λ, then $x_{X_i} \in G^\lambda$ $(i < \omega)$ is a summable family and

$$\psi : G^\omega \to G^\lambda \quad \Big(\sum_{i<\omega} a_i e_i \mapsto \sum_{i<\omega} a_i x_{X_i}\Big)$$

is a well–defined homomorphism taking e_i to x_{X_i}, *since G is an R-algebra.* In particular, $X_i\varphi^* = e_i\psi\varphi = 0$ for almost all $i < \omega$, because G is self–slender. Thus

$$\Phi : G^\omega \to G \quad \Big(\sum_{i<\omega} a_i e_i \mapsto \sum_{i<\omega} a_i(X_i\varphi^*)\Big)$$

is also well defined and Φ coincides with $\psi\varphi$ when restricted to $G^{(\omega)}$. Hence $\Phi - \psi\varphi \restriction G^{(\omega)} = 0$ and so $\Phi - \psi\varphi = 0$ follows from Lemma 3.1. Now it is immediate that

$$\Big(\bigcup_{i<\omega} X_i\Big)\varphi^* = \lambda\varphi^* = x_\lambda\varphi = x\varphi = \Big(\sum_{i<\omega} e_i\Big)\psi\varphi = \Big(\sum_{i<\omega} e_i\Big)\Phi = \sum_{i<\omega} X_i\varphi^*$$

and $\varphi^* : \mathfrak{P}(\lambda) \to G$ is σ–additive. This shows the first claim.

Now, following exactly the proof of [7, Theorem 1.4.13], one can show that the Boolean algebra $\mathfrak{B}_\lambda = (\mathfrak{B}_\lambda, \leq)$ defined there, satisfies the descending chain condition. Moreover the measure μ defined there is σ-additive, and vanishes on singletons because G is self–slender.

As noted above, self–slenderness is equivalent to "λ-self–slenderness" with the obvious interpretation of this latter term, and so the proof of [7, Corollary 1.4.14] carries over immediately since the set E defined there, is again finite. ∎

In fact the Theorem above can be generalized to all cardinals λ as below; the proof follows exactly that in [2] once one notes that G is an R-algebra and so the critical definition 3.2.4 there again makes sense.

Theorem 3.3 *An R-algebra G is self-slender if, and only if, for all λ, $\mathrm{Hom}_R(G^\lambda, G) = \bigoplus_{D\in\mathcal{D}(\lambda)} \mathrm{Hom}_R(G^\lambda/D, G)$. If there are no measurable cardinals, then G is self-slender if, and only if, G is strongly self-slender.*

If R is an $\mathbb{S}$-ring, G is an $\mathbb{S}$-reduced $\mathbb{S}$-torsion-free R-module and $\mathbb{S}'$ is a multiplicatively closed subset derived from $\mathbb{S} = \{s_i : i < \omega\}$ by replacing the terms $q_n = \prod_{i=1}^{n} s_i$ by the corresponding 'factorials' obtained by replacing s_i with $s_i z_i$, where $z_i \in \mathrm{End}_R(G)$, then we shall say that the resulting topology defined on G by replacing $\mathbb{S}$ by $\mathbb{S}'$, is an $\mathbb{S}$-*derived topology.* In the situation where G is an $\mathrm{E}(R)$-algebra, and thus the elements z_i may be considered as elements of the algebra G, the derived topology will be Hausdorff if the original $\mathbb{S}$-topology was Hausdorff.

Proposition 3.4 *Let G be an $\mathbb{S}$-reduced $\mathbb{S}$-torsion-free R-module which is also an E(R)-algebra, and which is not complete in any $\mathbb{S}$-derived topology, then G is self–slender.*

Proof. Suppose that $\sigma : \prod\limits_{n<\omega} G_n \to G$, where each G_n is isomorphic to G. Then, as G is an $\mathrm{E}(R)$-algebra, $\sigma \restriction G_n \to G$ can be identified as multiplication by an element of G, z_n say. Suppose, for a contradiction, that G is not self–slender so that infinitely many of the z_n are non-zero. By omitting those G_n for which z_n is zero, and working with the remaining infinite direct product, we may, without loss of generality, assume that $z_n \neq 0$ for all n. Note that $\phi = \sigma \restriction \bigoplus_n G_n$ acts as the summation map $(g_1, g_2, \dots g_n, 0, \dots) \mapsto z_1 g_1 + z_2 g_2 + \dots + z_n g_n$.

