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Aspects of Minimality in Abelian Groups

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Contents

Abstract

This thesis investigates those abelian groups which are minimal with respect to certain quasi-orders defined on Ab_κ, the category of abelian groups of a given infinite cardinality κ. Six such quasi-orders are defined and groups which are minimal with respect to these quasi-orders are called either quasi-minimal, with the associated concepts of purely and directly quasi-minimal groups, or simply minimal with the corresponding associated groups. A complete characterisation is derived for the quasi-minimal groups and, assuming GCH, for the purely quasi-minimal groups. Moreover, it is shown that the direct quasi-minimality of a group may be undecidable in ZFC. In the minimal case, consideration of torsion groups can be reduced to that of p-groups, and a criterion for the minimality of a p-group is found in terms of its Ulm invariants. The minimality of various classes of torsion-free groups is determined. In particular, a characterisation in terms of their critical typesets is found for all finite rank and for large classes of infinite rank completely decomposable groups. Several equivalent conditions are given for the minimality of general separable groups. The minimality of mixed groups is also investigated, particularly those of torsion-free rank 1.

Introduction

The notions of strong and weak quasi-minimality first appeared in the context of topological spaces (see e.g. [18]), building on observations from an important paper of Ginsburg and Sands in 1979 [12]. The essential ingredients were: a family $\mathcal{F}$ of topological spaces and the quasi-ordering of spaces obtained by saying, for $X, Y \in \mathcal{F}$, $X \preceq Y$ if X is homeomorphic to a subspace of Y. Then we say $X \in \mathcal{F}$ is *strongly quasi-minimal* if $Y \preceq X$, $Y \in \mathcal{F}$, implies X is homeomorphic to Y. The surprising result in this context is that if $\mathcal{F}$ is the family of all countably infinite topological spaces then the strongly quasi-minimal members of $\mathcal{F}$ are precisely the five spaces singled out by Ginsburg and Sands [12].

Some related concepts have been extended by Matier and McMaster (see [17]) to the more general setting of what they call a *sized category*. By this they mean a category $\mathcal{C}$ with an equivalence relation $\sim$ on the objects of $\mathcal{C}$, a quasi-order (i.e. a reflexive and transitive but, not necessarily anti-symmetric, binary relation) "sub" on $\mathcal{C}$, an assignment c which associates with each object $X \in \mathcal{C}$ a "size" $c(X)$ of X and another quasi-order $\leq$ on the range of c satisfying:

(1) X sub $Y, X \sim X', Y \sim Y'$ together imply X' sub Y',

(2) X sub Y implies $c(X) \leq c(Y)$,

(3) if $c(X) \leq c(Y)$ then there exists $Z \in \mathcal{C}$ such that X sub Z and $c(Y) = c(Z)$,

(4) if $c(X) \leq c(Y)$ then there exists $W \in \mathcal{C}$ such that W sub Y and $c(W) = c(X)$.

(In almost all cases of interest the size $c(X)$ will be the cardinality of the set X with $\leq$ the usual ordering on cardinals.)

We then say that $X \in \mathcal{C}$ is *strongly quasi-minimal* if Y sub X, $Y \in \mathcal{C}$, implies $X \sim Y$. [1]

The categories of interest to Matier and McMaster derive mainly from general topology. Our interest, however, lies in another direction and we focus exclusively on the category Ab_κ of all abelian groups of the fixed but arbitrary cardinal κ. The relation "sub" on objects $X, Y \in Ab_\kappa$ may take a number of different forms:

[1]Note that the quasi-order sub induces another quasi-order, again called sub, on the equivalence classes of objects in $\mathcal{C}$ where we say $\mathcal{X}$ sub $\mathcal{Y}$ if and only if X sub Y for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, for equivalence classes $\mathcal{X}$ and $\mathcal{Y}$. In the case of abelian groups, however, where the equivalence relation is isomorphism, a natural concept, this plays no significant role.

(i) X sub Y if X is isomorphic to a subgroup of Y,

(ii) X $\text{sub}_* Y$ if X is isomorphic to a pure subgroup of Y,

(iii) X $\text{sub}_{\sqsubset} Y$ if X is isomorphic to a direct summand of Y,

(iv) X $\text{sub}_{f.i.} Y$ if X is isomorphic to a subgroup of finite index of Y,

(v) X $\text{sub}_{f.i.*} Y$ if X is isomorphic to a pure subgroup of finite index of Y,

(vi) X $\text{sub}_{f.i.\sqsubset} Y$ if X is isomorphic to a direct summand of finite index of Y.

In each case the equivalence relation $\sim$ will be isomorphism and $c(X)$ will denote the cardinality of X. Thus we have six different concepts of strong quasi-minimality, corresponding to the various forms of "sub" above. Henceforth we omit the adjective "strong". Specifically, we say:

(a) X is *quasi-minimal* in Ab_κ if X is quasi-minimal with respect to *sub*,

(b) X is *purely quasi-minimal* in Ab_κ if X is quasi-minimal with respect to sub_*,

(c) X is *directly quasi-minimal* in Ab_κ if X is quasi-minimal with respect to $sub_{\sqsubset}$.

(d) X is *minimal* in Ab_κ if X is quasi-minimal with respect to $sub_{f.i.}$,

(e) X is *purely minimal* in Ab_κ if X is quasi-minimal with respect to $sub_{f.i._*}$,

(f) X is *directly minimal* in Ab_κ if X is quasi-minimal with respect to $sub_{f.i._{\sqsubset}}$.

Note that a group is purely minimal if and only if it is directly minimal since a pure subgroup of finite index is always a summand. Also note that a group is quasi-minimal if and only if it is isomorphic to all its subgroups of the same cardinality, with similar statements for the other five types of minimality. In practice this is the more useful viewpoint of minimality to take. We will make the corresponding definitions as required.

In the context of topological spaces it was also of interest to have a notion of weak quasi-minimality: the essential difference here being that $Y \preceq X$, $Y \in \mathcal{F}$, now implies $X \preceq Y$ only. Since the order $\preceq$ is not necessarily anti-symmetric, this is, a priori, a different concept to strong quasi-minimality. In a similar way we introduce the corresponding concept in Ab_κ. Just as in topological spaces, the orders introduced in (i) –

(vi) are not anti-symmetric as the following examples show:

(a) Let $P = \prod_{n<\omega} \mathbb{Z}e_n$, $S = \bigoplus_{n<\omega} \mathbb{Z}e_n$, $P_{o/e} = \prod_{n \text{ odd/even}} \mathbb{Z}e_n$, $S_{o/e} = \bigoplus_{n \text{ odd/even}} \mathbb{Z}e_n$ and $G = S_o \oplus P_e$, where e_n is the n^{th} unit vector. Then G sub_*P and P sub_*G but $G \not\cong P$:

obviously G is a pure subgroup of P and also $P \cong P_e$ $(n \longrightarrow 2n)$, a pure subgroup of G. However, $G \not\cong P$ since $G^* \cong S_0^* \oplus P_e^*$ is uncountable (see [10, Theorem 43.1]) and so is not isomorphic to $P^* \cong S$ which is countable (see [11, Lemma 94.1 and Proposition 94.2]).

(b) (Corner, [5]) There exists a group X such that $X \cong X \oplus X \oplus X$ but $X \not\cong X \oplus X$. Letting $G = X \oplus X$, we get that G $\text{sub}_{\sqsubset}X$ and X $\text{sub}_{\sqsubset}G$ but $X \not\cong G$.

(c) We will see later, in chapter V, that there exist weakly minimal groups which are not minimal.

Observe that the failure of anti-symmetry displayed above is not, in itself, a guarantee that there will be e.g. weakly quasi-minimal groups that are not quasi-minimal. Of course quasi-minimality (of any type) will always imply the corresponding weak concept.

The principal objective of this thesis is to investigate these concepts with a view to obtaining characterisations of the groups involved. A complete characterisation of quasi-minimal groups and, assuming the generalised continuum hypothesis, of purely quasi-minimal groups is obtained. An independence result is established for directly quasi-minimal groups. The problem of determining minimal and purely minimal groups is considerably more difficult but we obtain complete characterisations for some restricted classes of groups. It is perhaps worth noting that the concept we have described as "minimality" has been studied in the context of general, not necessarily abelian, groups where such groups are referred to as "hc-groups"; the terminology derives from considerations of connectedness of manifolds. Further details may be found in Robinson and Timm [24].

The notation used throughout the thesis is standard and follows that of [10] and [11] except that maps are written on the right.

I Preliminaries

In this first chapter we list the basic definitions and results from abelian group theory which are used in the thesis. All of these can be found in the standard textbooks of L. Fuchs ([10] and [11]).

§1 General Abelian Groups

All groups in this work are assumed to be abelian. If G is any abelian group the torsion subgroup, tG, of G is the subgroup of G consisting of elements of finite order. If $G = tG$ we say G is a *torsion* group and if $tG = 0$ we say G is *torsion-free.* G is a *mixed* group if $0 \neq tG \neq G$. We say a mixed group G *splits* if tG is a direct summand of G.

Definition 1.1 *The* torsion-free rank *of an arbitrary group G, $r_0(G)$, is the cardinality of any maximal linearly independent subset of G consisting of torsion-free elements.*

It is not difficult to see that $r_0(G) = r_0(G/tG)$, so we need only consider torsion-free groups when looking at the torsion-free rank and in this case we simply write $r(G)$ for $r_0(G)$.

The study of torsion groups is reduced to that of p-groups, p a prime, i.e. groups all of whose elements have order a power of p, by the following structure theorem.

Lemma 1.2 *Every torsion group G can be written in the form*

$G = \bigoplus_{p \in \Pi} G_p$ *where Π is the set of primes and G_p is a p-group. The groups G_p are called the primary components of G.*

Proof: See [10, Theorem 8.4.]. □

An important concept in the study of abelian groups is divisibility. If G is any group, then an element $g \in G$ is *divisible by $n \in \mathbb{Z}$ in G* if there exists some $g' \in G$ such that $g = ng'$. A group D is *divisible* if every element in D is divisible by n for all $n \in \mathbb{Z}$. If p is any prime then D is *p-divisible* if $p^n D = D$ for all $n \in \mathbb{N}$. It is easily seen that D is divisible if and only if D is p-divisible for all p. Every group has a maximal divisible subgroup and every group can be embedded in a minimal divisible group, called a *divisible hull* of the group.

We say a group C is *reduced* if C has no divisible subgroups. The structure theorem on divisible groups says that every divisible group G can be written as $G = \bigoplus_{p \in \Pi} \bigoplus_{I_p} \mathbb{Z}(p^\infty) \oplus \bigoplus_{I} \mathbb{Q}$ for some index sets I_p and I

(see [10, Theorem 23.1]).

Lemma 1.3 *Every group A is the direct sum of a divisible group D and a reduced group C where D is the maximal divisible subgroup of A and is uniquely determined.*

Proof: See [10, Theorem 21.3]. □

Another important concept is that of an exact sequence:

Definition 1.4 *A sequence of groups A_i and homomorphisms α_j*

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \ldots A_{k-1} \xrightarrow{\alpha_k} A_k \quad (k \geq 2)$$

is exact *if* $\mathrm{Im}(\alpha_j) = \mathrm{Ker}(\alpha_{j+1})$, *for* $j = 1, \ldots, k-1$.

In particular, an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \qquad (1)$$

is called a *short exact sequence*. Here α is monic, β is epic and $C \cong B/\mathrm{Im}(\alpha)$. In this case B is called an *extension* of A by C. We identify two such short exact sequences

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

and

$$0 \longrightarrow A \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C \longrightarrow 0$$

if there exists a homomorphism $\phi : B \longrightarrow B'$ such that $\alpha' = \alpha\phi$ and $\phi\beta' = \beta$. The 5-Lemma [10, Lemma 2.3] tells us that ϕ must, in fact, be an isomorphism. In this way we get an equivalence relation on these short exact sequences. The equivalence classes form a group $\text{Ext}(C, A)$, called the *group of extensions of A by C*. We refer to e.g. [10, IX] for more details.

The group $A \oplus C$ is called the *splitting extension* of A by C. The equivalence class of this splitting extension is the zero element in the group $\text{Ext}(C, A)$. We say the short exact sequence (1) *splits* if there exists a homomorphism $\gamma : C \longrightarrow B$ such that $\gamma\beta = I_C$ or, equivalently, there exists a homomorphism $\delta : B \longrightarrow A$ such that $\alpha\delta = I_A$. All such sequences are equivalent to the splitting extension since, in this case, $B = A\alpha \oplus C\gamma$. The following lemma is an important result from homological algebra involving the functors Hom and Ext.

Lemma 1.5 *(Cartan and Eilenberg) For the exact sequence (1) and an arbitrary group G the induced sequences*

$$0 \longrightarrow Hom(C, G) \longrightarrow Hom(B, G) \longrightarrow Hom(A, G) \longrightarrow$$

$$\longrightarrow Ext(C, G) \longrightarrow Ext(B, G) \longrightarrow Ext(A, G) \longrightarrow 0$$

and

$$0 \longrightarrow Hom(G, A) \longrightarrow Hom(G, B) \longrightarrow Hom(G, C) \longrightarrow$$

$$\longrightarrow Ext(G,A) \longrightarrow Ext(G,B) \longrightarrow Ext(G,C) \longrightarrow 0$$

with the usual connecting homomorphisms, are exact.

Proof: See [10, Theorem 51.3]. □

An alternative approach to divisibility is the idea of injectivity:

Definition 1.6 *A group D is* injective *if for every exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B$$

and every homomorphism $\phi : A \longrightarrow D$ there exists a homomorphism $\psi : B \longrightarrow D$ such that $\alpha\psi = \phi$.

Injective groups are easily characterised:

Lemma 1.7 *For any group D the following are equivalent:*

(i) *D is injective;*

(ii) *D is divisible;*

(iii) *D is a summand of every group containing D.*

Proof: See [10, Theorem 24.5]. □

We now come to another essential idea in the theory of abelian groups, namely the so-called pure subgroups. If G is any group we say a subgroup H of G is *pure* in G, written $H \leq_* G$, if $nH = nG \cap H$ for all $n \in \mathbb{Z}$, i.e. if $h = ng$ for some $h \in H, g \in G$ and $n \in \mathbb{Z}$ then there exists $h_1 \in H$ such that $h = nh_1$. If p is any prime, then H is *p-pure* in G if $p^n H = p^n G \cap H$ for all $n \in \mathbb{N}$. Again H is pure in G if and only if it is p-pure in G for all p. The main facts concerning pure subgroups used in this thesis are the following:

Lemma 1.8 *If B, C are subgroups of A such that $C \leq B \leq A$, then we have:*

(i) *If C is pure in B and B is pure in A, then C is pure in A;*

(ii) *If B is pure in A, then B/C is pure in A/C;*

(iii) *If C is pure in A and B/C is pure in A/C, then B is pure in A.*

Proof: See [10, Lemma 26.1]. □

Lemma 1.9 *A bounded pure subgroup is a direct summand where a group G is* bounded *if there exists $n \in \mathbb{N}$ such that $nG = 0$.*

Proof: See [10, Theorem 27.5]. □

Lemma 1.10 *If C is a pure subgroup of A such that A/C is a direct sum of cyclic groups, then C is a direct summand of A.*

Proof: See [10, Theorem 28.2]. □

Returning to short exact sequences we say the sequence (1) is *pure exact* if $A\alpha \leq_* B$. This leads to the following definition of a pure injective group.

Definition 1.11 *A group G is* pure injective *if it is injective with respect to all pure exact sequences, i.e. given a pure exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

and a homomorphism $\phi : A \longrightarrow G$ there exists a homomorphism $\psi : B \longrightarrow G$ such that $\alpha\psi = \phi$.

These groups can be characterised in different ways. First we need:

Definition 1.12 *A group G is* algebraically compact *if whenever S is a system of equations over G with coefficients in $\mathbb{Z}$ such that every finite subsystem of S has a solution in G, then S has a solution in G.*

Lemma 1.13 *For any group G the following are equivalent:*

(i) *G is pure injective;*

(ii) *G is algebraically compact;*

(iii) *G is a summand of every group that contains G as a pure subgroup.*

Proof: See [10, Theorem 38.1]. □

For a torsion-free group A an important notion is the *purification* of any subset S of A, written $\langle S\rangle_{*_A}$ or simply $\langle S\rangle_*$, if A is understood, where $\langle S\rangle_*$ is the smallest pure subgroup of A containing S, i.e. the intersection of all pure subgroups of A which contain S. It is easy to see that $\langle S\rangle_* = \{a \in A : na \in S \text{ for some } n \in \mathbb{Z}\}$.

Next we consider the idea of a p-basic subgroup of a group A.

Definition 1.14 *Let p be any prime. A subgroup B of a group A is called a* p-basic subgroup *of A if B satisfies the following conditions:*

(i) *B is a direct sum of cyclic p-groups and infinite cyclic groups;*

(ii) *B is p-pure in A;*

(iii) *A/B is p-divisible.*

Lemma 1.15 *Every group has p-basic subgroups, for every prime p. Moreover, for a given p, all p-basic subgroups of a group are isomorphic.*

Proof: See [10, Theorem 32.3] and [10, Theorem 35.2]. □

If B is a p-basic subgroup of A, then B can always be written in the form $B = B_0 \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots$ where $B_0 = \bigoplus_I \mathbb{Z}$ and $B_n = \bigoplus_{I_n} \mathbb{Z}(p^n)$ for $n \geq 1$ and some index sets I, I_n.

Lemma 1.16 *Let A be any group and B as above. Then, for any $n \geq 1$, $A = B_1 \oplus \cdots \oplus B_n \oplus (B_n^* + p^n A)$ where $B_n^* = B_0 \oplus \bigoplus_{i>n} B_i$.*

Proof: See [10, Theorem 32.4]. □

If G is any group we can define a linear topology on G by taking $\{nG : n \in \mathbb{N}\}$ as a fundamental system of neighbourhoods of 0. This topology is called the *$\mathbb{Z}$-adic topology* on G. If we take $\{p^n G : n \in \mathbb{N}\}$ as our fundamental system, where p is some prime, then we get the *p-adic topology* on G. We say that G is *complete* in its $\mathbb{Z}$-adic (or p-adic) topology if the topology is Hausdorff and if every Cauchy sequence in G has a limit in G. Every group G, that is Hausdorff in some linear topology, can be embedded as a pure dense subgroup of a complete group $\widehat{G}$, called a completion of G, and any two such completions are homeomorphically isomorphic.

The $\mathbb{Z}$-adic completion of $\mathbb{Z}$ is denoted by $\widehat{\mathbb{Z}}$ and, if p is any prime, the p-adic completion of $\mathbb{Z}$ is denoted by $\widehat{\mathbb{Z}}_p$, the ring of p-adic integers.

$\widehat{\mathbb{Z}}_p$ may be taken as the set of all elements of the form

$$x = b_0 + b_1 p + b_2 p^2 + \ldots + b_n p^n + \ldots$$

where $0 \leq b_n \leq p-1$ and $b_n \in \mathbb{Z}$ for all n. If $y \in \widehat{\mathbb{Z}}_p$ is given by $y = c_0 + c_1 p + c_2 p^2 + \ldots + c_n p^n + \ldots$ with $0 \leq c_n \leq p-1$ and $c_n \in \mathbb{Z}$ for all n then $x+y$ is given by $s_0 + s_1 p + s_2 p^2 + \ldots + s_n p^n + \ldots$ with $0 \leq s_n \leq p-1$ and $s_n \in \mathbb{Z}$ for all n where $s_0 = b_0 + c_0 - k_0 p$, $s_n = b_n + c_n + k_{n-1} - k_n p$ for all $n \geq 1$; the product xy is given by $q_0 + q_1 p + q_2 p^2 + \ldots + q_n p^n + \ldots$ with $0 \leq q_n \leq p-1$ and $q_n \in \mathbb{Z}$ for all n where $q_0 = b_0 c_0 - m_0 p$ and $q_n = b_0 c_n + b_1 c_{n-1} + \ldots + b_n c_0 + m_{n-1} - m_n p$ for $n \geq 1$; $k_0, \ldots, k_n, \ldots$, $m_0, \ldots, m_n \ldots$ are uniquely determined by the fact that b_n and c_n are between 0 and $p-1$ for all n.

The additive group of $\widehat{\mathbb{Z}}_p$ is denoted by J_p and is called the group of p-adic integers.

The next result characterises groups which are complete in their $\mathbb{Z}$-adic topologies.

Lemma 1.17 *A group G is complete in its $\mathbb{Z}$-adic topology if and only if G is reduced and algebraically compact.*

Proof: See [10, Theorem 39.1]. □

Finally, we have a characterisation for reduced algebraically compact groups in terms of p-adic complete groups.

Lemma 1.18 *A reduced group A is algebraically compact if and only if it is of the form $A = \prod_{p\in\Pi} A_p$ where each A_p is complete in its p-adic topology. In particular, $\widehat{\mathbb{Z}} \cong \prod_{p\in\Pi} J_p$.*

Proof: See [10, Proposition 40.1]. □

To finish this section we consider another important class of groups, namely the cotorsion groups. It is easily seen that a group G is torsion if and only if $\mathrm{Hom}(G, J) = 0$ for all torsion-free groups J. The definition of cotorsion groups is, in a sense, the dual of this.

Definition 1.19 *A group G is* cotorsion *if* $\mathrm{Ext}(J, G) = 0$ *for all torsion-free groups J. In other words G is a summand of every group that contains it with a torsion-free quotient.*

It is not difficult to show (see [10, p.232]) that for a group to be cotorsion it is sufficient that $\mathrm{Ext}(\mathbb{Q}, G) = 0$. The main results concerning cotorsion groups that we shall need are given in the following two lemmas.

Lemma 1.20 *A group is cotorsion if and only if it is an epimorphic* *image of an algebraically compact group. Especially, a torsion-free group is cotorsion if and only if it is algebraically compact.*

Proof: See [10, Proposition 54.1] and [10, Corollary 54.5]. □

Lemma 1.21 *Every group A can be embedded in a cotorsion group G, called its* cotorsion completion, *such that G/A is torsion-free and divisible. If A is reduced, then G can also be taken as reduced. In fact, the group G can be taken as* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, A)$.

Proof: See [10, Theorem 58.1]. □

§2 p-Groups

Let G be a p-group and let Ord be the class of all ordinals. We define a decreasing sequence of subgroups $p^\sigma G, \sigma \in Ord$, of G by

$$p^0 G = G,$$

$$p^{\sigma+1} G = p(p^\sigma G), \quad \text{if } \sigma + 1 \text{ is a successor ordinal,}$$

$$p^\sigma G = \bigcap_{\tau<\sigma} p^\tau G, \quad \text{if } \sigma \text{ is a limit ordinal.}$$

Obviously, there exists a least ordinal λ such that $p^\lambda G = p^{\lambda+1} G$. Then $p^\lambda G$ is the maximal divisible subgroup of G and so G is reduced if and only if $p^\lambda G = 0$. This λ is called the *length* of the group G.

Definition 2.1 *Let G be a p-group and $g \in G$. The* height *of g in G, written $h^G(g)$, is defined by*

$$h^G(g) = \sigma \textit{ if } g \in p^\sigma G \setminus p^{\sigma+1}G$$

and

$$h^G(g) = \infty \textit{ if } g \in \bigcap_{\sigma \in Ord} p^\sigma G.$$

Next we introduce the idea of the socle of a p-group. For any p-group G the *socle* of G, $G[p]$, is defined to be the subgroup of G consisting of those elements of order p, i.e. $G[p] = \{g \in G : pg = 0\}$. Obviously, we can consider $G[p]$ as a vector space over $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$. The socle plays a very important role in the theory of p-groups.

We now come to the definition of the Ulm invariants of a p-group G. The descending chain of subgroups of G, $\{p^\sigma G\}$, gives rise to a descending chain of $\mathbb{Z}(p)$-vector spaces, $\{p^\sigma G[p]\}$ where $p^\sigma G[p] = (p^\sigma G)[p]$. Hence, for each $\sigma \in Ord$, we have the vector space decomposition $p^\sigma G[p] = S_\sigma \oplus p^{\sigma+1}G[p]$ where S_σ consists of those elements in $G[p]$ of height σ.