Let $\mathbb{S}'$ be the multiplicatively closed set obtained as the multiplicative closure of $\{s_i z_i : s_i \in \mathbb{S}\}$. As observed above the resulting $\mathbb{S}'$-topology will be Hausdorff.

Now, by assumption, G is not complete in its $\mathbb{S}'$-topology. Thus there is an element $x = \sum\limits_{n<\omega} w_n q_n g_n$ which lies in $\widehat{G}^{\mathbb{S}'} \setminus G$; here w_n is divisible by z_n and so we write $w_n = v_n z_n$. Take $g \in G^\omega$ to be the element with nth coordinate $v_n q_n g_n$, so that $g \in G^\omega \cap \widehat{G^{(\omega)}}^{\mathbb{S}'}$. Since the maps σ and ϕ agree on a dense subset of $\widehat{G^{(\omega)}}^{\mathbb{S}'}$, it follows by continuity that $g\sigma = g\widehat{\phi}$. Now $\widehat{\phi}$ is just the extension of the summation map, so $g\widehat{\phi} = \sum\limits_{i<\omega} g_i z_i = g \notin G$, but $g\sigma \in G$ – contradiction. ■

The final step required to establish the existence of non-slender, self–slender modules is to exhibit suitable non-slender $\mathrm{E}(R)$-algebras. Our construction is based on the construction of such modules using the Strong Black Box and follows closely the development in [7, Theorem 13.4.1]. Whilst we will not repeat all the details of that theorem, concentrating instead on the crucial first step which establishes that the outcoming module is not slender, it is important to point out that one must actually rework the proof; it is not sufficient to simply replace the ring R with a non-slender one R' and use the established version of the theorem since this will exhibit $\mathrm{E}(R')$-algebras rather than $\mathrm{E}(R)$-algebras. Note that at this point it is necessary to assume that R^+ is actually torsion-free and not just $\mathbb{S}$-torsion-free.

Theorem 3.5 *Let R be a domain which is an $\mathbb{S}$-cotorsion-free $\mathbb{S}$-ring with R^+ torsion-free. Let λ and μ be infinite cardinals such that $\mu^{\aleph_0} = \mu$, $\lambda = \mu^+$ and $|R|^{\aleph_0} \leq \lambda$. Then there exists an E(R)-algebra A of cardinality λ which is not slender and is not $\mathbb{S}'$-complete for any $\mathbb{S}'$ derived from $\mathbb{S}$.*

Proof. The construction of an $\mathrm{E}(R)$-algebra via the Strong Black Box is described in [7, Construction 13.4.3]; in essence one constructs inductively a sequence A^β of $\mathbb{S}$-pure, $\mathbb{S}$-cotorsion-free R-submodules of the $\mathbb{S}$-completion $\widehat{B}$, where B is the polynomial ring in λ commuting variables, $B = R[x_\alpha | \alpha < \lambda]$. We need to replicate this construction but in the process modify it in such a way that the final module $A = \bigcup_{\alpha<\lambda} A^\alpha$ is not slender. This can be achieved by modifying the initial step so that a copy of the product R^ω is embedded in A^0; note that this will suffice since the final module

A constructed is a union of a chain of modules and hence will necessarily contain the embedded copy of R^ω. This modification can be easily achieved: let $B = R[x_\alpha | \alpha < \lambda]$ and observe that $B = \bigoplus_{m \in \mathfrak{M}} Rm$, where $\mathfrak{M}$ denotes the set of monomials in the variables x_α. If $q_i = \prod_{j<i} s_j$, set

$$P' = \{\sum_{i<\omega} r_i q_i x_i | r_i \in R\}$$

and note that $P' \cong R^\omega$. If now we choose A^0 as the $\mathbb{S}$-purification of the subalgebra of $\widehat{B}$ generated by B and P', $A^0 \leq \widehat{B} \bigcap \prod_{m \in \mathfrak{M}} Rm$ and so A^0 is a torsion-free, $\mathbb{S}$-cotorsion-free domain which is not slender. The remainder of the construction is exactly as in [7, Construction 13.4.3]. The proof of [7, Theorem 13.4.1] carries through to give the desired $\mathrm{E}(R)$-algebra.