Definition 2.2 *For each $\sigma \in Ord$ the σ^{th}* Ulm invariant *of G, $f_\sigma(G)$, is the dimension of S_σ as a vector space over $\mathbb{Z}(p)$, i.e.*

$$f_\sigma(G) = \dim_{\mathbb{Z}(p)}(p^\sigma G[p]/p^{\sigma+1}G[p]).$$

If two p-groups are isomorphic then their Ulm invariants are equal for all σ. A p-group G is said to be *totally projective* if $p^\sigma \mathrm{Ext}(G/p^\sigma G, C) = 0$ for all $\sigma \in Ord$ and all groups C. The Ulm invariants form a complete system of invariants for the totally projective p-groups.

Finally, in the case of a p-group G, a q-basic subgroup of G must be 0 for all $q \neq p$, so for p-groups we may, without fear of ambiguity, refer to their p-basic subgroups simply as basic subgroups.

§3 Torsion-Free Groups

In this final section we consider torsion-free groups.

A fundamental concept in the theory of torsion-free groups is the type of an element. In order to define type we must first consider the idea of height in a torsion-free group. To this end let A be any torsion-free group and $a \in A$. If p is any prime, then the *p-height* of a in A, denoted by $h_p^A(a)$ or simply $h_p(a)$, if A is understood, is the largest $k \in \mathbb{N}$ such that p^k divides a. If no such k exists we put $h_p(a) = \infty$.

If $\{p_1, p_2, \ldots\}$ is the set of primes in increasing magnitude then the sequence of p-heights $(h_{p_1}(a), h_{p_2}(a), \ldots)$ is called the *height-sequence* or *characteristic* of a in A, $\chi_A(a)$. Any sequence $(k_1, k_2, \ldots)$, where each k_i is either a non-negative integer or the symbol ∞, represents a characteristic, namely the characteristic of 1 in the subgroup R of $\mathbb{Q}$ generated by $\{p_i^{-l_i} : l_i \leq k_i, i \in \mathbb{N}\}$ where, if $k_i = \infty$, we use $\{p_i^{-l} : l \in \omega\}$.

The set of all characteristics can be partially ordered by defining $(k_1, k_2, \ldots) \geq (l_1, l_2, \ldots)$ if and only if $k_i \geq l_i$ for all $i \in \mathbb{N}$ and it becomes a lattice under the operations $(k_1, k_2, \ldots) \wedge (l_1, l_2, \ldots) = (min(k_1, l_1), min(k_2, l_2), \ldots)$ and $(k_1, k_2, \ldots) \vee (l_1, l_2, \ldots) = (max(k_1, l_1), max(k_2, l_2), \ldots)$, with minimum element $(0, 0, \ldots)$ and maximum element $(\infty, \infty, \ldots)$.

For every torsion-free group A the following facts can readily be proved:

(i) If $B \leq A$, then $\chi_B(b) \leq \chi_A(b)$ for all $b \in B$

(ii) If $B \leq_* A$, then $\chi_B(b) = \chi_A(b)$ for all $b \in B$

(iii) $\chi(b + c) \geq \chi(b) \wedge \chi(c)$ for all $b, c \in A$ where we have equality if $A = B \oplus C$ with $b \in B$ and $c \in C$

(iv) For any homomorphism $\phi : A \longrightarrow B$ between torsion-free groups A and B we have $\chi_A(a) \leq \chi_B(a\phi)$ for all $a \in A$.

We define two characteristics $(k_1, k_2, \ldots)$ and $(l_1, l_2, \ldots)$ to be equivalent if $|\{i \in \omega : k_i \neq l_i\}|$ is finite and for these i both k_i and l_i are finite. This is easily seen to be an equivalence relation on the set of characteristics and an equivalence class of characteristics is called a *type*. If $\chi_A(a)$

belongs to the type t then we write $t_A(a) = t$ or just $t(a) = t$, if A is understood, and we say that the type of a (in A) is t.

There is an induced partial order defined on the set of types. If $\chi_1 = (k_1, k_2, \ldots)$ and $\chi_2 = (l_1, l_2, \ldots)$ are any two characteristics in the types t and s, respectively, then we define $t \geq s$ if $k_i < l_i$ for only finitely many i and, for these i, both k_i and l_i are finite. Again we get a lattice where $t \wedge s$ and $t \vee s$ are the types given by the characteristics $\chi_1 \wedge \chi_2$ and $\chi_1 \vee \chi_2$ respectively. It is now straightforward to check that the properties (i), (ii), (iii), and (iv) above also hold for types.

If A is any torsion-free group, then the *typeset* of A, $T(A)$, is the set of types of elements of A, i.e. $T(A) = \{t : t = t(a) \text{ for some } a \in A\}$. If t is any type we can define two fully invariant subgroups of A, $A(t)$ and $A^*(t)$, by $A(t) = \{a \in A : t(a) \geq t\}$ and $A^*(t) = \langle a \in A : t(a) > t\rangle$. It is easily seen that $A^*(t) \leq A(t)$ and that $A(t)$ is pure in A.

Definition 3.1 *A torsion-free group A is* homogeneous *if all of its elements have the same type. In this case we can meaningfully speak about the type of the group.*

A rank 1 torsion-free group is obviously homogeneous and, in fact, rank 1 torsion-free groups are characterised by their types.

Lemma 3.2 *Two rank 1 torsion-free groups are isomorphic if and only if they have the same type. Every type is realised by some rational group, i.e. a subgroup of* $\mathbb{Q}$.

Proof: See [11, Theorem 85.1]. □

A very important class of torsion-free groups is the class of *completely decomposable* groups where a group is completely decomposable if it is a direct sum of rank 1 groups. Trivial examples of such groups are free groups and torsion-free divisible groups. Completely decomposable groups can be characterised by the number of summands of each type that occur in the decomposition.

Lemma 3.3 *Any two direct decompositions of a completely decomposable group into direct sums of rank 1 groups are isomorphic and the number of rank 1 summands of any type* t *in such a decomposition is given by the rank of* $A_t = A(t)/A^*(t)$.

In particular, the ranks $r(A_t)$ *taken for all types* t *form a complete and independent system of invariants for completely decomposable groups.*

Proof: See [11, Proposition 86.1]. □

Another very useful result on completely decomposable groups is Baer's Lemma:

Lemma 3.4 *(Baer'sLemma)* *If C is a pure subgroup of the torsion-free group A such that*

(a) *A/C is completely decomposable and homogeneous of type t,*

(b) *all the elements in $A \setminus C$ are of type t,*

then C is a summand of A.

Proof: See [11, Theorem 86.5]. □

The next result gives conditions for a subgroup of a homogeneous completely decomposable group to be completely decomposable.

Lemma 3.5 *If A is a homogeneous completely decomposable group of type t, then every subgroup C of A which is homogeneous of type t (in particular, every pure subgroup of A) is completely decomposable.*

Proof: See [11, Theorem 86.6].

Finally, we have a fundamental result on completely decomposable groups.

Lemma 3.6 *A direct summand of a completely decomposable group is again completely decomposable.*

Proof: See [11, Theorem 86.7]. □

A more general concept than a completely decomposable group is that of a separable group where we say a torsion-free group A is *separable* if every finite subset of A is contained in a completely decomposable direct summand of A. Completely decomposable groups are obviously separable and in the countable case the converse is true, as the following result shows.

Lemma 3.7 *Every countable torsion-free separable group is completely decomposable.*

Proof: See [11, Theorem 87.1]. □

In the case of homogeneous groups we have a characterisation of separable groups.

Lemma 3.8 *A homogeneous group A is separable if and only if every finite rank pure subgroup of A is a summand of A.*

Proof See [11, Proposition 87.2]. □

Finally we have a corresponding result to Lemma 3.6 for separable groups.

Lemma 3.9 *A direct summand of a separable group is separable.*

Proof: See [11, Theorem 87.5]. □

II Quasi-Minimal Groups

Our study of groups with minimal properties begins with the idea of quasi-minimality and its associated concepts of weak and pure quasi-minimality. To this end let κ be an infinite cardinal and let Ab_κ be the set of all abelian groups of cardinality κ. We first consider the quasi-minimal groups in Ab_κ.

§1 Quasi-Minimal Groups

Definition 1.1 *A group $G \in Ab_\kappa$ is* quasi-minimal *if G is isomorphic to all its subgroups of cardinality κ.*

In the following we achieve a complete characterisation of quasi-minimal groups in Ab_κ.

Lemma 1.2 *If G is quasi-minimal, then G is either torsion-free or a p-group.*

Proof: If $|tG| = |G|$, then $G \cong tG$, since G is quasi-minimal. Now, $tG = \bigoplus_{p\in\Pi} G_p$ where G_p is a p-group for all p. Choose p such that $G_p \neq 0$. Then $tG = G_p \oplus \bigoplus_{q\neq p} G_q$, so $|G| = |G_p|$ or $|G| = |\bigoplus_{q\neq p} G_q|$. If $|G| = |\bigoplus_{q\neq p} G_q|$, then $G \cong \bigoplus_{q\neq p} G_q$, a contradiction, since $G_p \neq 0$. Therefore $|G| = |G_p|$

and so $G \cong G_p$. Now suppose $|tG| < |G|$. Then $|G/tG| = |G| = \kappa$. If $\kappa = \aleph_0$, then tG is finite and so $tG \sqsubset G$, by I, Lemma 1.9, since tG is pure in G. Then $G = tG \oplus C$ where C is torsion-free and $|G| = |C| = \aleph_0$, and so $G \cong C$ (and $tG = 0$), i.e. G is torsion-free. If $\kappa > \aleph_0$ then $r(G/tG) = |G/tG|$, so we can choose κ linearly independent elements $\overline{g}_\alpha = g_\alpha + tG, \alpha < \kappa$. Let $C = \langle g_\alpha : \alpha < \kappa \rangle \leq G$. C is torsion-free since, if $c = \sum_{\alpha<\kappa} k_\alpha g_\alpha \in tG$ where $k_\alpha = 0$ for almost all α, then $\sum_{\alpha<\kappa} k_\alpha g_\alpha + tG = 0$ in G/tG, so $\sum_{\alpha<\kappa} k_\alpha \overline{g}_\alpha = 0$ and hence $k_\alpha = 0$ for all α, i.e. $c = 0$. Now $|G| = |C|$ and so $G \cong C$, i.e. G is torsion-free. □

Lemma 1.3 *If G is quasi-minimal and torsion-free, then*

(i) *for $\aleph_0 = \kappa = |G|$, $G \cong \mathbb{Z}$, and*

(ii) *for $\aleph_0 < \kappa = |G|$, $G \cong \bigoplus_\kappa \mathbb{Z}$.*

Proof: First $\mathbb{Z}$ is quasi-minimal since the only non-zero subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z} \cong \mathbb{Z}$ where $n \in \mathbb{N}$. Also $\bigoplus_\kappa \mathbb{Z}$ is quasi-minimal for $\kappa > \aleph_0$ since if $H \leq \bigoplus_\kappa \mathbb{Z}$ $(\kappa > \aleph_0)$ with $|H| = \kappa$ then H is free of rank κ and so $H \cong G$. It remains to show that these are the only torsion-free quasi-minimal groups.

(i) Let $0 \neq g \in G$ and consider $\langle g \rangle$. Since G is torsion-free $|\langle g \rangle| = \aleph_0$ and so $G \cong \langle g \rangle \cong \mathbb{Z}$.

(ii) In this case $r(G) = \kappa$ so G has κ linearly independent elements $\{g_\alpha : \alpha < \kappa\}$. Now $\langle g_\alpha : \alpha < \kappa \rangle = \bigoplus_{\alpha<\kappa} \langle g_\alpha \rangle$ and $| \bigoplus_{\alpha<\kappa} \langle g_\alpha \rangle | = \kappa$. Therefore $G \cong \bigoplus_{\alpha<\kappa} \langle g_\alpha \rangle \cong \bigoplus_{\kappa} \mathbb{Z}$. □

Lemma 1.4 *If G is quasi-minimal, then G is either divisible or reduced.*

Proof: Let G be any quasi-minimal group. By I, Lemma 1.3, we may write G as $G = D \oplus R$ where D is divisible and R is reduced. We clearly have $|G| = |D|$ or $|G| = |R|$. If $|G| = |D|$, then $G \cong D$ and G is divisible. If $|D| < |G|$, then $|G| = |R|$ and so $G \cong R$ and G is reduced. □

Lemma 1.5 *If G is quasi-minimal and divisible, then $G \cong \mathbb{Z}(p^\infty)$ for some prime p.*

Proof: The group $\mathbb{Z}(p^\infty)$ is obviously quasi-minimal since its only subgroup of cardinality $\aleph_0$ is itself. So let G be a divisible quasi-minimal group. Then G is either torsion-free or a p-group, by Lemma 1.2. The group G is not torsion-free, by Lemma 1.3. Therefore G is a divisible p-group, i.e. $G \cong \bigoplus_I \mathbb{Z}(p^\infty)$, for some index set I and for some prime p, by [10, Theorem 23.1]. If $|I| \geq 2$, then G contains a subgroup $H \cong \mathbb{Z}(p) \oplus \bigoplus_J \mathbb{Z}(p^\infty)$ where $I = J \cup \{i_0\}$, say. Now $|H| = |G| \Rightarrow G \cong H$, a contradiction, since H is not divisible. Therefore $|I| = 1$ and $G \cong \mathbb{Z}(p^\infty)$.

□

Lemma 1.6 *If G is a reduced quasi-minimal p-group, then $G \cong \bigoplus_{\kappa} \mathbb{Z}(p)$ where $|G| = \kappa$.*

Proof: First, $\bigoplus_{\kappa} \mathbb{Z}(p)$ is quasi-minimal since any subgroup is of the form $\bigoplus_{\lambda} \mathbb{Z}(p)$ and if the cardinalities are equal then $\lambda = \kappa$. Let G be a reduced quasi-minimal p-group of cardinality $\kappa \geq \aleph_0$. We claim that $|G| = |G[p]|$ where $G[p]$ is the socle of G.

Let B be a basic subgroup of $G, B = \bigoplus_{n \in \mathbb{N}} \bigoplus_{i \in I_n} \langle x_i \rangle$ with $o(x_i) = p^n$ for $i \in I_n$ where $I_n = \emptyset$ is allowed.

First suppose $|G| > \aleph_0$. If $|B| = |G|$, then some I_n has cardinality κ and so $|B[p]| = | \bigoplus_{n \in \mathbb{N}} \bigoplus_{I_n} \langle p^{n-1} x_i \rangle | = \kappa$ and hence $|G[p]| \geq |B[p]| = \kappa = |G|$. If $|B| < |G|$, then $|G/B| = |G|$ and G/B is divisible, so $G/B = \bigoplus_{j \in J} \mathbb{Z}(p^\infty)$ where $|J| = \kappa$. There exists $C \leq G$, containing B, such that $C/B = \bigoplus_{j \in J} \mathbb{Z}(p)$. Then $C \cong B \oplus \bigoplus_{j \in J} \mathbb{Z}(p)$, by I, Lemma 1.10, since B is pure in C, and so $C[p] = B[p] \oplus \bigoplus_{j \in J} \mathbb{Z}(p)$. Therefore $|G| = |G/B| = \kappa = |C[p]| \leq |G[p]|$. So, in both cases, $|G[p]| = |G|$.

If $\kappa = \aleph_0$, then $|B| = |G|$, since if B is finite then $B \sqsubset G$, $G = B \oplus D$, say, with $D \cong G/B$, divisible. Now, G is reduced, so $D = 0$ and hence G is finite, a contradiction. Again we conclude $|G[p]| = \aleph_0 = |G|$.

We have that $|G[p]| = |G|$ in all cases and, since G is quasi-minimal, we get $G \cong G[p] \cong \bigoplus_{\kappa} \mathbb{Z}(p)$. □

We summarise what we have proved about quasi-minimal groups in the following theorem:

Theorem 1.7 *If G is a quasi-minimal group of cardinality κ, then*

(i) $\kappa = \aleph_0$ $\quad G = \mathbb{Z}, \quad \mathbb{Z}(p^\infty)$ *or* $\bigoplus_{\aleph_0} \mathbb{Z}(p)$;

(ii) $\kappa > \aleph_0$ $\quad G = \bigoplus_{\kappa} \mathbb{Z}$ *or* $\bigoplus_{\kappa} \mathbb{Z}(p)$.

§2 Weakly Quasi-Minimal Groups

The condition on a group G of cardinality κ necessary for quasi-minimality may be relaxed in various different ways. One way is to require that G be isomorphic only to a subgroup of each of its subgroups of cardinality κ. This idea leads us to the concept of weak quasi-minimality. Another way is to insist that G must be isomorphic only to its pure subgroups of cardinality κ. In this case G is said to be purely quasi-minimal. In this section we deal with the weakly quasi-minimal groups and in the next section we consider the purely quasi-minimal case.

Definition 2.1 *A group $G \in Ab_\kappa$ is* weakly quasi-minimal *if, whenever $H \leq G$ with $|H| = \kappa$, then H contains a subgroup which is isomorphic* *to* G.

By definition, every quasi-minimal group is weakly quasi-minimal. We wish to investigate if there are weakly quasi-minimal groups which are not quasi-minimal.

Since a subgroup of a p-group is a p-group and a subgroup of a torsion-free group is a torsion-free group, Lemma 1.2 is true for weakly quasi-minimal groups, with a similar proof. Also Lemma 1.3 holds for weakly quasi-minimal groups since a subgroup of a free group is free and the rank of a free group is equal to its cardinality if the cardinality is uncountable. However Lemma 1.4 need not be true because a subgroup of a divisible group is not necessarily divisible, but we do have the following lemma.

Lemma 2.2 *If* G *is a weakly quasi-minimal p-group, then* $G \cong \mathbb{Z}(p^\infty)$ *or* $G \cong \bigoplus_{\kappa} \mathbb{Z}(p)$ *where* $|G| = \kappa$.

Proof: Let B be a basic subgroup of G, possibly $B = 0$.

If $|G| = \kappa > \aleph_0$, then, as in Lemma 1.6, $|G[p]| = |G|$ and so $G \cong \bigoplus_{\lambda} \mathbb{Z}(p)$, for some λ, where $|\bigoplus_{\lambda} \mathbb{Z}(p)| = \kappa$, and so $\lambda = \kappa$.

If $|G| = \aleph_0$ and $|B| = \aleph_0$, then, again as in Lemma 1.6, $G \cong \bigoplus_{\aleph_0} \mathbb{Z}(p)$.

Finally, in this case, if B is finite, then $B \sqsubset G$, i.e. $G = B \oplus D$, where

$D \cong G/B$ and so is divisible. Now, $D = \bigoplus_I \mathbb{Z}(p^\infty)$ for some index set I, so $\mathbb{Z}(p^\infty) \leq G$ and, since G is weakly quasi-minimal and $\mathbb{Z}(p^\infty)$ has no proper countable subgroups, we get $G \cong \mathbb{Z}(p^\infty)$. □

So, in fact, we get nothing new in the weakly quasi-minimal case since the class of weakly quasi-minimal groups is the same as the class of quasi-minimal groups.

§3 Purely Quasi-Minimal Groups

Next we consider the purely quasi-minimal groups in Ab_κ.

Definition 3.1 *A group $G \in Ab_\kappa$ is* purely quasi-minimal *if G is isomorphic to all its pure subgroups of cardinality κ.*

Lemma 3.2 *If G is purely quasi-minimal, then G is either divisible or reduced.*

Proof: The arguments are the same as in Lemma 1.4, using the fact that direct summands are pure. □

Lemma 3.3 *For a divisible purely quasi-minimal group G one of the following is true*

(i) $G \cong \mathbb{Q} \quad (\kappa = \aleph_0)$,

(ii) $G \cong \mathbb{Z}(p^\infty)$, *for some* $p \quad (\kappa = \aleph_0)$,

(iii) $G \cong \bigoplus_\kappa \mathbb{Q} \quad (\kappa > \aleph_0)$,

(iv) $G \cong \bigoplus_\kappa \mathbb{Z}(p^\infty)$, *for some* $p \quad (\kappa > \aleph_0)$.

Proof: First note that $\mathbb{Q}, \mathbb{Z}(p^\infty), \bigoplus_\kappa Q, \bigoplus_\kappa \mathbb{Z}(p^\infty)$ are indeed purely quasi-minimal since a pure subgroup of a divisible group is again divisible.

Let G be divisible and purely quasi-minimal. Then $G = \bigoplus_I \mathbb{Q} \oplus \bigoplus_{p \in \Pi} \bigoplus_{I_p} \mathbb{Z}(p^\infty)$, for some index sets I, I_p. If $|G| = \aleph_0$, then G must be isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$, for some p. Now suppose $|G| = \kappa > \aleph_0$. Then, if $|I| = \kappa$, we have $G \cong \bigoplus_\kappa \mathbb{Q}$, whereas if $|I| < \kappa$, then some $|I_p| = \kappa$ and we get $G \cong \bigoplus_{I_p} \mathbb{Z}(p^\infty)$. □

Theorem 3.4 *If G is a reduced purely quasi-minimal group, then G is either a homocyclic p-group, i.e. $G = \bigoplus_I \mathbb{Z}(p^n)$ for some index set I and some $n \in \mathbb{N}$, or G is torsion-free.*

Proof: First we show that a homocyclic group is purely quasi-minimal. Let $G = \bigoplus_{i<\kappa} y_i\mathbb{Z}(p^n)$, for some n and some infinite cardinal κ, and let $H \leq_* G$ with $|H| = |G|$. Now, H is again a direct sum of cyclics, by [10, Theorem 18.1], $H = \bigoplus_I \langle x_i \rangle$, say, with $|I| = \kappa$ since $|H| = \kappa$. Suppose $o(x_i) = p^m$ for some $m < n$ and some $i \in I$. We have $x_i = \sum_{j<\kappa} k_j y_j$ where

$k_j = 0$ for almost all j. Since $p^m x_i = 0$ it follows that $0 = \sum_{j<\kappa} p^m k_j y_j$ and so $p^m k_j y_j = 0$ for all j. Therefore p^n divides $p^m k_j$ for all j, so p divides k_j for all j and hence $x_i = p \sum_{j<\kappa} r_j y_j$ where $k_j = pr_j$ for all j. We get that p divides x_i in G and so p divides x_i in H since H is pure in G, a contradiction, since x_i is a generator of $\langle x_i \rangle$ in H. We conclude that $o\langle x_i \rangle = p^n$ for all $i \in I$ and so $H \cong G$.

Now let G be a reduced purely quasi-minimal group. If $tG = 0$, then G is torsion-free, so suppose $tG \neq 0$. Choose p such that $G_p \neq 0$ and let B be a p-basic subgroup of G. $B \neq 0$ since $0 \neq G_p$ which is not divisible and so G is not p-divisible. I will show that the group B cannot be torsion-free. Since G_p is not divisible there exists some $x \in G_p[p]$ with finite p-height (see [10, 20(c)]), and since G/B is p-divisible we have $x = p^n g_n + b_n$, for each n, where $g_n \in G$ and $b_n \in B$. Now $px = 0$ implies $p^{n+1} g_n + pb_n = 0$, so $p^{n+1} g_n = -pb_n \in p^{n+1} G \cap B = p^{n+1} B$ and hence $pb_n = p^{n+1} b'_n$ where $b'_n \in B$. If B is torsion-free then we get $b_n = p^n b'_n$ and so $x = p^n(g_n + b'_n)$, i.e. p^n divides x for all n, a contradiction.

Therefore $B = B_0 \oplus \bigoplus_{n=1}^{\infty} B_n$ where B_0 is free and not all $B_n = 0$. Let $k > 0$ be the smallest integer with $B_k \neq 0$. Then $G = B_k \oplus (B^* + p^k G)$ where $B^* = B_0 \oplus \bigoplus_{n>k} B_n$, by I, Lemma 1.16. If $|G| = |B_k|$ then $G \cong B_k$ and so $G \cong \bigoplus_{\kappa} \mathbb{Z}(p^k)$ and hence G is homocyclic. Otherwise $|G| = |B^* + p^k G|$

and hence $G \cong B^* + p^k G = H$, say.

We claim that B^* is a p-basic subgroup of H.

By definition, B^* is a direct sum of cycles. Also B^* is pure in B which is pure in G and so B^* is pure in G and hence in H. Finally $B^* = B \cap H$ since if $b \in B \cap H$ then $b = b^* + p^k g$ where $b^* \in B^*$ and $g \in G$, and so $b - b^* \in p^k G \cap B = p^k B \leq B^*$, which means that $B \cap H \leq B^*$. The converse inclusion is obvious. Therefore $H/B^* = H/(B \cap H) \cong (H + B)/B = G/B$ which is p-divisible.

Since $H \cong G$ we get $B^* \cong B$, a contradiction, since $B_k \neq 0$.

Hence $|G| = |B_k|$ and G is a homocyclic group. □

It remains to consider the reduced torsion-free case. Before the characterisation can be established we need some general results on reduced torsion-free groups. First we give a definition.

Definition 3.5 *Let A be a subgroup of a torsion-free group G. A subgroup K of G is an A-*high subgroup *of G if $A \cap K = 0$ and if $K' \supseteq K$ such that $A \cap K' = 0$, then $K' = K$, i.e. K is maximal with respect to the property $A \cap K = 0$.*

Lemma 3.6 *Let A be a subgroup of a torsion-free group G and let K be* *an A-high subgroup of G. Then $K \leq_* G$ and $G/(A \oplus K)$ is torsion.*

Proof: Suppose that $mg \in K$ for some $g \in G \setminus K$ and some $m \in \mathbb{Z}$. Then $\langle K, g\rangle \cap A \neq 0$, so there exists some non-zero $c \in A$ with $c = k + ng$ for some $k \in K$ and $n \in \mathbb{Z}$. Therefore $mc = mk + mng \in A \cap K = 0$ and so $c = 0$, since A is torsion-free, a contradiction. Hence $mg \notin K$ if $g \notin K$ and so $K \leq_* G$.

For the second part consider $g \in G \setminus A \oplus K$. Then $g \in G \setminus K$ and hence $\langle K, g\rangle \cap A \neq 0$. Therefore $-k + ng = c$ for some $k \in K, c \in A$ and $n \in \mathbb{Z}$. We have $ng = c + k$ and so $n(g + A \oplus K) = 0$ in $G/(A \oplus K)$ and hence $G/(A \oplus K)$ is torsion. □

Lemma 3.7 *If G is a reduced torsion-free purely quasi-minimal group, then G is t-homogeneous for some type t.*

Proof: Let $t \in T(G)$, the typeset of G. Then there exists some $g \neq 0$ in G such that $t(g) = t$. Consider $G(t) = \{g \in G : t(g) \geq t\} \neq 0$. $G(t) \leq_* G$ and we claim that $|G(t)| = |G|$.

Let K be a $G(t)$-high subgroup of G. Lemma 3.6 now tells us that $K \leq_* G$ and $G/(K \oplus G(t))$ is torsion. Therefore $G = (K \oplus G(t))_*$ and hence $|G| = |K \oplus G(t)| \cdot \aleph_0 = |K| \cdot |G(t)| \cdot \aleph_0 = |K|$ or $|G(t)|$. If $|G| = |K|$, then $G \cong K$, since $K \leq_* G$, and we get $G(t) \cong K(t) \leq G(t) \cap K = 0$, a contradiction, since $G(t) \neq 0$. Therefore $|G| = |G(t)|$ and hence $G \cong G(t)$ whenever $G(t) \neq 0$. Now, if $t, s \in T(G)$, then there exist $a, b \in G$ such that $t(a) = t$ and $t(b) = s$. Since $G \cong G(t)$ and $G \cong G(s)$ we get $G(t) \overset{\phi}{\cong} G(s)$. Now $t(a) = t(a\phi)$ and $t(b) = t(b\phi^{-1})$ imply that $t = s$ and therefore G is t-homogeneous. □

Definition 3.8 *A linearly independent subset S of a torsion-free group G is* quasi-pure independent *if $\bigoplus_{x \in S} \langle x \rangle_*$ is a pure subgroup of G where $\langle x \rangle_* = \langle x \rangle$ whenever $\langle x \rangle_*$ is cyclic.*

Note that every torsion-free group has quasi-pure independent subsets and Zorn's Lemma implies that any quasi-pure independent set is contained in a maximal one.

Next we state some results concerning quasi-pure independent subsets of a torsion-free group G.

Lemma 3.9 *Let G be any torsion-free group. Then*

(i) *If T, S are two infinite maximal quasi-pure independent sets of G, then $|T| = |S|$;*

(ii) *If S a maximal quasi-pure independent subset of G, then $|G| \leq (|S| + 1)^{\aleph_0}$.*

Proof: See [13, Corollary 125 and Theorem 126]. □

Note that If $|G| > 2^{\aleph_0}$ then it can easily be deduced from Lemma 3.9 that any two maximal quasi-pure independent subsets of G have the same cardinality.

Definition 3.10 *A subgroup H of a torsion-free group G is* pure essential *in G if $H \leq_* G$ and if $A \leq G$ with $A \cap H = 0$ and $A \oplus H \leq_* G$, then $A = 0$, in other words, $G/(A \oplus H)$ is not torsion-free for any such non-zero $A \leq G$.*

Theorem 3.11 *(See [13, Theorem 129]) Every torsion-free group G has a completely decomposable pure essential subgroup C such that $|G| \leq |C|^{\aleph_0}$.*

Proof: Let S be a maximal quasi-pure independent subset of G and let $C = \bigoplus_{x \in S} \langle x \rangle_*$. Then Lemma 3.9 implies that $|G| \leq (|S|+1)^{\aleph_0} = |C|^{\aleph_0}$. C is obviously completely decomposable so it remains to show that C is pure essential in G. First of all $C \leq_* G$. If $0 \neq A \leq G$ such that $A \cap C = 0$ and $C \oplus A \leq_* G$, then choose some $a \neq 0$ in A and consider $\langle a \rangle_*$ (purification in A). We have $C \oplus \langle a \rangle_* \leq_* C \oplus A \leq_* G$. Also $\langle a \rangle_* \leq_* A \leq_* C \oplus A \leq_* G$, so the purification of $\langle a \rangle$ in G is the same as its purification in A. Therefore $S \cup \{a\}$ is again a quasi-pure independent subset of G, which contradicts the maximality of S. Thus we can conclude that C is pure essential in G. $\square$

We are now ready to establish the characterisation of reduced torsion-free purely quasi-minimal groups.

Theorem 3.12 (GCH) *If $G \in Ab_\kappa$ is a torsion-free reduced purely quasi-minimal group, then either $G \cong R$ $(\kappa = \aleph_0)$, or $G \cong \bigoplus_\kappa R$ $(\kappa > \aleph_0)$, for some rank 1 group R.*

Obviously, we do not need GCH *in the countable case.*

Proof: First we show that R and $\bigoplus_\kappa R$ are purely quasi-minimal. If $0 \neq H \leq_* R$, then R/H is torsion-free. Now, if there exists $r \in R \setminus H$, then, for any $h \in H$, we can find non-zero integers m and n such that $mh + nr = 0$ and so $n(r + H) = 0$, a contradiction. Hence the only non-zero pure subgroup of R is R itself and so R is purely quasi-minimal. If $H \leq_* \bigoplus_\kappa R$, then H is also homogeneous completely decomposable of the same type as R, by I, Lemma 3.5. Therefore $H = \bigoplus_I R$ and if $|H| = \kappa > \aleph_0$ then $|I| = \kappa$ and so $H \cong \bigoplus_\kappa R$.

Now let G be a torsion-free reduced purely quasi-minimal group. Lemma

3.7 tells us that G is t-homogeneous for some type t. By Theorem 3.11 there exists a pure essential completely decomposable subgroup C of G such that $|G| \leq |C|^{\aleph_0}$. Let $C = \bigoplus_I R$, where R is a rank 1 group whose type must be t since $R \leq_* C \leq_* G$.

If $|G| = \aleph_0$, then $|R| = |C| = |G| = \aleph_0$ and hence $G \cong R$. So consider $|G| = \kappa > \aleph_0$. If $|C| = |G| = \kappa$, then $G \cong C$ and we are finished. We wish to prove that $|C| < |G|$ is impossible. Let us assume that $|C| < |G|$ to obtain a contradiction. First note that $|C| < |G|$ implies $2^{|C|} \leq |G|$, assuming GCH, and $|G| \leq |C|^{\aleph_0} \leq (2^{|C|})^{\aleph_0} = 2^{|C|}$, so $|G| = 2^{|C|}$. Now consider the short exact sequence $0 \longrightarrow C \xrightarrow{i} G \xrightarrow{\pi} G/C \longrightarrow 0$ where i is inclusion and π is canonical projection. The induced sequence $0 \longrightarrow \mathrm{Hom}(G/C, G) \longrightarrow \mathrm{Hom}(G, G) \longrightarrow \mathrm{Hom}(C, G)$ is exact, by I, Lemma 1.5. We claim that $\mathrm{Hom}(G/C, G) = 0$.

Let $g + C \in G/C$. Then $t(g + C) \geq t(g) = t$ since homomorphisms do not decrease types. If $t(g + C) = t$, then $R \cong \langle g + C \rangle_*$, by I, Lemma 3.2. Let $\langle g + C \rangle_* = B/C \leq_* G/C$. Then $C \leq_* B$ and B/C is homogeneous completely decomposable of type t and every element of $B \setminus C$ is of type t since $B \leq_* G$, so $C \sqsubset B$, by I, Lemma 3.4. Therefore $B = C \oplus R_1$ where $R_1 \cong R$, and $B \leq_* G$, but this contradicts the fact that C is pure essential in G. We conclude that $t(g + C) > t$ for all $g \in G$. Hence $\mathrm{Hom}(G/C, G) = 0$, again since homomorphisms do not decrease types. Therefore $0 \longrightarrow \mathrm{Hom}(G,G) \longrightarrow \mathrm{Hom}(C,G)$ is exact, so $\mathrm{Hom}(G,G)$ is isomorphic to a subgroup of $\mathrm{Hom}(C,G)$ and hence $|\mathrm{Hom}(G,G)| \leq |\mathrm{Hom}(C,G)| \leq |G|^{|C|} = (2^{|C|})^{|C|} = 2^{|C|} = |G|$. But $|G| > \aleph_0$ means that $r(G) = |G|$ and so there exists a maximal linearly independent set in G of cardinality κ. Then this set contains 2^{κ} different linearly independent subsets of G of cardinality κ (see [15, p.43]). Each of these subsets S generates a pure subgroup $\langle S \rangle_*$ of G. Furthermore, if $S_1 \neq S_2$, then $\langle S_1 \rangle_* \neq \langle S_2 \rangle_*$ since otherwise, for any $s \in S_1 \setminus S_2$, we have $s \in \langle S_2 \rangle_*$ and so there exist non-zero integers $n, n_1, \ldots, n_k$, for some k, such that $ns = n_1 x_1 + \ldots + n_k x_k$ with $x_1, \ldots, x_k \in S_2$; but this contradicts the fact that $S_1 \cup S_2$ is contained in a maximal linearly independent subset of G.

Now, if K_1 and K_2 are two such pure subgroups of G, then $G \cong K_1$ and $G \cong K_2$, since G is purely quasi-minimal. If $\phi_1 : G \longrightarrow K_1$ and $\phi_2 : G \longrightarrow K_2$ are isomorphisms, then $K_1 \neq K_2$ implies $\phi_1 \neq \phi_2$ and thus there exist at least 2^{κ} different endomorphisms of G. Therefore $2^{|G|} \leq |\mathrm{End}(G)| \leq |G|$ which is obviously a contradiction. Hence we can deduce that $|C| < |G|$ is impossible and so $|C| = |G|$ and G is homogeneous completely decomposable. $\square$

Note that, in Theorem 3.12, if $|G| \leq \aleph_\omega$ then it is enough to assume the continuum hypothesis (CH), i.e. $2^{\aleph_0} = \aleph_1$, as the following argument shows. The Hausdorff Formula (see [15, p.48]) tells us that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$ for all $\alpha, \beta \in Ord$ with $\beta < \alpha + 1$; so $\aleph_1^{\aleph_0} = \aleph_1 \cdot \aleph_0^{\aleph_0} = \aleph_1 \cdot 2^{\aleph_0} = \aleph_1$, using CH, and a simple induction argument now gives $\aleph_n^{\aleph_0} = \aleph_n$ for all $n > 1$. So, in Theorem 3.12, if:

(i) $|G| = \aleph_1$ and $|C| = \aleph_0$, then CH gives us that $|G| = 2^{|C|}$ and we get a contradiction as in Theorem 3.12;

(ii) $|G| = \aleph_\alpha$ where $1 < \alpha \leq \omega$ and $|C| = \aleph_n, n < \alpha$, then $|C|^{\aleph_0} = \aleph_n^{\aleph_0} = \aleph_n$ (CH) $< \aleph_\alpha = |G|$, a contradiction to $|G| \leq |C|^{\aleph_0}$.

I have not been able to establish whether Theorem 3.12 is true or not in ZFC; it may indeed be undecidable in ZFC.

As in the quasi-minimal case, we summarise what we have established concerning the purely quasi-minimal groups in the following theorem.

Theorem 3.13 *If $G \in Ab_\kappa$ is a purely quasi-minimal group, then*

(i) $\kappa = \aleph_0$ $\quad G = R, \quad \mathbb{Z}(p^\infty), \quad$ *or* $\bigoplus_{\aleph_0} \mathbb{Z}(p^k)$;

(ii) $\kappa > \aleph_0$ $\quad G = \bigoplus_\kappa R$ *(GCH),* $\quad \bigoplus_\kappa \mathbb{Z}(p^\infty), \quad$ *or* $\bigoplus_\kappa \mathbb{Z}(p^k)$,

where R is a rank 1 group, p is any prime and k is any positive integer.

§4 Weakly Purely Quasi-Minimal Groups

If we now relax the quasi-minimal condition in both ways, i.e. weakly and purely, we can define the concept of a weakly purely quasi-minimal group.

Definition 4.1 *A group $G \in Ab_\kappa$ is* weakly purely quasi-minimal *if whenever $H \in Ab_\kappa$ is a pure subgroup of G, then G is isomorphic to a pure subgroup of H.*

Any purely quasi-minimal group is obviously weakly purely quasi-minimal. As in section 2 we wish to investigate if there are other weakly purely quasi-minimal groups. Since a subgroup of a reduced group is obviously reduced and a pure subgroup of a divisible group is divisible Lemma 3.2 and Lemma 3.3 hold in the weakly purely quasi-minimal case. The following two lemmas are the analogues of Theorem 3.4 and Lemma 3.7.

Lemma 4.2 *If G is a reduced weakly purely quasi-minimal group, then G is either torsion-free or a homocyclic p-group.*

Proof: As in Theorem 3.4 choose some p such that $G_p \neq 0$ and let B be a p-basic subgroup of G. Then B is not torsion-free. Let $k > 0$

be the least integer such that $B_k \neq 0$; we have $G = B_k \oplus (B^* + p^k G)$ where $B^* = B_0 \oplus \bigoplus_{n>k} B_n$. If $|G| = |B_k|$, then G is isomorphic to a pure subgroup of B_k. We have shown in Theorem 3.4 that a pure subgroup of a homocyclic p-group is again homocyclic, so we can deduce that G is homocyclic. If $|G| > |B_k|$, then $|G| = |B^* + p^k G|$. Again letting $H = B^* + p^k G$ we get that G is isomorphic to some C, a pure subgroup of H. As before B^* is a p-basic subgroup of H. Now a p-basic subgroup of C can be extended to a p-basic subgroup of H since a p-pure independent subset of C is also p-pure independent in H. Therefore B is isomorphic to a summand of B^*, since all p-basic subgroups of C are isomorphic, and so we get the same contradiction as in Theorem 3.4 . □

Lemma 4.3 *If G is a torsion-free weakly purely quasi-minimal group, then G is t-homogeneous for some type t.*

Proof: Let $t \in T(G)$. As in Lemma 3.7, if $|G(t)| < |G|$ then $|G| = |K|$, where K is some $G(t)$-high subgroup of G. Therefore $G \cong K_1$ where K_1 is some pure subgroup of K. We have $G(t) \cong K_1(t) = \{k \in K_1 : t_{K_1}(k) \geq t\} = \{k \in K_1 : t_G(k) \geq t\} = K_1 \cap G(t) \leq K \cap G(t) = 0$ which is a contradiction since $G(t) \neq 0$. Therefore, as before, $|G| = |G(t)|$ and hence $G \cong G_1$, where G_1 is some pure subgroup of $G(t)$. Now, if

$G \cong G_1 \leq_* G(t)$ and $G \cong G_2 \leq_* G(s)$ where $t, s \in T(G)$, then there exist $g_1 \in G_1$ and $g_2 \in G_2$ with $t_{G_1}(g_1) = s$ and $t_{G_2}(g_2) = t$. Hence $t_G(g_1) = s$ and $t_G(g_2) = t$ and so we get $s \geq t$ and $t \geq s$, i.e. $s = t$ and so G is t-homogeneous. □

The characterisation theorem now follows.

Theorem 4.4 (GCH) *If $G \in Ab_\kappa$ is a torsion-free reduced weakly purely quasi-minimal group, then either $G \cong R$, if $\kappa = \aleph_0$, or $G \cong \bigoplus_\kappa R$, if $\kappa > \aleph_0$, for some rank 1 group R.*

Again, obviously, we do not need GCH *in the countable case.*

Proof As in Theorem 3.12, let $C = \bigoplus_I R$ be a pure essential completely decomposable subgroup of G such that $|G| \leq |C|^{\aleph_0}$. If $|G| = \aleph_0$, then $|R| = |C| = |G| = \aleph_0$ and so G is isomorphic to a pure subgroup of R. But we have seen in Theorem 3.12 that the only non-zero pure subgroup of R is R itself, so $G \cong R$.

Now suppose that $|G| = \kappa > \aleph_0$. If $|C| = |G| = \kappa$, then $G \cong H$ where H is some pure subgroup of C. By I, Lemma 3.5 we have H is also homogeneous completely decomposable of the same type as R, i.e. $H \cong \bigoplus_J R$ where $|J| = \kappa$, since $G \cong H$. Now suppose $|C| < |G| = \kappa$.

As before, $|G| = 2^{|C|}$, using GCH, and $|\mathrm{End}(G)| \leq |G|$ and there exists a maximal linearly independent subset X of G of cardinality κ. Assuming GCH, there exist 2^κ almost disjoint subsets of X of cardinality κ (see [15, p.252]), i.e. $|Y \cap Y'| < \kappa$ for any pair Y, Y' of such subsets. Each such subset Y generates a pure subgroup $K = \langle Y \rangle_*$ of G of cardinality κ. We claim that $\langle Y \cap Y' \rangle_* = \langle Y \rangle_* \cap \langle Y' \rangle_*$.

If $x \in \langle Y \cap Y' \rangle_*$, then there exists $n \in \mathbb{Z}$ such that $nx \in \langle Y \cap Y' \rangle$ so $nx \in \langle Y \rangle \cap \langle Y' \rangle$ and thus $x \in \langle Y \rangle_* \cap \langle Y' \rangle_*$. Conversely, if $x \in \langle Y \rangle_* \cap \langle Y' \rangle_*$, then there exist non-zero $n, m \in \mathbb{Z}$ such that $nx \in \langle Y \rangle$ and $mx \in \langle Y' \rangle$. Therefore there exist non-zero integers $n_1, \ldots, n_k$ and $m_1, \ldots, m_r$ and elements $y_1, \ldots, y_k \in Y$ and $y'_1, \ldots, y'_r \in Y'$ such that $nx = n_1 y_1 + \ldots + n_k y_k$ and $mx = m_1 y'_1 + \ldots + m_r y'_r$. Then we get that $mnx = mn_1 y_1 + \ldots + mn_k y_k = nm_1 y'_1 + \ldots + nm_r y'_r$. Since G is torsion-free and X is a linearly independent subset of G we obtain that $k = r$ and each $y_i (i = 1, \ldots, k)$ must be equal to some $y'_j (j = 1, \ldots, r)$. Hence $y_i \in Y \cap Y'$ for all $1 \leq i \leq k$ and so $nx \in \langle Y \cap Y' \rangle$, i.e. $x \in \langle Y \cap Y' \rangle_*$. We conclude that $|K \cap K'| = |\langle Y \cap Y' \rangle_*| < \kappa$ where $K' = \langle Y' \rangle$. Now, K and K' are pure in G with $|K| = |G| = |K'|$, thus there exist monomorphisms $\phi : G \longrightarrow K$ and $\phi' : G \longrightarrow K'$. If $\phi = \phi'$, then $G \cong \mathrm{Im}(\phi) = \mathrm{Im}(\phi') \leq K \cap K'$ and so $|G| \leq |K \cap K'| < \kappa$, which is a contradiction. Therefore $|\mathrm{End}(G)| \geq 2^\kappa$ since in the proof of Theorem 3.12 we showed that we have 2^κ different such pure subgroups. We now have $2^\kappa \leq |\mathrm{End}(G)| \leq \kappa$, again a contradiction. Hence, as in Theorem 3.12, $|C| = |G|$ and this completes the proof. □

So, just as for the quasi-minimal case, the class of weakly purely quasi-minimal groups is, in fact, the same as the class of purely quasi-minimal groups. However, in this case, we needed GCH to show the equality of the two classes. It is an open question whether this fact is true in ZFC.

§5 Directly Quasi-Minimal Groups

The final type of quasi-minimal group G we consider is where we require that G be isomorphic only to all its direct summands of the same cardinality as itself. In this case G is called directly quasi-minimal.

Definition 5.1 *A group $G \in Ab_\kappa$ is* directly quasi-minimal *if G is isomorphic to all its summands of cardinality κ.*

Lemma 5.2 *If G is directly quasi-minimal, then G is either divisible or reduced.*

Proof: The arguments are the same as in Lemma 1.4. □

We can immediately characterise the divisible directly quasi-minimal groups.

Lemma 5.3 *For a divisible directly quasi-minimal group G one of the following is true:*

(i) $G \cong \mathbb{Q} \quad (\kappa = \aleph_0)$,

(ii) $G \cong \mathbb{Z}(p^\infty)$, *for some* $p \quad (\kappa = \aleph_0)$,

(iii) $G \cong \bigoplus_\kappa \mathbb{Q} \quad (\kappa > \aleph_0)$,

(iv) $G \cong \bigoplus_\kappa \mathbb{Z}(p^\infty)$, *for some* $p \quad (\kappa > \aleph_0)$.

Proof: The arguments are similar to those in Lemma 3.3. □

Theorem 5.4 *If G is a reduced directly quasi-minimal group, then G is either a homocyclic p-group or G is torsion-free.*

Proof: The same arguments as in Theorem 3.4. □

It remains to consider the torsion-free reduced case. Every indecomposable torsion-free reduced group is trivially directly quasi-minimal. Such groups exist in abundance: Shelah [26] has shown that, for each infinite cardinal κ, there exist 2^κ non-isomorphic indecomposable groups of cardinality κ.

For a countable decomposable directly quasi-minimal group we have the following lemma.

Lemma 5.5 *If G is a countable torsion-free directly quasi-minimal group which is decomposable, then*

(i) $G \cong \bigoplus_n G$, *for all $n \in \mathbb{N}$ and so G must have infinite rank;*

(ii) $G^* = \mathrm{Hom}(G, \mathbb{Z}) = 0$.

Proof: **(i)** Since G is decomposable $G = A \oplus B$ where $|A| = |B| = |G| = \aleph_0$. Therefore $G \cong A$ and $G \cong B$ and so $G \cong G \oplus G$. A straightforward induction now completes the proof.

(ii) Stein's Theorem (see [10, Corollary 19.3]) tells us that $G = N \oplus F$ where F is free and N has no free quotient groups (or equivalently, $N^* = 0$). If $F \neq 0$, then $G \cong F$ (and $N = 0$), so $G = \mathbb{Z}$ since G is directly quasi-minimal, a contradiction. Therefore $F = 0$ and $G \cong N$ and hence $G^* = 0$. □

Properties (i) and (ii) of Lemma 5.5 are not sufficient to characterise the countable directly quasi-minimal decomposable torsion-free groups as the following example shows:

Corner [5] has given an example of a countable group G with countable endomorphism ring where $G \cong \bigoplus_n G$ for all n but $G \not\cong \bigoplus_{\aleph_0} G$. Stein's Theorem tells us that $G = N \oplus F$ where N and F are as in Lemma 5.5. Therefore $G^* \cong N^* \oplus F^* = F^*$ (see [10, Theorem 43.1]). Since $G^* \cong G^* \oplus G^*$ we get that $F^* \cong F^* \oplus F^*$ and so either $F^* = 0$ or F^* has infinite rank. Hence the same must be true for F, since F is free. Now, if F has infinite rank, then $|F^*| = 2^{\aleph_0}$ and so $|G^*| = 2^{\aleph_0}$. But $G^* \leq \text{End}(G)$, since G is torsion-free, so G^* is countable. We conclude that $F^* = G^* = 0$. Now, if we take two such groups $G_1 \not\cong G_2$, then $G_1 \oplus G_2$ clearly satisfies properties (i) and (ii) but is obviously not directly quasi-minimal.