The final statement in the theorem follows by a minor extension of the proof of Theorem 13.4.1 in [7, p. 486] using the construction of I. Using the existence of such a set $I = \{\alpha_1 < \alpha_2 < \dots < \alpha_n < \dots\}$ such that

$$I \cap [a] \text{ is finite for all } a \in A, \tag{3.1}$$

we consider the element $g = \sum_{i<\omega} q_i z_i x_{\alpha_i}$ in an $\mathbb{S}'$-topology on A given by $\langle s_i z_i \mid i < \omega \rangle$ as explained above. Then clearly $g \in \widehat{A}$, the $\mathbb{S}'$-completion of A, but $g \notin A$ by (3.1). Thus A is not $\mathbb{S}'$-complete. ∎

Putting these results together we can deduce the existence of non-slender self–slender modules. Thus we have:

Corollary 3.6 *Let R be a domain which is an $\mathbb{S}$-cotorsion-free $\mathbb{S}$-ring with R^+ torsion-free and let λ and μ be infinite cardinals such that $\mu^{\aleph_0} = \mu$, $\lambda = \mu^+$ and $|R|^{\aleph_0} \leq \lambda$. Then there exists a non-slender, self–slender R-module G of cardinality λ.*

Proof. Let G be the $\mathrm{E}(R)$-algebra given by Theorem 3.5. Then G is not slender, but by Proposition 3.4 it is self–slender. ∎

Corollary 3.7 *Let R be a domain which is an $\mathbb{S}$-cotorsion-free $\mathbb{S}$-ring with R^+ torsion-free and let λ and μ be infinite cardinals such that $\mu^{\aleph_0} = \mu$, $\lambda = \mu^+$ and $|R|^{\aleph_0} \leq \lambda$. If there are no measurable cardinals then there exists a non-slender, strongly self-slender R-module G.*

Proof. Choose G as in Corollary 3.6. It is a self-slender R-algebra, so by Theorem 3.2 and our assumption, it is strongly self-slender. ∎

Corollary 3.8 *For any cardinal μ with $\mu^{\aleph_0} = \mu$, there exists a non-slender self–slender Abelian group of cardinal μ^+. In particular there exists a non-slender self-slender Abelian group of cardinality $(2^{\aleph_0})^+$.*

It is now easy to address the question raised by Faticoni in [3], again using our assumption that no measurable cardinals exist. We state the result for Abelian groups although it clearly holds in our more general module setting.

Corollary 3.9 *If there are no measurable cardinals, then for any cardinal μ with $\mu^{\aleph_0} = \mu$, there exists a self–small, strongly self–slender Abelian group of cardinal μ^+, which is not slender.*

Proof. Choose $\mathbb{S} = \{p^n | n < \omega\}$ and $R = \mathbb{Z}$ in Corollary 3.5 above. Then the E-ring so obtained is a domain and hence the finite topology on its endomorphism ring is discrete. Thus the group is self–small – see [1, Corollary 2.1]. ■

Finally we consider a seemingly innocent question with a surprising answer: does there exist a self-slender, but not slender Abelian group of cardinality $\aleph_n$? The answer to this is independent of ZFC. In any model of (ZFC + CH), $\aleph_n^{\aleph_0} = \aleph_n$ for all $n \geq 1$, and so it follows from Corollary 3.8 that there exists such a group of power $\aleph_n$ for all $n \geq 2$. However, in any model of (ZFC + $2^{\aleph_0} \geq \aleph_{\omega+1}$), any reduced torsion-free group of cardinality $\aleph_n$ is slender by a famous result of Sąsiada. It is interesting to note that the non-existence of measurable cardinals does not eliminate the independence phenomenon here: both (ZFC + CH + there are no measurable cardinals) and (ZFC + $2^{\aleph_0} \geq \aleph_{\omega+1}$ + there are no measurable cardinals) are relatively consistent. It would be interesting to know whether in ZFC, self-slenderness implies slenderness for groups of power $\aleph_1, \aleph_\omega$, or $\aleph_{\omega+1}$.

References

[1] D. Arnold and C.E. Murley, *Abelian groups, A, such that* Hom$(A, -)$ *preserves direct sums of copies of A,* Pacific J. Math. **56** (1975) 7–20.

[2] P. Eklof and A. Mekler, *Almost Free Modules, Set-theoretic Methods,* North-Holland, 2002.

[3] T.G. Faticoni, *Modules and point set topological spaces,* in *Abelian Groups, Rings, Modules, and Homological Algebra*, Lecture Notes in Pure and Applied Mathematics, **249** Chapman & Hall 2006, 87 - 105.

[4] T.G. Faticoni, *A Duality for Self-slender Modules,* to appear in Comm. Algebra.

[5] L. Fuchs, *Infinite Abelian Groups,Vol.I,* Academic Press 1970.

[6] L. Fuchs, *Infinite Abelian Groups, Vol.II,* Academic Press 1974.

[7] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules,* Expositions in Mathematics, **41**, De Gruyter, Berlin 2006.