I have not been able to establish whether the group G above is directly quasi-minimal or not.

Turning to the uncountable case, every purely quasi-minimal group is, of course, directly quasi-minimal. The following lemma gives an example of an uncountable decomposable torsion-free reduced directly quasi-minimal group which is not purely quasi-minimal.

Lemma 5.6 *The Baer-Specker group* $\prod_{\aleph_0} \mathbb{Z}$ *is directly quasi-minimal.*

Proof Let $P = \prod_{\aleph_0} \mathbb{Z}$. If $P = A \oplus B$, then both A and B are products of $\mathbb{Z}$ (see [8, IX, Theorem 1.4]) and either $|A| = |P| = 2^{\aleph_0}$ or $|B| = |P| = 2^{\aleph_0}$ or both. Suppose that $A = \prod_I \mathbb{Z}$ and $|A| = |P|$. Then $|I| = \lambda$, say, with $\lambda \leq \aleph_0$ and $2^\lambda = 2^{\aleph_0}$. Then $\lambda = \aleph_0$ and hence $A \cong P$. □

However, if we consider $G = \prod_\kappa \mathbb{Z}$ where $\kappa > \aleph_0$, then the direct quasi-minimality of G may be undecidable in ZFC, as is shown by the next lemma.

Lemma 5.7 *Let* $\kappa > \aleph_0$ *and let* $G = \prod_\kappa \mathbb{Z}$. *Then:*

(i) *Assuming* GCH, G *is directly quasi-minimal;*

(ii) *Assuming* $\neg$ CH *and* $2^\kappa = 2^{\aleph_0}$ *for all* $\aleph_0 < \kappa < 2^{\aleph_0}$ *then, for all such* κ, G *is not directly quasi-minimal.*

Proof **(i)** Suppose $G = A \oplus B$ with $|A| = |G| = 2^\kappa$, say. Then, as above, $A = \prod_I \mathbb{Z}$ with $|I| = \lambda$, for some $\lambda \leq \kappa$, and so $2^\lambda = 2^\kappa$. Assuming GCH we get $\lambda = \kappa$ and so $A \cong G$ and hence G is directly quasi-minimal.

(ii) Now assume that the assumption $\neg$ CH and $2^{\aleph_0} = 2^\kappa$ for all $\aleph_0 < \kappa < 2^{\aleph_0}$ holds. Let $G = \prod_\kappa \mathbb{Z}$ where $\aleph_0 < \kappa < 2^{\aleph_0}$. Then $G = \prod_{\aleph_0} \mathbb{Z} \oplus B$, say, with $|\prod_{\aleph_0} \mathbb{Z}| = |G|$ but $\prod_{\aleph_0} \mathbb{Z} \not\cong G$, since $\bigoplus_{\aleph_0} \mathbb{Z} = (\prod_{\aleph_0} \mathbb{Z})^* \not\cong G^* = \bigoplus_\kappa \mathbb{Z}$, and so G is not directly quasi-minimal. □

Since both GCH and the assumption that $\neg$ CH and $2^{\kappa} = 2^{\aleph_0}$ for all $\aleph_0 < \kappa < 2^{\aleph_0}$ holds can be shown to be consistent with ZFC (see [27]) we can deduce that the direct quasi-minimality of e.g. $G = \prod_{\aleph_1} \mathbb{Z}$ is not decidable in ZFC.

III General Minimal Groups and Finite Index Subgroups

In this chapter we consider some properties of minimal groups in general while in later chapters we examine the torsion and torsion-free cases separately. Some facts concerning the finite index subgroups of a group are also proved.

§1 General Minimal Groups

We begin with the required definitions.

Definition 1.1 *Let G be any group. A subgroup H of G is of* finite index *in G if $|G/H|$ is finite.*

Definition 1.2 *G is* minimal *if $G \cong H$ for all finite index subgroups H of G.*

Clearly a finite group is minimal if and only if it is the trivial group and so it makes sense to consider infinite groups only. In the following we prove some general results on minimal groups.

Lemma 1.3 *Every divisible group is minimal.*

Proof: Let G be divisible and let H be of finite index in G. There exists $n \in \mathbb{N}$ such that $n(G/H) = 0$, i.e. $nG \leq H$. But $nG = G$, so $G \leq H$. Hence $G = H$ and the only finite index subgroup of G is G itself. Therefore G is minimal. $\square$

The next theorem shows that in considering minimal groups only reduced groups have to be looked at.

Theorem 1.4 *Let $G = D \oplus M$ where D is the maximal divisible subgroup of G and M is reduced. Then G is minimal if and only if M is minimal.*

Proof: First suppose M is minimal. Let H be of finite index in G. Then $H \cap D \leq D$ and $D/(H \cap D) \cong (D+H)/H \leq G/H$ is finite. Therefore $H \cap D = D$ by Lemma 1.3. We have $D \leq H$ and so $H = D \oplus N$ where N is reduced. Now $M \cong G/D$ and $N \cong H/D$ and $M/N \cong (G/D)/(H/D) \cong G/H$ is finite, so $H/D \cong G/D$ since $G/D \cong M$ is minimal. Therefore $N \cong M$ and hence $H \cong G$.

Conversely, if G is minimal and N is of finite index in M then $D \oplus N$ is of finite index in $D \oplus M$, so $D \oplus M \overset{\phi}{\cong} D \oplus N$. Therefore $M \cong G/D \cong (D \oplus N)/D\phi$. Now, $D\phi \leq D$ and $D\phi$ is divisible, so $D\phi \sqsubset D$, *i.e.* $D = D\phi \oplus E$ with E also divisible. Hence $M \cong E \oplus N$. Since M is a reduced group we get that E must be 0 and so $M \cong N$. □

If p is any prime we can define a local version of minimality at p in the following way:

Definition 1.5 *We say a group G is p-*minimal *if G is isomorphic to all its subgroups of index p.*

The following theorem shows that in investigating the minimality of a group it suffices to consider the local case.

Theorem 1.6 **(i)** *A group G is minimal if and only if G is p-minimal for all primes p;*

(ii) *A group G is p-minimal if and only if G is isomorphic to all its subgroups of index a power of p;*

(iii) *A p-group is minimal if and only if it is p-minimal.*

Proof: **(i)** If G is minimal, then obviously G is p-minimal for all p. Conversely, suppose that G is p-minimal for all p and that H is any finite index subgroup of G. We show that G is isomorphic to H by induction on the order of G/H. If $|G/H| = 1$, then $G = H$. Now, assume that G is isomorphic to all its subgroups of index less than some integer n and let $|G/H| = n$. If n is prime, then $G \cong H$ by assumption. If n is not prime, then $n = r \cdot s$ for some $r, s < n$. Now, G has a subgroup G_1 containing H such that $|G_1/H| = s$. Then $|G/G_1| = r$ and so the induction hypothesis gives us $G \cong G_1$ and hence G_1 is isomorphic to all its subgroups of index less than n since this property is an isomorphic invariant. Therefore $G_1 \cong H$ and so $G \cong H$, i.e. G is minimal.

(ii) The same argument as in (i), where the induction now is on the power of p.

(iii) This follows from (ii) and the fact that a p-group has no subgroups of index q for any prime $q \neq p$ since it is q-divisible. □

Finally, we show that in the case of torsion minimal groups we may restrict ourselves to the study of p-groups.

Theorem 1.7 *A torsion group is minimal if and only if all of its primary components are minimal.*

Proof: Let $G = \bigoplus_{p \in \Pi} G_p$. If G is minimal and H_p is of finite index in G_p, then $\bigoplus_{q \neq p} G_q \oplus H_p$ is of finite index in G and so $\bigoplus_{q \neq p} G_q \oplus H_p \overset{\phi}{\cong} G$. Therefore $\phi \restriction H_p : H_p \longrightarrow G_p$ is an isomorphism.

Conversely, suppose that G_p is minimal for all p and H is of index q in G, for some prime q. Then $H = \bigoplus_{p \in \Pi} H_p$ with $H_p = H \cap G_p$ for all p

and $\mathbb{Z}(q) \cong G/H \cong \bigoplus_{p\in\Pi}(G_p/H_p)$. Therefore $H_p = G_p$ for all $p \neq q$ and $\mathbb{Z}(q) \cong G_q/H_q$. Now, since G_q is minimal, we get that $G_q \cong H_q$ and so $H \cong G$. Hence G is minimal, by Theorem 1.6. □

§2 Finite Index Subgroups

Before we can study minimal groups in more detail we need some results on finite index subgroups.

Proposition 2.1 *Every infinite reduced group G has non-trivial finite index subgroups.*

Proof: Since G is not divisible, by assumption, there exists a prime p such that $pG \neq G$. Now, G/pG is a $\mathbb{Z}(p)$-vector space and thus $G/pG \cong \bigoplus_{i\in I}\mathbb{Z}(p)$ for some index set I. If $pG = 0$, then $G = \bigoplus_{i\in I}\mathbb{Z}(p)$. Therefore we deduce in this case that I must be infinite and obviously G has finite index subgroups. If $pG \neq 0$, then, for each $j \in I$, there exists $H_j \leq G$ such that $H_j/pG \cong \bigoplus_{i\neq j}\mathbb{Z}(p)$. Then $G/H_j \cong (G/pG)/(H/pG) \cong \mathbb{Z}(p)$ and so H_j is a non-trivial finite index subgroup of G. □

We now get a lower bound for the number of finite index subgroups of a reduced group G. For this purpose we introduce the notation F.I.(G)

to denote the set of all the finite index subgroups of a given group G.

Theorem 2.2 *Let G be a reduced group and $G/pG \cong \bigoplus_{I_p} Z(p)$ for each p. Then $|\mathrm{F.I.}(G)| \geq \sum_{p\in\Pi} |I_p|$, if I_p is finite for all p, or else $|\mathrm{F.I.}(G)| \geq sup\{2^{|I_p|} : I_p$ is infinite, $p \in \Pi\}$.*

Proof: Since G is reduced there exists some p such that $pG \neq G$ and $G/pG \cong \bigoplus_{I_p} Z(p)$, i.e. $I_p \neq \emptyset$ for at least one prime p. If $|I_p|$ is finite for all p, then, for each p, we have at least $|I_p|$ different finite index subgroups of G and so the first part of the theorem follows. If $|I_p|$ is infinite for some p, then G/pG is a vector space over $\mathbb{Z}(p)$ of dimension $|I_p|$. Then, if H/pG is a vector subspace of G/pG of finite codimension, we have that $G/H \cong (G/pG)/(H/pG)$ is a finite group, and so H is of finite index in G. Hence $|\mathrm{F.I.}(G)|$ is not less than the number of subspaces of G/pG of codimension 1. It is well-known that if $V = \bigoplus_{i\in I} \langle e_i \rangle$ is an infinite-dimensional vector space over a field F then the number of subspaces of V of codimension 1 is not less than $\mathrm{rank}(V^*)$ where $V^* = \mathrm{Hom}(V, F)$ (see e.g. [16, Chapter 4, 13.6]). Now, $V^* = \mathrm{Hom}(V, F) = \mathrm{Hom}(\bigoplus_{i\in I} e_i F, F) \cong \prod_{i\in I} \mathrm{Hom}(F, F)$, and so $|V^*| = |\prod_{i\in I} \mathrm{Hom}(F, F)| = 2^{|I|}$ if $\mathrm{Hom}(F, F)$ is countable which is the case for $F = \mathbb{Z}(p)$. Therefore $\mathrm{rank}(V^*) = 2^{|I|}$ and so there exists at least $2^{|I_p|}$

subgroups of finite index of G. Since this is true for all p with I_p infinite, the result follows. □

When G is a minimal group we can also get an upper bound for $|\text{F.I.}(G)|$, as the next proposition shows.

Proposition 2.3 *If G is minimal, then $|\text{F.I.}(G)| \leq |\text{End}(G)|$ where $\text{End}(G)$ is the endomorphism ring of G.*

Proof: If G is minimal and H is of finite index in G, then there exists an isomorphism $\phi_H : G \longrightarrow H$. If $H_1 \neq H_2$, then $\phi_{H_1} \neq \phi_{H_2}$ and so $|\text{F.I.}(G)| \leq |\text{End}(G)|$. □

Finally, we consider residually finite groups. If G is an arbitrary group, then the set $\text{F.I.}(G)$ induces a linear topology on G called the *finite index topology.*

Definition 2.4 *The group G is* residually finite *if G is Hausdorff in the finite index topology i.e. $\bigcap_{i \in I} H_i = 0$ where $\text{F.I.}(G) = \{H_i : i \in I\}$.*

Lemma 2.5 *An infinite reduced residually finite group G has infinitely many finite index subgroups.*

Proof: If G has only finitely many finite index subgroups $H_1, \ldots, H_m$ then, for each $i (1 \leq i \leq m)$, there exists $n_i \in \mathbb{N}$ such that $n_i G \leq H_i$. Letting $n = n_1 \cdots n_m$ we get $nG \leq \bigcap_{i=1}^{i=m} H_i = 0$, so G is bounded, and hence is a direct sum of cycles. Now, since G has only finitely many finite index subgroups, G must be finite, a contradiction. □

Note that if $H_1, \ldots, H_m$ are finite index subgroups of G then an easy induction argument shows that so is $\bigcap_{i=1}^{i=m} H_i$ and thus, in the previous lemma, 0 is a finite index subgroup of G if G has only finitely many finite index subgroups, and hence G is finite.

The next result gives a characterisation of a residually finite group in terms of the $\mathbb{Z}$-adic topology on the group.

Theorem 2.6 *A group G is residually finite if and only if it is Hausdorff in its $\mathbb{Z}$-adic topology i.e. $G' = 0$ where $G' = \bigcap_{n \in \omega} nG$. In fact $G' = \bigcap_{i \in I} H_i$ where again* F.I.$(G) = \{H_i : i \in I\}$.

Proof: First suppose that G is residually finite. For each $H_i \in$ F.I.(G), there exists some $n \in \omega$ such that $nG \leq H_i$ and so $G' \leq \bigcap_{i \in I} H_i$. Hence G is Hausdorff in its $\mathbb{Z}$-adic topology.

Conversely, suppose that G is Hausdorff in its $\mathbb{Z}$-adic topology. Note that the closure of a subgroup A in the finite index topology is given by $\bigcap_{i\in I}(A+H_i) = \bigcap_{\{i\in I: A\leq H_i\}} H_i$, since if H_i is of finite index in G, then so is $A+H_i$. Using this we show that nG is closed in the finite index topology for all n:

if nG is not closed, for some n, then we can find $g \in \bigcap_{\{i\in I: nG\leq H_i\}} H_i$ such that $g \notin nG$. Now, $G/nG \cong \bigoplus_{j\in J} A_j$, for some index set J, where each of the A_j is a finite cyclic group. The element $g + nG \in G/nG$ has finite support J', say. Define $H \leq G$ by $H/nG \cong \bigoplus_{j\in J\setminus J'} A_j$. Then H is a finite index subgroup of G containing nG but not containing g, a contradiction. Since each nG is closed we get that $\bigcap_{n\in\omega} nG$ must be closed i.e. 0 is closed and so the finite index topology is Hausdorff.

Now, if G is any group, then G/G' is Hausdorff in its $\mathbb{Z}$-adic topology. The above argument then implies that $\bigcap_{i\in I}(H_i/G') = 0$ since H/G' is of finite index in G/G' if and only if H is of finite index in G. It now follows that $G' = \bigcap_{i\in I} H_i$. □

We conclude this chapter with a characterisation of a residually finite group as a subdirect sum of the quotient groups G/H_i. Recall that a subgroup G of the direct product $A = \prod_{i\in I} B_i$ is a *subdirect sum* of the B_i

if, for each i, the projection $\pi_i \upharpoonright G : G \longrightarrow B_i$ is epic (see [10, p.42]).

Proposition 2.7 *If G is residually finite then G is isomorphic to a subdirect sum of $\prod_{i \in I}(G/H_i)$.*

Proof: Define $\phi : G \longrightarrow \prod_{i \in I}(G/H_i)$ by $g\phi = (g + H_i)_{i \in I}$. Then ϕ is a homomorphism with $Ker(\phi) = \bigcap_{i \in I} H_i = 0$. Therefore $G \cong Im(\phi) = \{(g + H_i)_{i \in I} : g \in G\}$, which is obviously a subdirect sum of the G/H_i.

□

IV Torsion and Mixed Minimal Groups

We have already seen that the consideration of torsion minimal groups can be confined to the case of reduced p-groups. We now wish to characterise minimal p-groups. The characterisation is in terms of the Ulm invariants. Before we tackle the characterisation problem we need some properties of Ulm invariants and basic subgroups of p-groups. These properties are well-known but we include the proofs here for the sake of completeness.

§1 Ulm Invariants and Basic Subgroups

Lemma 1.1 *If G is any p-group then $f_n(G) = f_n(G/p^\omega G)$ for all $n < \omega$.*

Proof: Define $\phi : (p^n G/p^\omega G)[p] \longrightarrow p^n G[p]/p^{n+1}G[p]$ by the following: if $(p^n g + p^\omega G) \in (p^n G/p^\omega G)[p]$ then $p^{n+1}g \in p^\omega G$, so there exists $x \in pG$ such that $p^{n+1}g = p^{n+1}x$.

Then $p(p^n g - p^n x) = 0$ and so $p^n(g - x) \in p^n G[p]$. Let $(p^n g + p^\omega G)\phi = p^n(g - x) + p^{n+1}G[p]$. ϕ is well-defined since, if $(p^n g - p^n g_1) \in p^\omega G$ with $p^{n+1}g, p^{n+1}g_1 \in p^\omega G$ then, choosing $x, x_1 \in pG$ such that $p^{n+1}g = p^{n+1}x$ and $p^{n+1}g_1 = p^{n+1}x_1$, we get $p^n(g - x) - p^n(g_1 - x_1) = p^n(g - g_1) - p^n(x - x_1) \in p^{n+1}G$ and $p(p^n(g - x) - p^n(g_1 - x_1)) = 0$, so $p^n(g -$

$x) - p^n(g_1 - x_1) \in p^{n+1}G[p]$. ϕ is obviously an epimorphism. $\mathrm{Ker}(\phi) = (p^{n+1}G/p^\omega G)[p]$ since, if $p^n g + p^\omega G \in \mathrm{Ker}(\phi)$, then $p^n(g-x) \in p^{n+1}G[p]$ so $p^n(g-x) = p^{n+1}g_1$ where $p^{n+2}g_1 = 0$; hence $p^n g = p^n x + p^{n+1}g_1 = p^{n+1}x_1 + p^{n+1}g_1 = p^{n+1}(x_1+g_1)$, for some $x_1 \in G$ since $x \in pG$. Therefore $p^n g + p^\omega G \in (p^{n+1}G/p^\omega G)[p]$.

Conversely, if $p^{n+1}g + p^\omega G \in (p^{n+1}G/p^\omega G)[p]$, then $(p^{n+1}g + p^\omega G)\phi = (p^n(pg) + p^\omega G)\phi = p^n(pg - x) + p^{n+1}G[p] = 0$. We have $(p^n G/p^\omega G)[p]/(p^{n+1}G/p^\omega G)[p] \cong p^n G[p]/p^{n+1}G[p]$ for all $n < \omega$, i.e. $f_n(G/p^\omega G) = f_n(G)$ for all $n < \omega$. □

Lemma 1.2 *Let G be a reduced p-group and B one of its basic subgroups. If B is bounded then $G = B$.*

Proof: Since G/B is divisible we have $G = B + p^n G$ for all n. Thus $G/B = (B + p^n G)/B \cong p^n G/(B \cap p^n G) = p^n G/p^n B$ for all n. Now, there exists $m \in \mathbb{N}$ such that $p^m B = 0$, so $G/B \cong p^m G$. But G/B is divisible and $p^m G$ is reduced, so $G/B = p^m G = 0$, i.e. $G = B$. □

The next lemma tells us that for finite ordinals the Ulm invariants of a basic subgroup of a p-group G are in fact the same as the Ulm invariants of G. This result is essential in establishing the characterisation of minimal p-groups.

Lemma 1.3 *If B is basic in the reduced p-group G, then $f_n(G) = f_n(B)$ for all $n < \omega$.*

Proof: If $n \in \mathbb{N}$ we have $p^nB[p]/p^{n+1}B[p] = p^nB[p]/(p^{n+1}G \cap B)[p] = p^nB[p]/(p^{n+1}G \cap p^nG \cap B)[p] = p^nB[p]/(p^{n+1}G \cap p^nB)[p] = p^nB[p]/(p^{n+1}G[p] \cap p^nB[p]) \cong (p^nB[p] + p^{n+1}G[p])/p^{n+1}G[p]$. We claim that $p^nB[p] + p^{n+1}G[p] = p^nG[p]$.

Obviously $p^nB[p] + p^{n+1}G[p] \leq p^nG[p]$.

Conversely, if $g \in G[p]$, then, for any $n \in \mathbb{N}$, $g = b + g_1$ where $b \in B$ and $g_1 \in p^nG$, since $G = B + p^nG$. Now, $0 = pb + pg_1$, so $pg_1 \in B \cap p^{n+1}G = p^{n+1}B$. Therefore $pg_1 = pb_1$ where $b_1 \in p^nB$. Hence $g = b + b_1 + g_1 - b_1$ where $(b + b_1) \in B$ and $(g_1 - b_1) \in p^nG[p]$, so $(b + b_1) \in B[p]$. Therefore $G[p] \leq B[p] + p^nG[p]$ for any $n \in \mathbb{N}$. Now let $g \in p^nG[p]$; $g \in G[p] = B[p] + p^{n+1}G[p]$, so $g = b + g_1$ where $b \in B[p]$ and $g_1 \in p^{n+1}G[p]$. Therefore $b = (g - g_1) \in B \cap p^nG = p^nB$ and so $b \in p^nB[p]$. Hence $g \in p^nB[p] + p^{n+1}G[p]$ and thus $p^nG[p] \leq p^nB[p] + p^{n+1}G[p]$. Therefore $p^nB[p]/p^{n+1}B[p] \cong p^nG[p]/p^{n+1}G[p]$ and so $f_n(B) = f_n(G)$ for all $n < \omega$. □

Note that Lemma 1.3 is a special case of a more general result which gives criteria for the equality of the Ulm invariants of a p-group G and the Ulm invariants of certain of its subgroups (see [25, 2, Proposition 7.4]).

Now we establish some important properties of the Ulm invariants of minimal p-groups.

Proposition 1.4 *Let G be a minimal p-group such that $f_n(G) = 0$ for some $n < \omega$. Then $f_m(G) = 0$ for all $m \geq n$ and G is bounded.*

Proof: Let B be a basic subgroup of G. By Lemma 1.3 $f_m(B) = f_m(G)$ for all m. So if $f_{n+1}(G) \neq 0$, then B has a summand $\mathbb{Z}(p^{n+2})$ and, since $B \leq_* G$, so has G, $G = \mathbb{Z}(p^{n+2}) \oplus G_1$, say. Now $H = p\mathbb{Z}(p^{n+2}) \oplus G_1$ is a finite index subgroup of G and so $H \cong G$. We have $0 = f_n(G) = f_n(H) = 1 + f_n(G_1) = 1 + f_n(G) = 1$, which is a contradiction. Therefore $f_{n+1}(G) = 0$. A simple induction argument now gives that $f_m(G) = 0$ for all $m \geq n$. Also $f_m(B) = 0$ for all $m \geq n$ so B is bounded. Therefore $G = B$, by Lemma 1.2. □

Proposition 1.5 *Let G be a minimal p-group such that $f_n(G) \neq 0$ for some $n < \omega$. Then $f_n(G)$ is infinite.*

Proof: Let B be a basic subgroup of G. Then $f_n(B) \neq 0$. Therefore B, and hence G, has a summand $\mathbb{Z}(p^{n+1})$, i.e. $G = \mathbb{Z}(p^{n+1}) \oplus G_1$. Since G/G_1 is finite, $G_1 \cong G$ and so $f_n(G_1) = f_n(G)$; this means that $f_n(G) = 1 + f_n(G)$ and so $f_n(G)$ must be infinite. □

§2 The Characterisation of Minimal p-Groups

The following theorem, based on an argument due to Pierce (see [21, Lemma 16.5]), is the main tool used in characterising minimal p-groups. We could, of course, have restricted our considerations to subgroups of index p but we shall need the more general result later on.

Theorem 2.1 *If G is a p-group and H is a finite index subgroup of G, then there exists $K \leq H$ such that $K \sqsubset G$ and G/K is finite.*

Proof: The proof is by induction on the exponent of the order of G/H. First suppose $|G/H| = p$. If $H \leq_* G$, then $G = H \oplus A$ where $A \cong \mathbb{Z}(p)$, by I, Lemma 1.10. Thus $K = H$ gives the result. If $H \not\leq_* G$, then (see [10, p.114 (h)]) there exists some $y \in H[p]$ such that $h^H(y)$ is finite and $h^H(y) < h^G(y)$. If $h^H(y) = k - 1$, then $h^G(y) = k$ since if there exists $x \in G \setminus H$ such that $y = p^{k+1}x$ then $p^k(px) = y$ and $px \in H$, since $pG \leq H$, a contradiction. So consider x where $y = p^k x$. Then $\langle px \rangle \leq_* H$

(see [10, Corollary 27.2]) and is finite, so $\langle px\rangle \sqsubset H$, by I, Lemma 1.9, i.e. $H = K \oplus \langle px\rangle$. We claim that $G = K \oplus \langle x\rangle$.

We certainly have $G = \langle x\rangle + H = \langle x\rangle + K + \langle px\rangle = \langle x\rangle + K$. Also $\langle x\rangle \cap H = \langle px\rangle$ since if $nx \in \langle x\rangle \cap H$, then $n(x + H) = 0$ so p divides n and hence $nx \in \langle px\rangle$; obviously $\langle px\rangle \subseteq \langle x\rangle \cap H$. Therefore $\langle x\rangle \cap K = \langle x\rangle \cap K \cap H = K \cap (\langle x\rangle \cap H) = K \cap \langle px\rangle = 0$, and so $G = K \oplus \langle x\rangle$. Now suppose the result is true for all subgroups L of G such that $|G/L| \leq p^n$, and let $|G/H| = p^{n+1}$. There exists a subgroup H_1 of G such that $H \leq H_1 \leq G$ and $|H_1/H| = p$. Then $G/H_1 \cong (G/H)/(H_1/H)$ which has order p^n and so the hypothesis implies there exists $K_1 \leq H_1$ such that $K_1 \sqsubseteq G$ with G/K_1 finite. If $K_1 \leq H$ let $K = K_1$. If $K_1 \nleq H$, then $0 \neq K_1/(K_1 \cap H) \cong (K_1 + H)/H \leq H_1/H$ so $|K_1/(K_1 \cap H)| = p$. The first part of the proof now gives some $K \leq K_1 \cap H$ such that $K \sqsubset K_1$ and K_1/K is finite. We have $K \leq H$ and $K \sqsubset K_1 \sqsubset G$ with G/K finite since $G/K_1 \cong (G/K)/(K_1/K)$. $\square$

We are now ready to state and prove the characterisation.

Theorem 2.2 *If G is a reduced p-group, then G is minimal if and only if $f_n(G)$ is infinite for all $n < \omega$ or there exists some $\lambda < \omega$ such that $f_n(G)$ is infinite for all $n < \lambda$ and $f_n(G) = 0$ for all $n \geq \lambda$.*

Proof: If G is minimal the necessity of the condition is given by Proposition 1.4 and Proposition 1.5. For sufficiency consider H, a subgroup of index p in G. Theorem 2.1 tells us that there exists $K \leq H$ such that $G = K \oplus A$ where A is cyclic. Then $H = K \oplus (H \cap A) = K \oplus B$, say. We get $f_n(G) = f_n(K) + f_n(A)$ and $f_n(H) = f_n(K) + f_n(B)$, so $f_n(K)$ and $f_n(H)$ are both infinite when $f_n(G)$ is infinite. Also, if $f_n(G) = 0$, then $0 = f_n(G) = f_n(K) + f_n(A)$, so $f_n(K) = f_n(A) = 0$ and $f_n(H) = 0 + f_n(B) = 0$, since $B \leq A$ and A is a cyclic group. If B_K is a basic subgroup of K, then $f_n(B_K) = f_n(K) = f_n(G)$ for all n, so $B_K = \bigoplus_{I_1} \mathbb{Z}(p) \oplus \bigoplus_{I_2} \mathbb{Z}(p^2) \oplus \cdots \oplus \bigoplus_{I_n} \mathbb{Z}(p^n) \oplus \cdots$ with I_n infinite for all $n < \omega$ or I_n infinite for all $n <$ some $\lambda < \omega$ and I_n empty for all $n \geq \lambda$. Since A is cyclic we get that $A = \mathbb{Z}(p^n)$ for some n. B_K has a bounded summand $C_K = \bigoplus_{I_n} \mathbb{Z}(p^n)$ which is pure in K and so $C_K \sqsubset K, K = K_1 \oplus C_K$, say. Therefore $G = K \oplus A = K_1 \oplus C_K \oplus A \cong K_1 \oplus C_K = K$. Similarly we get that $H \cong K$ and so $H \cong G$. Therefore G is minimal, by III, Theorem 1.6. □

We finish this section with some consequences of Theorem 2.2.

Corollary 2.3 *A direct sum of cyclic p-groups is minimal if and only if it is of the form* $\bigoplus_{I_1} \mathbb{Z}(p) \oplus \bigoplus_{I_2} \mathbb{Z}(p^2) \oplus \cdots \oplus \bigoplus_{I_n} \mathbb{Z}(p^n) \oplus \cdots$ *where* I_n *is infinite for all* $n < \omega$ *or there exists* n *such that* I_j *is infinite for all*

$j < n$ and I_j is empty for all $j \geq n$.

Proof: If G is a direct sum of cyclic p-groups, then $f_n(G)$ is the number of copies of $\mathbb{Z}(p^{n+1})$ for each $n \geq 0$ and now the result is immediate from Theorem 2.2. □

Corollary 2.4 *A direct sum of minimal p-groups is minimal.*

Proof: If $G = \bigoplus_{i \in I} G_i$ where each G_i is a minimal p-group, then $f_n(G)$ is infinite if some $f_n(G_i)$ is infinite since Ulm invariants are additive. □

Note that a summand of a minimal p-group need not be minimal since if A is minimal, then $f_n(A \oplus B)$ is infinite if $f_n(A)$ is, for all p-groups B. Also note that the only Ulm invariants involved in the characterisation are the finite ones. This is reflected in the following corollary.

Corollary 2.5 *A reduced p-group G is minimal if and only if $G/p^{\omega}G$ is minimal.*

Proof: Lemma 1.1 shows that $f_n(G) = f_n(G/p^{\omega}G)$ for all $n < \omega$ and now the result follows trivially from Theorem 2.2. □

§3 Purely Minimal p-Groups

Having characterised the minimal p-groups, following the same procedure as in the quasi-minimal case, we now consider the class of purely minimal p-groups.

Definition 3.1 *A group G is* purely minimal *if G is isomorphic to all its pure subgroups of finite index.*

Note that if $H \leq_* G$ and G/H is finite, then G/H is a direct sum of cycles, and so $H \sqsubset G$, by I, Lemma 1.10. Therefore G is purely minimal if and only if G is isomorphic to all its direct summands of finite index, i.e. G is *directly minimal.*

We do not, however, have a corresponding reduction to subgroups of index p, in the case of purely minimal groups, as the following example shows:

Example $G = \bigoplus_{\aleph_0} \mathbb{Z}(p) \oplus \mathbb{Z}(p^2)$ is obviously not purely minimal; however, if $H \leq_* G$ of index p, then $H = \bigoplus_I \mathbb{Z}(p) \oplus \bigoplus_J \mathbb{Z}(p^2)$ where $|I| = \aleph_0$ and $|J| \leq 1$. Now, $G \cong H \oplus \mathbb{Z}(p)$ and so $|J| = 1$, as a consideration of the Ulm invariants will show. Thus $G \cong H$.

Lemma 3.2 *If a reduced p-group G is purely minimal and $f_n(G) \neq 0$ for some $n < \omega$ then $f_n(G)$ is infinite.*

Proof: The same arguments as in the proof of Proposition 1.5. $\square$

Theorem 3.3 *A reduced p-group G is purely minimal if and only if whenever $f_n(G) \neq 0$ then $f_n(G)$ is infinite.*

Proof: The necessity is given by the previous lemma.

For sufficiency, let $G = H \oplus A$, where A is finite. If $f_n(G)$ is infinite, then so is $f_n(H)$ and if $f_n(G) = 0$, then so are $f_n(H)$ and $f_n(A)$. Let B_H be a basic subgroup of H. For any n such that $f_n(A) \neq 0$ we have that $f_n(G)$ is infinite and so $f_n(B_H) = f_n(H)$ is also infinite. If $A = \bigoplus_{I_1} \mathbb{Z}(p^{n_1}) \oplus \bigoplus_{I_2} \mathbb{Z}(p^{n_2}) \oplus \cdots \oplus \bigoplus_{I_r} \mathbb{Z}(p^{n_r})$ where $I_1, I_2, \ldots I_r$ are finite, then B_H, and hence H, has a summand $C_H = \bigoplus_{J_1} \mathbb{Z}(p^{n_1}) \oplus \bigoplus_{J_2} \mathbb{Z}(p^{n_2}) \oplus \cdots \oplus \bigoplus_{J_r} \mathbb{Z}(p^{n_r})$ where $J_1, J_2, \ldots J_r$ are infinite, $H = H_1 \oplus C_H$, say. Thus $G = H \oplus A = H_1 \oplus C_H \oplus A \cong H_1 \oplus C_H = H$, and hence G is purely minimal. $\square$

Note that a minimal p-group is obviously purely minimal but the converse is not necessarily true as we see from the following example:

Example The group $G = \bigoplus_{\aleph_0} \mathbb{Z}(p) \oplus \bigoplus_{\aleph_0} \mathbb{Z}(p^3)$ is purely minimal but not minimal by Theorem 3.3 and Theorem 2.2.

§4 Weakly Minimal p-Groups

The concept of a weakly minimal group is a natural analogue to the concept of a weakly quasi-minimal group introduced in Chapter II. In this case, however, the weakly minimal condition gives us a larger class of groups.

Definition 4.1 *A group G is* weakly minimal *if, whenever H is a finite index subgroup of G, then G is isomorphic to a finite index subgroup of H.*

If we define a group G to be p-weakly minimal if every subgroup of index p in G contains a finite index copy of G, then, for weakly minimal groups, we do get a reduction to the prime case.

Theorem 4.2 *A group G is weakly minimal if and only if it is p-weakly minimal for all primes p.*

Proof: If G is weakly minimal, then it is obviously p-weakly minimal for all p. Conversely, suppose G is p-weakly minimal for all p. As in

III, Theorem 1.6, the proof is by induction. If $|G/H| = 1$ then $G = H$. As a basis for the induction assume that every subgroup of G of index less than some n contains a finite index copy of G and let $|G/H| = n$. If n is a prime, then H has a finite index copy of G, by assumption. If n is not prime, then $n = r \cdot s$ where $r, s < n$. As before there exists $G_1 \leq G$, containing H, such that $|G/G_1| = r$ and $|G_1/H| = s$. The induction hypothesis tells us that there exists some $K \leq G_1$ of finite index with $K \cong G$. If $K \leq H$, then we are finished. If $K \not\leq H$, then consider $K + H \leq G_1$. Since $K/K \cap H \cong (K + H)/H$ we have that $|K/K \cap H| = |(K + H)/H| \leq |G_1/H| = s$. Now, $K \cong G$ implies that $K \cap H$ contains a finite index subgroup which is isomorphic to G and thus so also does H since $H/K \cap H$ is finite. □

The corresponding results to Proposition 1.4 and Proposition 1.5 in the weakly minimal case are given in the following lemma.

Lemma 4.3 *If a reduced p-group is weakly minimal, then its Ulm invariants satisfy one of the following*

(i) *if* $f_n(G) \neq 0$ *and* $f_i(G) = 0$ *for all* $i > n$, *then* $f_n(G)$ *is infinite;*

(ii) *if* $f_n(G) \neq 0$ *for infinitely many* n, *then* $f_n(G)$ *is infinite for infinitely many* n.

Proof: **(i)** To prove (i) let $f_n(G) \neq 0$ and $f_i(G) = 0$ for all $i > n$. Assume for contradiction that $f_n(G)$ is finite. Then $G = G_1 \oplus \bigoplus_{I_n} \mathbb{Z}(p^{n+1})$ where $f_n(G) = |I_n|$ is finite, and $f_n(G_1) = 0$. Now, $|G/G_1|$ is finite and G is weakly minimal and so there exists some G_2 of finite index in G_1 such that $G_2 \cong G$. By Theorem 2.1 there exists K such that $G_1 = K \oplus A, G_2 = K \oplus B$ where $B \leq A$ and A is finite. Thus $0 = f_n(G_1) = f_n(K) + f_n(A)$, so $f_n(K) = f_n(A) = 0$ and $f_i(G_1) = 0$ for all $i > n$ gives $f_i(K) = f_i(A) = 0$ for all $i > n$. Therefore the finite group A has no summand of order $\geq p^{n+1}$ and so neither has B. Hence $f_n(B) = 0$ and so $f_n(G_2) = 0$, a contradiction, since $G_2 \cong G$.

(ii) Suppose $f_i(G) \neq 0$ for infinitely many i. Assume $f_i(G)$ is finite for all i greater than some n, and choose $m > n$. Then we have $f_m(G)$ is finite and so $G = G_1 \oplus \bigoplus_{I_m} \mathbb{Z}(p^{m+1})$ where $|I_m| = f_m(G)$ is finite and $f_m(G_1) = 0$. There exists some G_2 of finite index in G_1 such that $G_2 \cong G$. Again, by Theorem 2.1, there exists K such that $G_1 = K \oplus A, G_2 = K \oplus B$ with $B \leq A$ and A finite. Then $0 = f_m(G_1) = f_m(K) + f_m(A)$ and so $f_m(K) = f_m(A) = 0$. Also $f_m(G_2) = f_m(K) + f_m(B)$ and so $|I_m| = f_m(G_2) = f_m(K) + f_m(B) = f_m(B)$. Moreover, for all $i > m$, we have $f_i(G) = f_i(G_1)$ and $f_i(G) = f_i(G_2)$ and so $f_i(A) = f_i(B)$. Now $f_m(A) = 0, f_m(B) \neq 0$ and $f_i(A) = f_i(B)$ for all $i > m$ and hence B

must have more elements of order greater than p^m than A, contradicting $B \leq A$. □

As in Theorem 2.2 the following theorem characterises the weakly minimal p-groups in terms of their Ulm invariants.

Theorem 4.4 *A reduced p-group G is weakly minimal if and only if its Ulm invariants satisfy one of the following*

(i) *if $f_n(G) \neq 0$ and $f_i(G) = 0$ for all $i > n$, then $f_n(G)$ is infinite;*

(ii) *if $f_n(G) \neq 0$ for infinitely many n, then $f_n(G)$ is infinite for infinitely many n.*

Proof: If G is weakly minimal the necessity of the conditions is given by the previous lemma.

Conversely, suppose the Ulm invariants of G satisfy (i) or (ii). Let H be of index p in G. Then there exists K such that $G = K \oplus A, H = K \oplus B$ with $B \leq A$ and A cyclic.

(i) Suppose that $A = \mathbb{Z}(p^r)$ for some $r \leq n+1$. Since $f_n(G) = f_n(K) + f_n(A)$ implies that $f_n(K)$ is infinite, we get that $K = K_1 \oplus \bigoplus_I \mathbb{Z}(p^{n+1})$, where $|I|$ is infinite. Therefore $K = K_1 \oplus \mathbb{Z}(p^{n+1}) \oplus \bigoplus_J \mathbb{Z}(p^{n+1})$ where $|J| = |I|$. Hence K has a subgroup of finite index which is isomorphic to

$K_1 \oplus A \oplus \bigoplus_I \mathbb{Z}(p^{n+1}) = K \oplus A = G$. A finite index subgroup of K is of finite index in H since $H = K \oplus B$, with B finite, and so G is isomorphic to a finite index subgroup of H.

(ii) Let $G = K \oplus A$ where A is as in (i), for some n. By assumption we can choose $m > n$ such that $f_m(G)$ is infinite. Then $f_m(K)$ is infinite and so $K = K_1 \oplus \bigoplus_I \mathbb{Z}(p^{m+1})$ with $|I|$ infinite. As above, K has a finite index subgroup which is isomorphic to $K_1 \oplus A \oplus \bigoplus_I \mathbb{Z}(p^{m+1}) = K \oplus A = G$ and which is of finite index in H. □

Note that Theorem 3.3 and Theorem 4.4 tell us that a purely minimal p-group is weakly minimal but again the converse is not necessarily true as the next example shows:

Example The group $G = \bigoplus_{\aleph_0} \mathbb{Z}(p) \oplus \bigoplus_I \mathbb{Z}(p^2) \oplus \bigoplus_{\aleph_0} \mathbb{Z}(p^3)$, where I is finite, is weakly minimal but not purely minimal by Theorem 4.4 and Theorem 3.3.

§5 Weakly Purely Minimal p-Groups

To conclude our discussion of minimal p-groups, just as in Chapter II, we relax the minimality condition in both the weak and the pure directions and we now consider weakly purely minimal groups.

Definition 5.1 *A group G is* weakly purely minimal *if whenever H is a pure subgroup of finite index of G, then G is isomorphic to a pure subgroup of finite index of H.*

Again, in this case, we have an analogue of Proposition 1.5.

Lemma 5.2 *If a reduced p-group G is weakly purely minimal and $f_n(G) \neq 0$, for some $n < \omega$, then $f_n(G)$ is infinite.*

Proof: As in Proposition 1.5, $G = \mathbb{Z}(p^{n+1}) \oplus G_1$. Since G/G_1 is finite we get that $G_1 = K \oplus A$ for some finite A and some $K \cong G$. Then $f_n(G) = 1 + f_n(G_1) = 1 + f_n(K) + f_n(A) = 1 + f_n(G) + f_n(A)$ and so $f_n(G)$ must be infinite. □

As the next theorem shows, we get nothing new in the weakly purely minimal case, since every weakly purely minimal p-group is in fact purely minimal.

Theorem 5.3 *A reduced p-group G is weakly purely minimal if and only if G is purely minimal.*

Proof: If G is weakly purely minimal, then Lemma 5.2 and Theorem 3.3 tell us that G is purely minimal. The converse is obvious from the definitions. □

Having obtained a complete characterisation of minimal p-groups we now proceed, in the final section of this chapter, to see how this can be used to investigate mixed minimal groups.

§6 Mixed Minimal Groups

We now consider mixed minimal groups. First we have some results on mixed minimal groups in general and then we concentrate on mixed groups of torsion-free rank 1.

Theorem 6.1 *Let G be a mixed group. If G is minimal, then both tG and $G/t(G)$ are minimal where $t(G)$ is the torsion subgroup of G.*

Proof: First we show that $t(G)$ is minimal. Let S be of finite index in tG. Then $G/tG \cong (G/S)/(tG/S)$ so $(G/S)/(tG/S)$ is torsion-free and hence $tG/S \leq_* G/S$. But tG/S is finite, especially it is bounded, and hence $tG/S \sqsubset G/S$, $G/S = tG/S \oplus K/S$, say. Now, $G/K \cong (G/S)(K/S) \cong tG/S$ is finite and so $K \cong G$ since G is minimal. Therefore $tG \cong tK$. Also $K/S \cong (G/S)/(tG/S) \cong G/tG$ is torsion-free, so $S = tK$, i.e. $S \cong tG$ as required.

It remains to show that $G/t(G)$ is minimal. Let H/tG be of finite index in G/tG. Then $G/H \cong (G/tG)/(H/tG)$ is finite and so $G \cong H$, since G is minimal. Therefore $G/tG \cong H/tH$. But $tH = H \cap tG = tG$ since $tG \leq H$ and so $G/tG \cong H/tG$. □

The next theorem shows that if G splits then the converse is true. First we prove a lemma, due to Procházka [22], but the proof given here is based on Theorem 2.1.

Lemma 6.2 *Let G be a mixed group and H a finite index subgroup of G. Then, if G splits, so does H.*

Proof If G/H is finite, then $tG/tH = tG/(H \cap tG) \cong (tG + H)/H \leq G/H$, so tG/tH is finite. Now, $tG = \bigoplus_{p\in\Pi} G_p$, $tH = \bigoplus_{p\in\Pi} H_p$, and $tG/tH \cong \bigoplus_{p\in\Pi}(G_p/H_p)$, so $H_p = G_p$ for almost all p. Suppose $H_p \neq G_p$ for $p = p_1, p_2, \ldots, p_n$ and we have equality for all other p. For each $p_i (i = 1, 2, \ldots, n)$, by Theorem 2.1, there exists $K_{p_i} \leq H_{p_i}$ such that $G_{p_i} = K_{p_i} \oplus A_{p_i}$ and $H_{p_i} = K_{p_i} \oplus B_{p_i}$, where $B_{p_i} \leq A_{p_i}$ and A_{p_i} is finite. Therefore $tG = \bigoplus_{p\neq p_1,\ldots,p_n} G_{p_i} \oplus \bigoplus_{i=1}^{n}(K_{p_i} \oplus A_{p_i}) = K \oplus \bigoplus_{i=1}^{n} A_{p_i}$ and $tH =$

$\bigoplus_{p \neq p_1,\ldots,p_n} G_{p_i} \oplus \bigoplus_{i=1}^{n}(K_{p_i} \oplus B_{p_i}) = K \oplus \bigoplus_{i=1}^{n} B_{p_i}$ where $K = \bigoplus_{p \neq p_1,\ldots,p_n} G_{p_i} \oplus \bigoplus_{i=1}^{n} K_{p_i}$. Now, G splits, by assumption, and so $G = tG \oplus G'$ where G' is torsion-free, i.e. $G = K \oplus A \oplus G'$ where $A = \bigoplus_{i=1}^{n} A_{p_i}$. Hence $H = H \cap G = K \oplus H \cap (A \oplus G') = K \oplus H'$ where $H' = H \cap (A \oplus G')$. But $t(H')$ is finite and so it is a summand of H'. Hence $H = K \oplus t(H') \oplus \overline{H}$ where $\overline{H}$ is torsion-free and so H also splits. □

Using Lemma 6.2 we can now establish the following partial converse to Theorem 6.1.

Theorem 6.3 *If G splits and tG and G/tG are both minimal, then G is minimal.*

Proof: Let H be of finite index in G. Then, as in Lemma 6.2, tH is of finite index in tG and thus $tG \cong tH$. Also $(G/tG)/((tG+H)/tG) \cong G/(tG+H) \cong (G/H)/((tG+H)/H)$ again is finite and so $G/tG \cong (tG+H)/tG \cong H/(H \cap tG) = H/tH$. Now, if G splits, then so does every finite index subgroup of G, by Lemma 6.2. Hence, if $G = tG \oplus K$ and $H = tH \oplus L$, say, then $K \cong G/tG \cong H/tH \cong L$ and $tG \cong tH$. Therefore $G \cong H$. □

The second part of Theorem 6.1 and the necessary part of III, Theorem 1.4 are examples of a more general result, concerning the idea of a preradical. To see this, following Charles [4], we recall the definition of a preradical.

Definition 6.4 *Let Ab denote the class of all abelian groups. A functor* $R : Ab \longrightarrow Ab$ *is a* preradical *if, for all* $A, B \in Ab$ *and for all homomorphisms* $\phi : A \longrightarrow B$*, we have* $RA \leq A$ *and* $(RA)\phi \leq RB$*.*

Note that it is conventional to write the preradical on the left. Note also that if R is a preradical then $R(RA) \leq RA$ for all A from the above definition.

Definition 6.5 *A preradical* R *is a* socle preradical *if* $R(RA) = RA$ *for all* A*.*

Obvious examples of socle preradicals are

(i) $R : Ab \longrightarrow Ab$ where $RA = t(A)$, the torsion subgroup of A, and

(ii) $R : Ab \longrightarrow Ab$ where $RA = D$, the maximal divisible subgroup of A.

Theorem 6.6 *If* $G \in Ab$ *is minimal and* R *is a socle preradical, then* G/RG *is also minimal.*

Proof: Let H/RG be of finite index in G/RG. Then H is of finite index in G and so $G \overset{\phi}{\cong} H$. We have $(RG)\phi \leq RH$ and $(RH)\phi^{-1} \leq RG$. Therefore $(RH)\phi^{-1}\phi \leq (RG)\phi$, i.e. $RH \leq (RG)\phi$ and so $(RG)\phi = RH$. Hence $G/RG \cong H/RH$. But $i : H \longrightarrow G$ is a homomorphism, so $(RH)i \leq RG$, i.e. $RH \leq RG$. Also $RG \leq H$, by assumption, so $R(RG) \leq RH$ (consider $i : RG \longrightarrow H$) and $R(RG) = RG$, so $RG \leq RH$. We have $RG = RH$ and hence $G/RG \cong H/RG$. □

Lemma 6.7 *Let χ be any subset of Ab. Then S_χ, defined by $S_\chi(A) = \sum\{\mathrm{Im}(\phi) : \phi \in \mathrm{Hom}(X, A), X \in \chi\}$, for all $A \in$ Ab, is a socle preradical. We write $S_G(A)$ for $S_\chi(A)$ if $\chi = \{G\}$.*

Proof: $S_\chi(A) \leq A$ for all A, by definition. If $\psi : A \longrightarrow B$ is a homomorphism, then $\phi\psi : X \longrightarrow B$ is a homomorphism for all $\phi : X \longrightarrow A$ and so $(S_\chi(A))\psi \leq S_\chi(B)$. It is obvious that $S_\chi(S_\chi(A)) = S_\chi(A)$. □

Examples (i) Let $G = \mathbb{Z}/p\mathbb{Z}$. Then $S_G(A) = A[p]$ is the p-socle of A. If A is minimal, then so is $A/A[p]$ for all p. If A is a p-group, then this is obvious by considering Ulm invariants and the characterisation of minimal p-groups, since $f_n(A/A[p]) = f_{n+1}(A)$ for all $n \geq 0$.

(ii) Let $G = \bigoplus_n \mathbb{Z}/p^n\mathbb{Z}$. Then $S_G(A) = A_p$ is the p-torsion subgroup of

A. If A is minimal, then so is A/A_p for all p.

(iii) Let $G = \bigoplus_{n,p} \mathbb{Z}/p^n\mathbb{Z}$. Then $S_G(A) = tA$ and we just get a restatement of the second part of Theorem 6.1.

We shall consider the socle preradical $S_G(A)$ where G is a rank 1 torsion-free group in the next chapter.

For the remainder of this section we concentrate on mixed groups of torsion-free rank 1. First we need the idea of the generalized p-height of an element in a group and some associated lemmas. To this end let A be an arbitrary group and p a prime.

Definition 6.8 *If $a \in A$ the* generalized p-height *of a in A, $h_p^{*A}(a)$ or just $h_p^*(a)$, if A is understood, is defined by $h_p^*(a) = \sigma$ if $a \in p^\sigma A \setminus p^{\sigma+1}A$ where $p^\sigma A$ is defined inductively as for p-groups A. If $p^\tau A = p^{\tau+1}A$, i.e. $p^\tau A$ is p-divisible, and $a \in p^\tau A$, we set $h_p^*(a) = \infty$ and we consider ∞ larger than every ordinal.*

Lemma 6.9 *If $a \in A$ and $n \in \mathbb{Z}$ and if p is a prime such that p does not divide n, then $h_p^*(a) = h_p^*(na)$.*

Proof: If $a \in p^\sigma A$, then $na \in p^\sigma A$, so $h_p^*(na) \geq h_p^*(a)$. If p does not divide n, then $(p,n) = 1$ so there exists $r, s \in \mathbb{Z}$ such that $rp + sn = 1$. Therefore $a = rpa + sna$ and so $h_p^*(a) \geq \min(h_p^*(rpa), h_p^*(sna))$; now, if $h_p^*(a) = \sigma$, then $h_p^*(rpa) \geq \sigma + 1$ and if $h_p^*(na) > \sigma$ we get $h_p^*(sna) > \sigma$ and so $\sigma \geq \sigma + 1$, a contradiction. □

Lemma 6.10 *For all $a \in A$ and for all $n \in \mathbb{N}, h_p^{*A}(a) \leq h_p^{*nA}(na)$.*

Proof: If $a \in p^\sigma A$, then $na \in np^\sigma A$, so it suffices to show $np^\sigma A \leq p^\sigma(nA)$ for all σ. We use transfinite induction on σ. It is obviously true for all $\sigma < \omega$. Suppose that $np^\sigma A \leq p^\sigma(nA)$ for all $\sigma < \rho$ and first consider $\rho = \tau + 1$. Then $np^\rho A = npp^\tau A = pnp^\tau A \leq pp^\tau(nA) = p^{\tau+1}(nA)$, by the induction hypothesis. If ρ is a limit ordinal, then $np^\rho A = n \bigcap_{\sigma<\rho} p^\sigma A \leq \bigcap_{\sigma<\rho} np^\sigma A \leq \bigcap_{\sigma<\rho} p^\sigma(nA) = p^\rho(nA)$, also by the induction hypothesis. □

Again let A be an arbitrary group. Each element $a \in A$ has an associated matrix, called the height-matrix of a, defined in the following way.

Definition 6.11 *The height-matrix of a is the $\omega \times \omega$ matrix (whose entries are ordinals or ∞s) $H(a) = (\sigma_{nk})$ where $\sigma_{nk} = h_{p_n}^*(p_n^k a)$ and*

$n = 1, 2, \ldots, k = 0, 1, \ldots$ *and the primes are arranged in order of magnitude.*

Definition 6.12 *Two $\omega \times \omega$ matrices, σ_{nk} and ρ_{nk}, whose entries are ordinals or ∞s, are equivalent if almost all their rows are identical and for each of the other rows there exist integers $l, m \geq 0$ such that $\sigma_{n,k+l} = \rho_{n,k+m}$ for all $k \geq 0$.*

It is easily seen that this is an equivalence relation on the class of such matrices.

Now, if A is a mixed group of torsion-free rank 1 and $a, b \in A$ are any two torsion-free elements, then there exist some $r, s \in \mathbb{Z}$ such that $ra = sb$. If p_n is any prime such that p_n does not divide rs, then the n^{th} rows of $H(a)$ and $H(b)$ are the same, by Lemma 6.9. If p_n divides rs, i.e. $r = p_n^l r_1, s = p_n^m s_1$ where $(p_n, r_1) = 1 = (p_n, s_1)$, then $h^*_{p_n}(ra) = h^*_{p_n}(sb)$, so $h^*_{p_n}(p_n^l a) = h^*_{p_n}(p_n^m b)$, again by Lemma 6.9, i.e. $\sigma_{nl} = \rho_{nm}$, where $H(a) = (\sigma_{nk})$ and $H(b) = (\rho_{nk})$. Similarly, for any $k > 0$, $r_1 p_n^{l+k} a = s_1 p_n^{m+k} b$ and so $h^*_{p_n}(p_n^{k+l} a) = h^*_{p_n}(p_n^{k+m} b)$, i.e. $\sigma_{n,k+l} = \rho_{n,k+m}$. Therefore $H(a)$ and $H(b)$ are equivalent and we denote this equivalence class simply as $H(A)$.

We now state a theorem which gives a condition, in terms of height matrices, for the isomorphism of groups in large classes of mixed groups of torsion-free rank 1 and we then apply this theorem to investigate minimality in these classes of groups.

Theorem 6.13 *Let A and C be countable mixed groups of torsion-free rank 1. Then $A \cong C$ if and only if*

(i) $t(A) \cong t(C)$ *and*

(ii) $H(A) = H(C)$.

Proof: See [11, Theorem 104.3]. □

This theorem can be extended ([29], [19]) to the cases where

(a) $t(A)$ and $t(C)$ are totally projective

(b) $t(A)$ and $t(C)$ are torsion-complete.

We first apply this result to prove a theorem concerning the minimality of local mixed groups of torsion-free rank 1 with divisible torsion-free quotient and then we use it to consider minimality in classes of mixed groups of torsion-free rank 1 whose height matrices have rows which are eventually gap-free.

For the rest of this section we will assume that G is a reduced mixed group of torsion-free rank 1 such that $t(G)$ belongs to one of the classes described in (a) or (b) above.

Theorem 6.14 *Let $t(G)$ be a p-group for some prime p and let $G/t(G) \cong \mathbb{Q}$. Then G is minimal if and only if tG is minimal, i.e. $f_n(G)$ is infinite for all $n < \omega$ or there exists $\lambda < \omega$ such that $f_n(G)$ is infinite for all $n < \lambda$ and $f_n(G) = 0$ for all $n \geq \lambda$, where $f_n(G)$ is the n^{th} Ulm invariant of $t(G)$.*

Proof: If G is minimal then $t(G)$ is minimal, by Theorem 6.1. Conversely, suppose that $t(G)$ is minimal. By III, Theorem 1.6 it suffices to consider subgroups of G of prime index. First, if A is a subgroup of G of some prime index $q \neq p$, then $t(G)/t(A) = t(G)/A \cap t(G) \cong (t(G) + A)/A \leq G/A$. Therefore $t(G) = t(A)$ since $t(G)/t(A)$ is a p-group and $G/A \cong \mathbb{Z}(q)$. Now, $G/A \cong (G/t(G))/(A/t(G))$ is both divisible and finite and so must be 0, a contradiction. Hence we need only consider subgroups A of G of index p. Then $pG \leq A$ and $t(G) \cong t(A)$ since $t(G)$ is minimal. Next note that if $t(G) \leq A$, then, as above, G/A is both divisible and finite and so must be 0. Therefore $t(G) \not\leq A$ and $(A+t(G))/A$ is a non-zero subgroup of G/A and so $(A+t(G))/A = G/A$, since G/A is a simple group. Hence there exists some $x \in t(G) \setminus A$ such that $G = \langle A, x \rangle$. If $g \in G$ is any element in G, then $g = a + rx$ for some $a \in A$ and some $0 \leq r < p$ and, since $t(G)$ is a p-group, there exists some $k \in \mathbb{N}$ such that $p^k x = 0$ and so $p^k g = p^k a \in A$ and therefore $p^k G = p^k A$. Now let $g \in G$ be any torsion-free element in G. Then $g_1 = p^k g \in p^k G = p^k A$. For all $n \geq 0$ we have $h_p^{*G}(p^n g_1) \geq h_p^{*A}(p^n g_1)$ and if $h_p^{*G}(p^n g_1) > h_p^{*A}(p^n g_1) = \sigma$, say, where $\sigma \geq k$, then $p^n g_1 = p g_2$ for some $g_2 \in p^\sigma G = p^\sigma A$ so $p^n g_1 = p a_1$ for some $a_1 \in p^\sigma A$, a contradiction to $h_p^{*A}(p^n g_1) = \sigma$. Therefore $h_p^{*G}(p^n g_1) = h_p^{*A}(p^n g_1)$ for all $n \geq 0$. If $q \neq p$, then $h_q^{*G}(q^n g_1) \leq h_q^{*pG}(pq^n g_1) \leq h_q^{*A}(pq^n g_1) = h_q^{*A}(q^n g_1)$ for all $n \geq 0$, by Lemma 6.9 and Lemma 6.10, and the converse inequality is true since $A \leq G$. We conclude that $H^G(g_1)$ is equivalent to $H^A(g_1)$ and so $H(A) = H(G)$. Hence $A \cong G$, by Theorem 6.13, and it follows that G is minimal, by III, Theorem 1.6. Now, Theorem 2.2 completes the proof. □

In our next result we do not need to assume that the torsion subgroup is a p-group.

Theorem 6.15 *Suppose that each row of $H(G)$ is eventually gap-free, i.e. given any torsion-free element a in G, for each prime p there exists some n_p such that $h_p^{*G}(p^{k+1}a) = h_p^{*G}(p^k a) + 1$ for all $k \geq n_p$. Then G*

is minimal if and only if tG is minimal, i.e. for each prime p, $f_n^p(G)$ *is infinite for all* $n < \omega$ *or there exists* $\lambda_p < \omega$ *such that* $f_n^p(G)$ *is infinite for all* $n < \lambda_p$ *and* $f_n^p(G) = 0$ *for all* $n \geq \lambda_p$*, where* $f_n^p(G)$ *is the* n^{th} *Ulm invariant of the p-primary component of G.*

Proof: If G is minimal, then $t(G)$ is minimal, again by Theorem 6.1. Conversely, suppose that $t(G)$ is minimal. As in Theorem 6.14 it suffices to consider a subgroup A of G of prime index p in G for any prime p. Then $pG \leq A$ and $t(A) \cong t(G)$, since $t(G)$ is minimal. Now, A contains torsion-free elements since if $A = tA \leq tG$ we have $G/tG \cong (G/tA)/(tG/tA) = (G/A)/(tG/A)$ which is finite, a contradiction. So A is also mixed of torsion-free rank 1. Let $a \in A$ be torsion-free and first consider $q \neq p$. Then, for any k, $h_q^{*G}(q^k a) \leq h_q^{*pG}(pq^k a) \leq h_q^{*A}(pq^k a) = h_q^{*A}(q^k a)$, using Lemma 6.9 and Lemma 6.10, and, since $A \leq G$, we have $h_q^{*A}(q^k a) \leq h_q^{*G}(q^k a)$. We conclude that $h_q^{*A}(q^k a) = h_q^{*G}(q^k a)$. Now suppose that $H^G(a)$ has no gaps after the k^{th} entry in the row corresponding to p. Then $h_p^{*G}(p^k a) \leq h_p^{*pG}(p^{k+1} a) \leq h_p^{*A}(p^{k+1} a) \leq h_p^{*G}(p^{k+1} a) \leq h_p^{*pG}(p^{k+2} a) \leq h_p^{*A}(p^{k+2} a) \leq h_p^{*G}(p^{k+2} a) \leq \ldots$. So, if $H^G(a) = (\sigma_{nm})$ and $H^A(a) = (\rho_{nm})$, then we get that $\sigma_{pk} \leq \rho_{p,k+1} \leq \sigma_{p,k+1} \leq \rho_{p,k+2} \leq \sigma_{p,k+2} \leq \ldots$ and, since both sequences are strictly increasing and $\sigma_{pk}, \sigma_{p,k+1}, \sigma_{p,k+2} \ldots$ has no gaps, we get that $H^G(a)$ and $H^A(a)$ are equivalent and so $H(A) = H(G)$ and hence $A \cong G$, by Theorem 6.13. Therefore G is minimal and now Theorem 2.2 again completes the proof. □

If G is any mixed group of torsion-free rank 1 Megibben [19] has observed that the rank 1 torsion-free group $G/t(G)$ can be recovered from $H(G)$. If (σ_{ij}) is any matrix in $H(G)$, then we define a sequence $(k_1, k_2, \ldots, k_n, \ldots)$ as follows:

$k_n = \infty$ if the n^{th} row of (σ_{ij}) contains an infinite ordinal or the symbol ∞ or has infinitely many gaps,

$k_n = \sigma_{nj} - j$ if the n^{th} row of (σ_{ij}) contains only integers and has no gaps after $\sigma_{n,j-1}$.

Then it is not difficult to see that $(k_1, k_2, \ldots, k_n, \ldots)$ is the characteristic of some element in $G/t(G)$ and hence determines $G/t(G)$.

If $t(G)$ is a p-group for some prime p, then let $p_n \neq p$ be any other prime and let $0 \neq g + t(G) = \overline{g} \in \overline{G} = G/t(G)$ have characteristic $(k_1, k_2, \ldots, k_n, \ldots)$. We have $h^{*G}_{p_n}(g) \leq h^{*\overline{G}}_{p_n}(\overline{g}) = k_n$.

If k_n is finite, then $\overline{g} = p_n^{k_n}\overline{g_1}$ and $g = p_n^{k_n} g_1 + t$, where $g_1 \in G$ and t is torsion, so $p^s g = p^s p_n^{k_n} g_1$, for some s, and hence $h^*_{p_n}(g) = h^*_{p_n}(p_n^{k_n} g_1) \geq k_n$. Therefore $h^*_{p_n}(g) = k_n$. Similarly, $h^{*G}_{p_n}(p_n^k g) = h^{*\overline{G}}_{p_n}(p_n^k \overline{g}) = k_n + k$ for all

$k \in \mathbb{N}$.

Now consider $k_n = \infty$. First note that $p_n^{\omega}\overline{G} = p_n^{\infty}\overline{G}$, since $\overline{G}$ is torsion-free. We show, by transfinite induction, that whenever $\overline{g} \in p_n^{\sigma}\overline{G}$ then $g \in p_n^{\sigma}G$. If $\sigma < \omega$, then it has been proved above. Assume that it is true for all ordinals $< \sigma$ and first consider $\sigma = \rho + 1$. Then if $\overline{g} \in p_n^{\sigma}\overline{G}$ we have $\overline{g} = p_n\overline{g_1}$, where $\overline{g_1} \in p_n^{\rho}\overline{G}$. If $\overline{g_1} = g_1 + t(G)$, then the induction hypothesis tells us that $g_1 \in p_n^{\rho}G$. But $g = p_n g_1 + t$, where $p^s t = 0$, for some $s \in \mathbb{N}$, so $h^*_{p_n}(g) \geq h^*_{p_n}(g_1) + 1 \geq \rho + 1 = \sigma$. If σ is a limit ordinal, then $\overline{g} \in p_n^{\rho}\overline{G}$ for all $\rho < \sigma$ so, again appealing to the induction hypothesis, we get that $g \in p_n^{\rho}G$ for all $\rho < \sigma$ and so $g \in p_n^{\sigma}G$. Therefore, if $k_n = \infty$, we must have that $h_{p_n}(g) = \infty$.

We can conclude that if $t(G)$ is a p-group then $G/t(G)$ determines all the rows of $H(G)$ except the row corresponding to p.

We now use this fact to establish our final result in this section.

Theorem 6.16 *Suppose that $t(G)$ is a p-group and the row corresponding to p in $H(G)$ is eventually gap-free. Then G is minimal if and only if both $t(G)$ and $G/t(G)$ are minimal.*

Proof: If G is minimal, then both $t(G)$ and $G/t(G)$ are minimal, by Theorem 6.1. Conversely, suppose that $t(G)$ and $G/t(G)$ are minimal.

Let A be a subgroup of G of prime index. Then, as in Theorem 6.3, $t(A) \cong t(G)$ and $A/t(A) \cong G/t(G)$. Let $a \in A$ be torsion-free. We wish to show that $H^A(a)$ is equivalent to $H^G(a)$. By what has been said above we need only consider the row corresponding to p. If $|G/A| = q \neq p$, then, for any $k \in \mathbb{N}$, we have $h_p^{*G}(p^k a) \leq h_p^{*qG}(qp^k a) \leq h_p^{*A}(qp^k a) = h_p^{*A}(p^k a)$, and so $h_p^{*G}(p^k a) = h_p^{*A}(p^k a)$. Hence $H(A) = H(G)$ and $A \cong G$. If $|G/A| = p$, then, since the row corresponding to p has only a finite number of gaps, proceeding as in Theorem 6.15, we get that $H^A(a)$ is equivalent to $H^G(a)$, and again $A \cong G$. We conclude that G is minimal.

□

V Torsion-Free Minimal Groups

In this final chapter we consider torsion-free minimal groups. To begin we need some properties of the torsion-free rank and the p-rank of a torsion-free group

§1 Rank and p-Rank of Torsion-Free Groups

In I, Definition 1.1 we defined the torsion-free rank of any group. Now we restrict our interest to torsion-free groups.

Definition 1.1 *If G is a torsion-free group then*

(i) *The* (torsion-free) rank *of G, $r(G)$, is the cardinality of any maximal linearly independent subset of G or, equivalently, the dimension of $\mathbb{Q} \otimes G$ as a vector space over $\mathbb{Q}$.*

(ii) *If p is a prime the* p-rank *of G, $r_p(G)$, is defined to be the dimension of G/pG or, equivalently, the dimension of $\mathbb{Z}(p) \otimes G$, as a vector space over $\mathbb{Z}(p)$.*

Note that, if G is a mixed group, then $\mathbb{Q} \otimes G = \mathbb{Q} \otimes (G/tG)$ and so the torsion-free rank of G is the same as $r(G/tG)$.

The following lemma establishes some facts concerning rank and subgroups of a given group. The proof of (ii) is from [1, Theorem 0.2].

Lemma 1.2 *If H is a subgroup of G, then*

(i) $r(G) = r(H) + r(G/H)$;

(ii) $r_p(G) \leq r_p(H) + r_p(G/H)$;

(iii) *If H is a pure subgroup of G, then $r_p(G) = r_p(H) + r_p(G/H)$;*

(iv) *If H is a finite index subgroup of G, then $r(H) = r(G)$ and $r_p(H) = r_p(G)$;*

(v) *If $r(G)$ is finite, then $r_p(G) \leq r(G)$.*

Proof: **(i)** Consider the short exact sequence $0 \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} G/H \longrightarrow 0$, where i is inclusion and π is canonical projection. Since $\mathbb{Q}$ is torsion-free the sequence $0 \longrightarrow \mathbb{Q} \otimes H \xrightarrow{i'} \mathbb{Q} \otimes G \xrightarrow{\pi'} \mathbb{Q} \otimes G/H \longrightarrow 0$ is also exact where $i' = id \otimes i$ and $\pi' = id \otimes \pi$ are vector space homomorphisms. Hence $\dim(\mathbb{Q} \otimes G) = \dim(\mathrm{Ker}(\pi')) + \dim(\mathrm{Im}(\pi'))$ and so $r(G) = r(H) + r(G/H)$.

(ii) Again, consider the short exact sequence $0 \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} G/H \longrightarrow 0$. This induces the sequence

$H/pH \xrightarrow{\alpha} G/pG \xrightarrow{\beta} (G/H)/p(G/H) \longrightarrow 0$ where $(h + pH)\alpha = h + pG$ and $(g + pG)\beta = (g + H) + p(G/H)$. This sequence is exact since obviously $\mathrm{Im}(\alpha) \leq \mathrm{Ker}(\beta)$ and β is onto and if $g + pG \in \mathrm{Ker}(\beta)$ then $g + H = p(g_1 + H)$, for some $g_1 \in G$, so $g + pG = h + pG$, for some

$h \in H$, i.e. $g + pG \in \mathrm{Im}(\alpha)$. Let $K = \mathrm{Ker}(\alpha) = (H \cap pG)/pH$.

Then $0 \longrightarrow K \xrightarrow{i} H/pH \xrightarrow{\alpha} G/pG \xrightarrow{\beta} (G/H)/p(G/H) \longrightarrow 0$ is exact and the homomorphisms i, α, β are $\mathbb{Z}(p)$-vector space homomorphisms. Therefore $\dim(G/pG) = \dim(\mathrm{Ker}(\beta)) + \dim(\mathrm{Im}(\beta)) = \dim(\mathrm{Im}(\alpha)) + \dim(\mathrm{Im}(\beta)) \leq \dim(H/pH) + \dim((G/H)/p(G/H))$, i.e. $r_p(G) \leq r_p(H) + r_p(G/H)$.

(iii) If $H \leq_* G$, then $K = (H \cap pG)/pH = pH/pH = 0$ and so we get that $0 \longrightarrow H/pH \xrightarrow{\alpha} G/pG \xrightarrow{\beta} (G/H)/p(G/H) \longrightarrow 0$ is a short exact sequence and $\dim(\mathrm{Im}\alpha) = \dim(H/pH)$ and so the result follows from (ii).

(iv) If G/H is finite, then $r(G/H) = 0$ and so (i) gives $r(G) = r(H)$. In (ii) we claim $K \cong (G/H)[p]$, the p-socle of G/H; define $\phi : (G/H)[p] \longrightarrow K$ by $(c+H)\phi = pc + pH$ where $(G/H)[p] = C/H = \{c + H : pc \in H\}$. If $c - c' \in H$, then $pc - pc' \in pH$ so ϕ is well-defined. Obviously ϕ is a homomorphism and ϕ is injective since $(c+H)\phi = 0$ implies $pc \in pH$ and hence $c \in H$ since H is torsion-free. Also ϕ is surjective since if $pg + pH \in K$ where $pg \in H \cap pG$, then $(g + H)\phi = pg + pH$ and $g + H \in C/H$. Therefore $K \cong (G/H)[p] \cong (G/H)/p(G/H)$ since G/H is a finite group and this isomorphism is a vector space isomorphism. Now, from (ii), we get $\dim(H/pH) = \dim(\mathrm{Ker}(\alpha)) + \dim(\mathrm{Im}(\alpha)) = \dim K + \dim(\mathrm{Im}(\alpha))$ and $\dim(G/pG) = \dim(\mathrm{Ker}(\beta)) + \dim(\mathrm{Im}(\beta)) = \dim(\mathrm{Im}(\alpha)) + \dim K$ and hence $r_p(H) = r_p(G)$.

(v) Let $\{g_i + pG\}$ be a linearly independent set in G/pG where $i \in I$ with $|I| = r_p(G)$. If $\sum_{1 \leq i \leq n} n_i g_i = 0$ for $n_1, n_2 \cdots n_r \in \mathbb{Z}$, not all 0, and $g.c.d.(n_1, n_2, \ldots, n_r) = 1$, then $\sum_{1 \leq i \leq r} n_i(g_i + pG) = 0$, and so $n_i g_i \in pG$ for $1 \leq i \leq n$. Therefore p divides n_i for each $i = 1, 2, \ldots, n$, which is a contradiction. □

§2 Torsion-Free Complete Groups and Groups with p-Rank at Most 1

Our first result shows that any torsion-free group that is complete in its $\mathbb{Z}$-adic topology is minimal.

Theorem 2.1 *Let G be a torsion-free group that is complete in its $\mathbb{Z}$-adic topology. Then G is minimal.*

Proof: First note that G is reduced and algebraically compact, by I, Lemma 1.17. Now let H be of index p in G where p is any prime. Then $p(G/H) = 0$ and so $\bigcap_{n \in \mathbb{N}} n(G/H) = 0$ and hence H is also reduced algebraically compact (see [10, Corollary 39.2]). Therefore, by I, Lemma 1.18, $G = \prod_{q \in \Pi} G_q$ and $H = \prod_{q \in \Pi} H_q$ where each G_q and H_q is q-adically complete and $H_q = \bigcap_{k \in \mathbb{N},\, r \neq q} r^k H \leq \bigcap_{k \in \mathbb{N},\, r \neq q} r^k G = G_q$, for all q, where $r \in \Pi$. Since G_q and H_q are complete in their respective q-adic topologies we get that $G_q = \widehat{\bigoplus_m J_q}$ and $H_q = \widehat{\bigoplus_n J_q}$, for some cardinals m and n. We have $\mathbb{Z}(p) \cong G/H = \prod_q G_q / \prod_q H_q \cong \prod_q (G_q/H_q)$ and since each G_q is divisible by p, for $q \neq p$, we must have $G_q = H_q$ for all $q \neq p$ and $|G_p/H_p| = p$.

Now, $G_p/pG_p \cong \widehat{\bigoplus_m J_p}/p(\widehat{\bigoplus_m J_p}) \cong (\bigoplus_m J_p)/p(\bigoplus_m J_p) = (\bigoplus_m J_p)/(\bigoplus_m pJ_p) \cong \bigoplus_m (J_p/pJ_p) \cong \bigoplus_m \mathbb{Z}/p\mathbb{Z}$ and similarly $H_p/pH_p \cong \bigoplus_n \mathbb{Z}/p\mathbb{Z}$. Therefore $m = r_p(G_p) = r_p(H_p) = n$, by Lemma 1.2 (iv), and so $G_p \cong H_p$. Hence $G \cong H$ and now III, Theorem 1.6 tells us that G is minimal. □

Corollary 2.2 *$\prod_{p \in \Pi} J_p$ is minimal.*

Proof: $\prod_p J_p \cong \widehat{\mathbb{Z}}$, the $\mathbb{Z}$-adic completion of $\mathbb{Z}$ by I, Lemma 1.18. □

Now we consider groups G with $r_p(G) \leq 1$ for all p. The following theorem has been proved by Procházka [23] for the finite rank case.

Theorem 2.3 *If G is torsion-free such that $r_p(G) \leq 1$, for all primes p, then G is minimal.*

Proof: As before it suffices to consider H of index p in G where p is any prime. If $|G/H| = p$, then $pG \leq H$ and $G/H \cong (G/pG)/(H/pG)$. If $r_p(G) = 0$, then $H = G$, a contradiction, so we may assume that

$r_p(G) = 1$. Then $|G/H||H/pG| = |G/pG|$, i.e. $p|H/pG| = p$. Therefore $|H/pG| = 1$ and so $pG = H$, i.e. the only subgroup of index p in G is pG. Since G is torsion-free $pG \cong G$ and hence $H \cong G$. □

Corollary 2.4 *Suppose that A is torsion-free with $r_p(A) \leq 1$ for all p. If $B \leq_* A$, then both B and A/B are minimal.*

Proof: By Lemma 1.2 (iii) we have that $r_p(A) = r_p(B) + r_p(A/B)$ for all p. Therefore $r_p(B) \leq 1$ and $r_p(A/B) \leq 1$ for all p and hence both B and A/B are minimal. □

Lemma 2.5 *$r_p(J) \leq 1$ for all p where J denotes the group $\prod\limits_{p\in\Pi} J_p$.*

Proof: $J/qJ = (\prod\limits_{p\in\Pi} J_p)/(q\prod\limits_{p\in\Pi} J_p) = (\prod\limits_{p} J_p)/(qJ_q \oplus \prod\limits_{p\neq q} J_p) \cong (J_q/qJ_q) \oplus \prod\limits_{p\neq q} J_p/J_p \cong \mathbb{Z}/q\mathbb{Z}$, for any prime q. Therefore $r_p(J) \leq 1$ for all p. □

Griffith [14] has characterised torsion-free reduced groups G, with $r_p(G) \leq 1$ for all p, as pure subgroups of J; to see this we first need a lemma.

Lemma 2.6 *Let G be a torsion-free group and p any prime. Then $r_p(G) \leq 1$ if and only if every p-basic subgroup of G has rank at most 1.*

Proof: If B be a p-basic subgroup of G, then $G/pG = (B + pG)/pG \cong B/(B \cap pG) = B/pB$ and so $r_p(G) = r_p(B) = r(B)$, since B is free. $\square$

Theorem 2.7 *If G is a torsion-free reduced group then $r_p(G) \leq 1$ for all p if and only if G is isomorphic to a pure subgroup of $J = \prod_{p \in \Pi} J_p$.*

Proof: If G is isomorphic to a pure subgroup of J then $r_p(G) \leq r_p(J) \leq 1$ for all p. Conversely, suppose that $r_p(G) \leq 1$ for all p. Then every p-basic subgroup of G is cyclic, by the previous lemma. Embed G in its cotorsion completion $E = Ext(\mathbb{Q}/\mathbb{Z}, G)$. E is reduced and E/G is torsion-free divisible (see I, Lemma 1.21), so $G \leq_* E$. Also E is torsion-free cotorsion and so E is algebraically compact (see I, Lemma 1.20). Hence $E = \prod_p A_p$ where each A_p is complete in its p-adic topology and so is a p-adic module. Since E/G is divisible and $G \leq_* E$ every p-basic subgroup of G is p-basic in E and so every p-basic subgroup of E is cyclic. But the p-basic subgroups of E and A_p are the same, so $r_p(A_p) \leq 1$ for all p. Therefore each A_p is indecomposable as a p-adic module, i.e. $A_p = 0$ or $A_p = J_p$ for each p and so the result follows. $\square$

We now have that pure subgroups of J_p, $\bigoplus_p J_p$ and $\prod_p J_p$ are minimal. This gives indecomposable minimal groups of any rank $\leq 2^{\aleph_0}$ since

p-pure and hence pure subgroups of J_p are indecomposable. It would be interesting to know if indecomposable minimal groups of arbitrary large cardinality can be shown to exist in ZFC.

Now we turn our attention to another class of torsion-free groups.

Definition 2.8 *A torsion-free group G is* strongly-indecomposable *if whenever $nG \leq A \oplus B \leq G$, for some $n \in \mathbb{N}$ and $A, B \leq G$, then either $A = 0$ or $B = 0$.*

Proposition 2.9 *If A is an indecomposable torsion-free group such that $r_p(A) \leq 1$ for all p, then A is strongly indecomposable. In particular, pure subgroups of J_p are strongly indecomposable.*

Proof: Let A be as above. First we show, by induction, that $|A/nA| \leq n$ for all $n \in \mathbb{N}$. If $n = 1$ this is obviously true. Suppose that $|A/mA| \leq m$ for all $m < n$. If n is prime, then $|A/nA| \leq n$, again by hypothesis. If n is not prime, then $n = r \cdot s$ where $r, s < n$. We have $A/rA \cong (A/nA)/(rA/nA)$ and $rA/nA \cong A/sA$, so $|A/nA| = |A/rA||A/sA| \leq r \cdot s = n$. Now if $nA \leq B \oplus C \leq A$, then $A/(B \oplus C) \cong (A/nA)/((B \oplus C)/nA)$ is finite; hence $A \cong B \oplus C$, so $B \oplus C$ is indecomposable, i.e. either $A = 0$ or $B = 0$. □

Note that Murley [20, p.656] proves this result for finite rank A.

Proposition 2.10 *If A is a finite rank minimal group, then A is indecomposable if and only if A is strongly indecomposable.*

Proof: If A is indecomposable and $nA \leq B \oplus C \leq A$, then $A/(B \oplus C) \cong (A/nA)/((B \oplus C)/nA)$, which is finite. Thus $A \cong B \oplus C$ and hence either $B = 0$ or $C = 0$. The converse is immediate. □

Beaumont and Pierce [2] have characterised minimal indecomposable torsion-free groups of rank 2 as those groups with p-rank at most 1 for all p. The proof given here uses the previous proposition and some other facts.

Theorem 2.11 *If G is torsion-free indecomposable of rank 2 then G is minimal if and only if $r_p(G) \leq 1$ for all p.*

Proof: If G is torsion-free with $r_p(G) \leq 1$ for all p, then G is minimal by Theorem 2.3. Conversely, if torsion-free G of rank 2 is indecomposable and minimal, then G is strongly indecomposable and this implies that End(G), the endomorphism ring of G, is commutative (see [1, Theorem 3.3]). Faticoni and Goeters have shown that if G is a torsion-free group of finite rank which is minimal with commutative endomorphism ring, then $r_p(G) \leq 1$ (see [9, Proposition (iii.5)]). □

Finally, we show that, for each prime p, there exists a minimal group of infinite p-rank which is not complete. Later in this chapter we will see that there exist non-complete (completely decomposable) minimal groups of any finite p-rank.

Theorem 2.12 $\bigoplus_I J_p$ *is minimal for any infinite index set* I.

Proof: Let $G = \bigoplus_I J_p$, where I is infinite, and let H be of prime index in G. Since G is divisible by all primes $q \neq p$ we may assume that H is of index p in G. We claim that H is a p-adic submodule of G.

If $\pi = \sum_{n<\omega} p^n a_n \in J_p$ and $h \in H$, then $\pi h + H = (\sum_{n \geq k} p^n a_n)h + H$, for all $k \geq 1$, so $\pi h + H = p^k(y_k + H)$, for some $y_k \in G$, for all $k \geq 1$. Hence $\pi h + H$ is divisible by p^k in G/H, for all k and so $\pi h \in H$, since G/H is a finite p-group. Therefore H is a submodule of the free p-adic module G. We conclude that H is free and G/H finite implies that $r(H) = r(G)$ and so $H \cong G$. □

§3 Weakly Minimal Torsion-Free Groups

Just as in Chapters II and IV we can consider the less restrictive condition of weak minimality in the torsion-free case. For finite rank torsion-free groups this concept is rather uninteresting in light of Proposition 3.2 below. However, in the infinite rank case, it is a useful idea as we see in Theorem 4.20 of this chapter concerning separable minimal groups. First recall the definition of a weakly minimal group.

Definition 3.1 *A group G is* weakly minimal *if, whenever H is a finite index subgroup of G, then H contains a finite index copy of G.*

Proposition 3.2 *Any torsion-free group of finite rank is weakly minimal.*

Proof: Let G be torsion-free of finite rank and let H be a finite index subgroup of G. Then there exists an integer n such that $nG \leq H$. Since G is of finite rank we have G/nG is finite and so H/nG, being a subgroup of G/nG, is also finite. Therefore G is weakly minimal since G torsion-free means that nG is isomorphic to G. □

Proposition 3.3 *If a torsion-free group G has finite p-rank for all p, then G is weakly minimal.*

Proof: We show, by induction, that G/nG is finite for all $n \in \mathbb{N}$. For $n = 1$ the result is obvious. Suppose G/mG is finite for all $m < n$. If n is prime, then G/nG is finite, by the hypothesis. If $n = r \cdot s$, then $G/rG \cong (G/nG)/(rG/r \cdot sG)$. Therefore $|G/nG| = |G/rG||G/sG|$ which is finite. Now, as in the previous proposition, if H is a finite index subgroup of G, then there exists an integer n such that $nG \leq H$ and $H/nG \leq G/nG$, which is finite. Now, $nG \cong G$ since G is torsion-free and hence G is weakly minimal. □

Proposition 3.4 *If A_i is a torsion-free weakly minimal group for $1 \leq i \leq n$, then $G = \bigoplus_{i \leq n} A_i$ is weakly minimal. In particular, a finite direct sum of minimal groups is weakly minimal.*

Proof: If H is of finite index in G, then $H \cap A_i$ is of finite index in A_i for all $i \leq n$. Therefore $H \cap A_i$ contains a finite index subgroup B_i such that $B_i \cong A_i$, for all $i \leq n$, since each A_i is weakly minimal. Now, $G/\bigoplus_{i \leq n} B_i \cong \bigoplus_{i \leq n}(A_i/B_i)$ whicb is finite and so we get that $\bigoplus_{i \leq n} B_i$ is a finite index subgroup of H which is isomorphic to G. Therefore G is weakly minimal. □

For the final section of this chapter we restrict our attention to two important classes of torsion-free groups, namely the completely decomposable groups and, more generally, the separable groups.

§4 Completely Decomposable and Separable Minimal Groups

Before we consider completely decomposable and separable minimal groups we need to establish some important facts concerning types and finite index subgroups of a general torsion-free group.

Let G be any torsion-free group, H a finite index subgroup of G, and let s be any type. There exists $n \in \mathbb{N}$ such that $nG \leq H$; if $h \in H$ then $t_G(h) = t_{nG}(nh) \leq t_H(nh) = t_H(h)$ and obviously $t_H(h) \leq t_G(h)$. Therefore $t_H(h) = t_G(h)$.

In Chapter I we have defined the two fully invariant subgroups $G(s) = \{g \in G : t(g) \geq s\}$ and $G^*(s) = \langle g \in G : t(G) > s \rangle$ of G. We will also need the subgroup $G^\sharp(s) = G^*(s)_*$ where the purification is taken in G. If we denote the rank of $G(s)/G^\sharp(s)$ by $r_G(s)$, i.e. $r_G(s) = r(G(s)/G^\sharp(s))$, then the next lemma shows that $r_H(s) = r_G(s)$ whenever H is a finite index subgroup of G.

Lemma 4.1 *If H is any finite index subgroup of the torsion-free group G and if s is any type, then $r_H(s) = r_G(s)$.*

Proof: Let G and H be as above. Then $H(s) = \{h \in H : t_H(h) \geq s\} = \{h \in H : t_G(h) \geq s\} = H \cap G(s)$ and $H^\sharp(s) = \langle h \in H : t_H(h) > s \rangle_{*H} = \langle h \in H : t_G(h) > s \rangle_{*H}$ and we show that $H^\sharp(s) = H \cap G^\sharp(s)$. First, if $h \in H^\sharp(s)$ then there exists $m \in \mathbb{Z}$ such that $mh = h_1 + h_2 + \ldots + h_n$ where $h, h_1, h_2, \ldots, h_n \in H$ and $t_G(h_i) > s$ for $i = 1, \ldots, n$ and therefore $h \in G^\sharp(s)$. Conversely, $h \in H \cap G^\sharp(s)$ implies that there exists $m \in \mathbb{Z}$ and $g_1, \ldots g_r$ in G with $t_G(g_i) > s$ $(i = 1, \ldots, r)$, such that $mh = g_1 + \ldots + g_r$. Now, there exists $n \in \mathbb{N}$ such that $nG \leq H$ and so $nmh = ng_1 + \ldots + ng_r = h_1 + \ldots + h_r$ where $h_i = ng_i \in H$ and $t_G(h_i) = t_G(g_i) > s$ $(i = 1, \ldots, r)$ and this means that $h \in H^\sharp(s)$.

Therefore we have $H(s)/H^\sharp(s) = (H \cap G(s))/(H \cap G^\sharp(s)) =$

$(H \cap G(s))/(H \cap G(s) \cap G^\sharp(s)) \overset{\phi}{\cong} (H(s) + G^\sharp(s))/G^\sharp(s) \leq G(s)/G^\sharp(s)$

and $(G(s)/G^\sharp(s))/((H(s) + G^\sharp(s))/G^\sharp(s)) \cong G(s)/(H(s) + G^\sharp(s)) \cong$

$(G(s)/H(s))/((H(s)+G^\sharp(s))/H(s))$ and $G(s)/H(s) = G(s)/(H \cap G(s)) \cong$

$(G(s) + H)/H \leq G/H$ is finite.

Hence $0 \longrightarrow H(s)/H^\sharp(s) \xrightarrow{\phi} G(s)/G^\sharp(s) \xrightarrow{\pi} (G(s)/G^\sharp(s))/\mathrm{Im}(\phi) \longrightarrow 0$ is a short exact sequence with $(G(s)/G^\sharp(s))/\mathrm{Im}(\phi)$ finite. Therefore $\mathbb{Q} \otimes (H(s)/H^\sharp(s)) \cong \mathbb{Q} \otimes (G(s)/G^\sharp(s))$, i.e. $r_H(s) = r_G(s)$. □

We will now apply this result to the case of a completely decomposable group G. First recall the definition of a completely decomposable group.

Definition 4.2 *A torsion-free group G is* completely decomposable *if it is a direct sum of rank 1 groups.*

Also recall the typeset, $T(G)$, of a torsion-free group G, i.e. $T(G) = \{t(g) : g \in G\}$. In the case of completely decomposable groups there is an important subset of the typeset, namely the set of critical types.

Definition 4.3 *If G is completely decomposable , $G = \bigoplus_{i \in I} G_i$ where each G_i is of rank 1, then the set of critical types of G, $T_{cr}(G)$, is $\{t(G_i) : i \in I\}$.*

If G is completely decomposable, then $G = \bigoplus_t A_t$ where A_t is homogeneous completely decomposable of type t and t ranges over all types in $T_{cr}(G)$. Now, $G(s) = \bigoplus_{t \geq s} A_t$ and $G^*(s) = G^\sharp(s) = \bigoplus_{t > s} A_t$. Therefore $r_G(s) = r(A_s)$ for all types s. If a finite index subgroup H of G is also completely decomposable, then $r_H(s) = r_G(s)$ for all types s by Lemma 4.1 and so, by I, Theorem 3.3, we have $H \cong G$.

Definition 4.4 *A completely decomposable group G is* cofinitely hereditarily completely decomposable *if all its finite index subgroups are completely decomposable.*

The following theorem is now immediate.

Theorem 4.5 *A completely decomposable group G is minimal if and only if it is cofinitely hereditarily completely decomposable.* □

Corollary 4.6 *Rank 1 groups are minimal.*

Proof: Any finite index subgroup of a rank 1 group is of rank 1 and hence is completely decomposable. □

Corollary 4.7 *Homogeneous completely decomposable groups are minimal.*

Proof: Let G be homogeneous completely decomposable and let H be of finite index in G. For all $h \in H$, $t_H(h) = t_G(h)$ and so H is also homogeneous of the same type as G. Now, I, Lemma 3.5 tells us that H is completely decomposable and so G is minimal. □

Corollary 4.8 *A direct summand of a minimal completely decomposable group is minimal.*

Proof: Let G be a minimal completely decomposable group and let $A \sqsubset G$, $G = A \oplus B$, say. If K is of finite index in A, then $K \oplus B$ is of finite index in G and so is completely decomposable. Since $K \sqsubset K \oplus B$ we deduce that K is also completely decomposable, by I, Lemma 3.6. □

Corollary 4.9 *If G is a completely decomposable group whose critical typeset $\{t_\mu : \mu < \alpha, \alpha$ some ordinal$\}$ is inversely well-ordered, i.e. $t_\mu < t_\nu$ for $\mu > \nu$, then G is minimal.*

Proof: The proof is by transfinite induction. If $T_{cr}(G) = \{t_0\}$, then G is homogeneous completely decomposable and so is minimal by Corollary 4.7. Suppose completely decomposable groups with inversely well-ordered critical typesets $\{t_\mu : \mu < \gamma\}$ are minimal for all $\gamma < \alpha$ and let G have critical typeset $\{t_\mu : \mu < \alpha\}$. $G = \bigoplus_{\mu<\alpha} G_\mu$ where G_μ is homogeneous completely decomposable of type t_μ. For each $\mu < \alpha$ let $K_\mu = \bigoplus_{\nu<\mu} G_\nu$. K_μ is minimal, by assumption, for all $\mu < \alpha$. If H is a finite index subgroup of G, then $H_\mu = H \cap K_\mu$ is of finite index in K_μ since $K_\mu/H_\mu = K_\mu/(H \cap K_\mu) \cong (K_\mu + H)/H \leq G/H$ is finite. Therefore $H_\mu \cong K_\mu$ since K_μ is minimal. Also, for all $\mu < \alpha, H_{\mu+1}/H_\mu = H_{\mu+1}/(H \cap K_\mu) = H_{\mu+1}/(H \cap K_{\mu+1} \cap K_\mu) = H_{\mu+1}/(H_{\mu+1} \cap K_\mu) \cong (H_{\mu+1}+K_\mu)/K_\mu \leq K_{\mu+1}/K_\mu \cong G_\mu$ and $(K_{\mu+1}/K_\mu)/((H_{\mu+1}+K_\mu)/K_\mu) \cong$

$K_{\mu+1}/(H_{\mu+1}+K_\mu) \cong (K_{\mu+1}/H_{\mu+1})/((H_{\mu+1}+K_\mu)/H_{\mu+1})$ is finite. Hence $H_{\mu+1}/H_\mu \cong (H_{\mu+1}+K_\mu)/K_\mu \cong K_{\mu+1}/K_\mu$ since G_μ is minimal. If $h \in H_{\mu+1} \setminus H_\mu$, then $h \in K_{\mu+1} \setminus K_\mu$ and so $t_{K_{\mu+1}}(h) = t_\mu$ since h must have a non-zero G_μ-component. Therefore $t_{H_{\mu+1}}(h) = t_\mu$ since $H_{\mu+1}$ is of finite index in $K_{\mu+1}$. Also $H_\mu \leq_* H_{\mu+1}$ since $H_{\mu+1}/H_\mu \cong G_\mu$ is torsion-free. By I, Lemma 3.4 we have that $H_\mu \sqsubset H_{\mu+1}$, i.e. $H_{\mu+1} = H_\mu \oplus B_\mu$ where $B_\mu \cong H_{\mu+1}/H_\mu \cong K_{\mu+1}/K_\mu \cong G_\mu$.

If α is a successor ordinal, $\alpha = \gamma+1$, say, then $G = \bigoplus_{\mu<\gamma} G_\mu \oplus G_\gamma = K_\gamma \oplus G_\gamma$ and, by assumption, K_γ is minimal. Also $H = H_{\gamma+1} = H_\gamma \oplus B_\gamma$ with $H_\gamma \cong K_\gamma$ and $B_\gamma \cong G_\gamma$; therefore $H \cong G$. If α is a limit ordinal, then $H = \bigcup_{\mu<\alpha} H_\mu$ is completely decomposable since each H_μ is and $H_\mu \sqsubset H_{\mu+1}$ for all $\mu < \alpha$. Hence G is minimal. □

Beaumont and Pierce (see [3, Corollary 9.6]) give a method of extending any finite rank torsion-free minimal group to a larger minimal group by adding on a finite rank completely decomposable group whose critical typeset is a chain (and which is of course minimal by Corollary 4.9). The proof given here is by induction on the rank of the completely decomposable part.

Theorem 4.10 *Let C be a finite rank torsion-free minimal group and*

$G_1, \ldots, G_n$ are rank 1 groups such that $t(G_1) \leq t(G_2) \leq \ldots \leq t(G_n) \leq t(c)$ for all $c \in C$. Then $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$ is also minimal.

Proof: If H is of finite index in G, then $kH \leq kG \leq H$ for some $k \in \mathbb{N}$. Since $kG \cong G$ it suffices to prove that if $kH \leq G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C \leq H$ then $H \cong G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus C$. First suppose that $kH \leq G_1 \oplus C \leq H$. Letting $B = C_{*_H}$, we have that H/B is torsion-free of rank 1 since $k(H/B) = (kH + B)/B \leq (G_1 + B)/B \leq H/B$ where $(G_1 + B)/B \cong G_1/(G_1 \cap B) \cong G_1$ and $k(H/B) \cong H/B$. In fact $H/B \cong (G_1 + B)/B \cong G_1$, by I, Lemma 3.3, since their types must be the same. Now, if $h \in H \setminus B$ then $kh \in kH \leq G_1 \oplus C \leq H$ and so $t_H(h) = t_{kH}(kh) \leq t_{G_1 \oplus C}(kh) \leq t_H(kh) = t_H(h) = t(G_1) = t(H/B)$ since $kh \notin C$ because $h \notin B$. Therefore I, Lemma 3.4 implies that B is a summand of H, $H = B \oplus D$, say, where $D \cong H/B \cong G_1$. Also $kB \leq C \leq B$ since, if $b \in B$ with $kb = g_1 + c$ where $g_1 \in G_1$ and $c \in C$, then $g_1 = kb - c \in G_1 \cap B = 0$. Since B/kB is finite we get that kB is of finite index in C and hence $B \cong kB \cong C$. Therefore $H \cong G_1 \oplus C$.

Now suppose the result is true for n and let $kH \leq G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus G_{n+1} \oplus C \leq H$ with C minimal and $t(G_1) \leq t(G_2) \leq \ldots \leq t(G_n) \leq t(G_{n+1}) \leq t(c)$ for all $c \in C$. Let $B = \langle G_2 \oplus \cdots \oplus G_{n+1}, C \rangle_{*_H}$. Again $H/B \cong G_1$. A similar argument to the first case gives $H = B \oplus D$ with

$D \cong H/B \cong G_1$. Now $kB \leq G_2 \oplus \cdots \oplus G_n \oplus G_{n+1} \oplus C \leq B$, so, by induction, $B \cong G_2 \oplus \cdots \oplus G_n \oplus G_{n+1} \oplus C$ and the result is proved. □

Note that complete decomposability is not sufficient to guarantee minimality as the following example shows:

Example Let e_1, e_2 be independent elements in a $\mathbb{Q}$-vector space and let p_1, p_2, q be distinct primes. Let $E_1 = \langle p_1^{-\infty} e_1 \rangle, E_2 = \langle p_2^{-\infty} e_2 \rangle$ and $G_1 = \langle E_1, q^{-1}e_1 \rangle, G_2 = \langle E_2, q^{-1}e_2 \rangle$. Then E_1, E_2, G_1, G_2 are rank 1 groups where $G_1 \cong E_1$, with type given by the characteristic $(0, 0, \ldots, 0, \infty, 0, \ldots)$, with ∞ in the p_1^{th} place, and $G_2 \cong E_2$, with type given by the characteristic $(0, 0, \ldots, \infty, 0, \ldots)$, with ∞ in the p_2^{th} place. Defining $\phi : \mathbb{Z} \longrightarrow G_1/E_1$ by $(z)\phi = zq^{-1}e_1 + E_1 (z \in \mathbb{Z})$ it is easily seen that ϕ is a surjective homomorphism with kernel $q\mathbb{Z}$. Therefore $G_1/E_1 \cong \mathbb{Z}/q\mathbb{Z}$ and so $|G_1/E_1| = q$. Similarly $|G_2/E_2| = q$. Now, $(G_1 \oplus G_2)/(E_1 \oplus E_2) \cong (G_1/E_1) \oplus (G_2/E_2)$, so $E_1 \oplus E_2$ is of index q^2 in $G_1 \oplus G_2$.

Let $A = \langle E_1 \oplus E_2, q^{-1}(e_1 + e_2) \rangle \leq G_1 \oplus G_2$. A is indecomposable (see [11, p.123, Example 2]) and so $A \not\cong G_1 \oplus G_2$. However, $(G_1 \oplus G_2)/A \cong ((G_1 \oplus G_2)/(E_1 \oplus E_2))/(A/(E_1 \oplus E_2))$ and so A is of finite index in $G_1 \oplus G_2$ and hence $G_1 \oplus G_2$ is not minimal.

Note that A is also not minimal, since $qA \leq E_1 \oplus E_2 \leq A$ and A/qA is finite since A has finite rank.

More generally, if t_1 and t_2 are two incomparable types whose characteristics have a common finite entry at some prime p, then letting $E_1 = \left\langle p_k^{-n_k} e_1 : p_k \neq p \right\rangle$ and $E_2 = \left\langle p_k^{-m_k} e_2 : p_k \neq p \right\rangle$ where t_1 and t_2 are given by the characteristics $(\ldots, n_k, \ldots, 0, \ldots)$ and $(\ldots, m_k, \ldots, 0, \ldots)$, respectively, with 0 in the p^{th} place and $0 \leq n_k, m_k \leq \infty$ for all $k \in \mathbb{N}$, we get that $\{E_1, E_2\}$ is a rigid system (see [11, p.124]).

Now, if $G_1 = \langle E_1, p^{-1}e_1 \rangle, G_2 = \langle E_2, p^{-1}e_2 \rangle$ and $A = \langle E_1 \oplus E_2, p^{-1}(e_1 + e_2) \rangle$ then A is indecomposable (see [11, Lemma 88.2]), and, as above, $|(G_1 \oplus G_2)/(E_1 \oplus E_2)| = p^2$ and so A is of finite index in $G_1 \oplus G_2$. Hence $G_1 \oplus G_2$ is not minimal. □

The next proposition gives necessary and sufficient conditions for a completely decomposable group of rank 2 with incomparable critical types to be minimal.

Proposition 4.11 *Let $G = G_1 \oplus G_2$ be completely decomposable of rank 2 where $t_1 = t(G_1)$ is incomparable to $t_2 = t(G_2)$. Then G is minimal if and only if $r_p(G_1) r_p(G_2) = 0$ for all primes p.*

Proof: If $r_p(G_1)r_p(G_2) = 0$ for all p, then $r_p(G) \leq 1$ for all p, so G is minimal by Theorem 2.3. Conversely, if $r_p(G_1) = 1 = r_p(G_2)$ for some p, then t_1 and t_2 have a common finite entry in the p^{th} place since a rank 1 group is p-divisible if and only if its p-rank is 0. Now the previous example completes the proof. □

Theorem 4.12 *Let* $G = \bigoplus_{i \in I} G_i$ *be completely decomposable, with* $r(G_i) = 1$ *for all* $i \in I$*, such that its critical typeset is an antichain, i.e. each pair in the critical typeset is incomparable. Then* G *is minimal if and only if* $r_p(G_i)r_p(G_j) = 0$ *for all* $i, j \in I$ *and for all primes* p.

Proof: If $r_p(G_i)r_p(Gj) = 0$ for all $i, j \in I$ and for all primes p, then $r_p(G) \leq 1$ for all p and so G is minimal by Theorem 2.3. Conversely, if G is minimal, then each $G_i \oplus G_j$ is minimal by Corollary 4.8 and so $r_p(G_i)r_p(G_j) = 0$ for all p by Proposition 4.11. □

Note that the index set I for a completely decomposable minimal group $G = \bigoplus_{i \in I} G_i$ with $r(G_i) = 1$ for all $i \in I$, whose critical typeset is an antichain, is at most countable since if the types of the G_i are labelled as $(a_{in}), i \in I, n \in \mathbb{N}$ and if $M_i = \{n \in \mathbb{N} : a_{in} \text{ is finite}\}$ then $M_i \cap M_j = \emptyset$, for $i \neq j$, and $\bigcup_{i \in I} M_i \subseteq \mathbb{N}$. An example of such a countable collection of types is given by $\{(a_{in}) : a_{nn} = 0, a_{in} = \infty\}$.

We now consider the case of general finite rank completely decomposable groups. The next result follows from a remark of Warfield (see [30, p.148]).

Proposition 4.13 *Let A and B be torsion-free minimal groups such that* $\mathrm{Ext}(A, B)$ *is torsion-free. Then $A \oplus B$ is minimal.*

Proof: Suppose H is of finite index in $A \oplus B$. First $B/(B \cap H) \cong (B + H)/H \leq (A \oplus B)/H$ is finite. Therefore $B \overset{\alpha}{\cong} B \cap H$.
Also $H/(B \cap H) \cong (H + B)/B \leq (A \oplus B)/B \cong A$ and
$((A \oplus B)/B)/((H + B)/B) \cong (A \oplus B)/(H + B) \cong$
$((A \oplus B)/H)/((H + B)/H)$ again is finite. Hence $H/(B \cap H) \cong A$ and so $0 \longrightarrow B \cap H \overset{i}{\longrightarrow} H \overset{\beta}{\longrightarrow} A \longrightarrow 0$ is exact, for some β, where i is inclusion. We show that this short exact sequence is torsion in $\mathrm{Ext}(A, B \cap H)$. Walker [28], (see [11, Lemma 102.1]), has shown that the exact sequence $0 \longrightarrow B \overset{\alpha}{\longrightarrow} C \overset{\beta}{\longrightarrow} A \longrightarrow 0$ is torsion in $\mathrm{Ext}(A, B)$ if and only if it is quasi-splitting, i.e. the sequence $0 \longrightarrow B \overset{\alpha}{\longrightarrow} nC + B\alpha \overset{\beta}{\longrightarrow} nA \longrightarrow 0$ is splitting for some $n \in \mathbb{N}$. Now, there exists $n \in \mathbb{N}$ such that $n(A \oplus B) \leq H$ and for this n consider the exact sequence

$$0 \longrightarrow B \cap H \xrightarrow{i} nH + B \cap H \xrightarrow{\beta} nA \longrightarrow 0 \qquad \text{(i)}.$$

We have that $nH \leq n(A \oplus B) \leq H \leq A \oplus B$. Now, $n(A \oplus B) + B \cap H = nA \oplus (B \cap H)$ since $n(a+b)+b' = na + (nb+b') \in nA + B \cap H$ where $a \in A, b \in B$ and $b' \in B \cap H$ and obviously $nA \oplus (B \cap H) \leq n(A \oplus B) + B \cap H$. We have $nH + B \cap H \leq n(A \oplus B) + B \cap H = (B \cap H) \oplus nA$, so the modular law implies $nH + B \cap H = (B \cap H) \oplus [nA \cap (nH + B \cap H)]$. Therefore the sequence (i) splits. Hence the sequence $0 \longrightarrow B \xrightarrow{\alpha} nH + B\alpha \xrightarrow{\beta} nA \longrightarrow 0$ splits and so the sequence $0 \longrightarrow B \xrightarrow{\alpha} H \xrightarrow{\beta} A \longrightarrow 0$ is torsion and hence is 0 since $\mathrm{Ext}(A, B)$ is torsion-free. Therefore this sequence splits and so $H \cong A \oplus B$ and hence $A \oplus B$ is minimal $\square$

We also need the following proposition. Recall that $S_A(G)$ is defined by $S_A(G) = \sum\{\mathrm{Im}(\phi) : \phi \in \mathrm{Hom}(A, G)\}$ for any groups A and G (see IV, Lemma 6.7).

Proposition 4.14 *Let A and G be torsion-free groups with $r(A) = 1$ and $t(A) = t$ and $r(G)$ finite. Then $G(t) = S_A(G)$.*

Proof: Since $t(a\phi) \geq t(a)$ for all $a \in A$ we get that $t(g) \geq t$ for all $g \in S_A(G)$. Hence $S_A(G) \subseteq G(t)$. Conversely, let $g \in G(t)$ and let $a \in A$. Then $h_p(a) \leq h_p(g)$ for almost all p and so there exist $\overline{a}, \overline{g}$, with

$a = m\overline{a}$, $g = n\overline{g}$ for some $m, n \in \mathbb{N}$, such that $h_p(\overline{a}) \leq h_p(\overline{g})$ for all p and $h_p(\overline{a}) = h_p(\overline{g}) = 0$ for all p for which $h_p(a) > h_p(g)$. Since $t(\overline{g}) \geq t$, there exists a homomorphism $\phi : A \longrightarrow < \overline{g} >_*$ such that $\overline{a}\phi = \overline{g}$. Now, $(n\overline{a})\phi = n(\overline{a}\phi) = n\overline{g} = g$ and so $g \in S_A(G)$, i.e. $G(t) \subseteq S_A(G)$. □

This proposition has the following corollary which gives a sufficient condition for the minimality of a summand of a minimal torsion-free group.

Corollary 4.15 *Let $G = A \oplus B$ be a minimal group. If* $\sup\{s : s \in T(A)\} < \inf\{s : s \in TB)\}$ *then A is also minimal.*

Proof: Let $t = \inf\{s : s \in T(B)\}$. Then $G(t) = B$ and $A \cong G/B = G/G(t)$. But $G(t) = S_U(G)$ where U is a rank 1 torsion-free group of type t. Therefore A is minimal by IV, Theorem 6.6. □

The following is due to Warfield.

Lemma 4.16 *Let A and B be torsion-free groups with $r(A) = 1$ and $r(B)$ finite. Then* $\operatorname{Ext}(A, B)$ *is torsion-free if and only if $r_p(A)r_p(B/S_A(B)) = 0$ for all p.*

Proof: See [30, Theorem 3]. □

The above result of Warfield, together with Proposition 4.13, enables us to give a satisfactory characterisation of minimal finite rank completely decomposable groups. Notice that the calculation in the theorem is straightforward to carry out for a given group.

Theorem 4.17 *Let* $G = \bigoplus_{i=1}^{i=n} G_i$ $(n \geq 2)$ *be a completely decomposable group where* $r(G_i) = 1$ $(i = 1, \ldots n)$. *Then* G *is minimal if and only if* $r_p(G_i)r_p(G_j) = 0$ *for all pairs* G_i, G_j *of incomparable type and for all* p.

Proof: First suppose that G is minimal and that G_i and G_j have incomparable type. Then $G_i \oplus G_j$ is minimal, by Corollary 4.8. Now, Proposition 4.11 tells us that $r_p(G_i)r_p(G_j) = 0$. The converse is proved by induction on n. When $n = 2$ the result has already been established since if $G = A \oplus B$ with $r(A) = r(B) = 1$, then for $t(A) = t(B)$ see Corollary 4.7, for $t(A)$ and $t(B)$ comparable see Corollary 4.9 and, finally, if $t(A)$ and $t(B)$ are incomparable see Proposition 4.11. Now, suppose the result is true for all such groups of rank less than some $n > 2$ and let $G = \bigoplus_{i=1}^{i=n} G_i$ satisfy the hypotheses. Moreover, let $t_i = t(G_i)$ for $i = 1, \ldots, n$. If all the types $t_1, t_2, \ldots, t_n$ are mutually comparable, then we are finished, by Corollary 4.9. Thus we may assume that there are incomparable types among the $t_1, t_2, \ldots, t_n$. Choose some type t from these types such that $t \leq t_i$ or t, t_i are incomparable for all $i = 1, \ldots, n$. By relabelling, if necessary, let this type be $t_1 = t(G_1)$. Let $A = G_1$ and $B = \bigoplus_{i=2}^{i=n} G_i$. Then $G = A \oplus B$ with $S_A(B) = B(t_1)$, by Lemma 4.14, and A, B minimal by Corollary 4.6 and the induction hypothesis.

If t_1 is not comparable to any other type, then $S_A(B) = B(t_1) = 0$ and so $B/S_A(B) \cong \bigoplus_{i=2}^{i=n} G_i$ and thus the hypotheses tell us that $r_p(G_1) r_p(B/S_A(B))$ $= 0$ for all p.

If $t_1 \leq t_i$ for all $2 \leq i < k$ and t_1 is not comparable to t_i for all $k \leq i \leq n$, where $2 < k < n$, then $S_A(B) = B(t_1) = \bigoplus_{i=2}^{i=k-1} G_i$ and $B/S_A(B) \cong \bigoplus_{i=k}^{i=n} G_i$. Again, the hypotheses tell us that $r_p(G_1) r_p(G_i) = 0$ for all $k \leq i \leq n$ and for all p and so $r_p(G_1) r_p(B/S_A(B)) = 0$ for all p. Now Theorem 4.16 implies that $\operatorname{Ext}(A, B)$ is torsion-free and hence $A \oplus B$ is minimal by Proposition 4.13, i.e. G is minimal. □

We now turn our attention to separable groups. First recall the definition of a separable group.

Definition 4.18 *A torsion-free group G is* separable *if every finite subset of G is contained in a completely decomposable summand of G, which* *clearly can be taken to be of finite rank.*

For a general separable group we introduce the analogue of Definition 4.4.

Definition 4.19 *A torsion-free separable group G is* cofinitely hereditarily separable *if all its subgroups of finite index are separable.*

The following theorem gives various conditions for the minimality of a general separable group.

Theorem 4.20 *If G is a torsion-free separable group, then the following are equivalent:*

(i) *G is minimal;*

(ii) *G is weakly minimal and every finite rank summand of G is minimal;*

(iii) *For every finite index subgroup H of G there exists a separable subgroup C of H which is of finite index in G and every finite rank summand of G is minimal;*

(iv) *G is cofinitely hereditarily separable and every finite rank summand of G is minimal.*

Proof: **(i) $\Rightarrow$ (ii)**

G is obviously weakly minimal. Let A be a finite rank summand of G and let B be a finite index subgroup of A. Then $G = A \oplus C$, say, and $B \oplus C$ is of finite index in G. Hence $B \oplus C \cong G$ and so $B \oplus C$ is separable. Therefore B is separable, by I, Lemma 3.9, and hence B is completely decomposable since it is of finite rank. Thus A is cofinitely hereditarily completely decomposable and hence is minimal by Theorem 4.5.

(ii) $\Rightarrow$ (iii)

If H is a finite index subgroup of G, then there exists a finite index subgroup C of H such that $C \cong G$, since G is weakly minimal. C is obviously separable.

(iii) $\Rightarrow$ (iv)

Let H be a finite index subgroup of G. There exists a separable subgroup C of H such that C is of finite index in G. Then $G = C + K$ for some finitely generated K. Since C is separable there exists a finite rank summand E of C such that $C \cap K \subseteq E$, i.e. $C = D \oplus E$, say. Therefore $G = C + K = D + E + K = D + F$ where $F = E + K$. Now, if $d \in D \cap F$ then $d = e + k$ where $e \in E$ and $k \in K$ and so $d - e \in C \cap K$ which is a subgroup of E. Hence $d \in D \cap E = 0$, i.e. $d = 0$. Therefore $G = D \oplus F$ where D is a subgroup of H. The modular law now gives

$H = H \cap G = D \oplus (H \cap F)$. Now F is a finite rank summand of G, so, by hypothesis, F is minimal. Also, since $F/(H \cap F) \cong (F + H)/H \leq G/H$, we get that $H \cap F$ is of finite index in F and so $H \cap F \cong F$. We conclude that $H \cong G$ and thus H is separable.

(iv) $\Rightarrow$ (i)

If H is a finite index subgroup of G, then H is separable. Now taking $H = C$ in the previous part gives the result. □

We continue with some consequences of the above theorem. The first corollary is the analogue of Corollary 4.8.

Corollary 4.21 *A direct summand of a minimal separable torsion-free group is minimal.*

Proof: Let $G = A \oplus B$ be a minimal separable torsion-free group and let H be of finite index in A. Now, A is separable, by I, Lemma 3.9, and $H \oplus B$ is a finite index subgroup of G. Hence $H \oplus B$ is separable and so H is separable, again by I, Lemma 3.9. We conclude that A is cofinitely hererditarily separable. Also, if X is a summand of A of finite rank, then X is minimal since X is a summand of G. Therefore A is minimal, by Theorem 4.20. □

For a homogeneous separable torsion-free group Theorem 4.20 simplifies considerably, as the next corollary shows.

Corollary 4.22 *For a torsion-free separable and homogeneous group G the following are equivalent:*

(i) *G is minimal;*

(ii) *G is weakly minimal;*

(iii) *For every finite index subgroup H of G there exists a separable finite index subgroup C of G such that C is contained in H;*

(iv) *G is cofinitely hereditarily separable.*

Proof: If G is separable homogeneous then any finite rank summand of G is homogeneous completely decomposable and hence is minimal, by Corollary 4.7. Now Theorem 4.20 completes the proof. □

Corollary 4.23 *Let G be torsion-free separable of finite p-rank for all primes p. Then G is minimal if and only if every finite rank summand of G is minimal.*

Proof: By Theorem 3.3, if $r_p(G)$ is finite for all p then G is weakly minimal. Theorem 4.20 now gives the result. □

Corollary 4.24 *Let $G = \bigoplus_{i=1}^{n} X_i$ be a torsion-free separable group where each X_i is minimal. Then G is minimal if and only if every finite rank summand of G is minimal.*

Proof: The group $G = \bigoplus_{i=1}^{n} X_i$ is weakly minimal by Theorem 3.4. Again an application of Theorem 4.20 means that G is minimal. □

Before we prove the next corollary we give another definition.

Definition 4.25 *A group is hereditarily separable if all its subgroups are separable.*

Obviously hereditarily separable implies cofinitely hereditarily separable.

Recall a group G is a Whitehead group if $\text{Ext}(G, \mathbb{Z}) = 0$. Whitehead groups are separable and homogeneous of type $\mathbb{Z}$ (see [8, IV Theorem 2.1]).

Corollary 4.26 *Whitehead groups are minimal.*

Proof Let G be a Whitehead group. We show that G is hereditarily separable. Every subgroup of a Whitehead group G is a Whitehead group: if $H \leq G$, then we get the short exact sequence $0 \longrightarrow H \stackrel{i}{\longrightarrow}$

$G \xrightarrow{\pi} G/H \longrightarrow 0$ which now gives, by I, Lemma 1.5, the exact sequence $0 = \text{Ext}(G, \mathbb{Z}) \longrightarrow \text{Ext}(H, \mathbb{Z}) \longrightarrow 0$ and hence $\text{Ext}(H, \mathbb{Z}) = 0$, so H is also a Whitehead group. Thus G is (cofinitely) hereditarily separable and so is minimal, by Corollary 4.22. □

The question arises as to whether every cofinitely hereditarily separable group is hereditarily separable. In fact, this statement cannot be deduced in ZFC. We will show, using the concept of minimality, that every coseparable group is cofinitely hereditarily separable. But, using CH, it is possible to construct a coseparable group which is not hereditarily separable. To define the idea of coseparability we first need to consider $\aleph_1$-free groups.

Definition 4.27 *A torsion-free group G is* $\aleph_1$-free *if all its countable subgroups are free.*

Definition 4.28 *If G is a torsion-free group, then a collection of subgroups of G, $\{G_\mu : \mu < \aleph_1\}$, is an* $\aleph_1$-filtration *of G if*

(i) *G_μ is countable for all $\mu < \aleph_1$,*

(ii) *$G_\mu < G_\nu$ for all $\mu < \nu$,*

(iii) *if ν is a limit ordinal, then $G_\nu = \bigcup_{\mu<\nu} G_\mu$,*

(iv) $G = \bigcup_{\mu < \aleph_1} G_\mu$.

$\aleph_1$-free groups of cardinality $\leq \aleph_1$ can be characterised in terms of the existence of $\aleph_1$-filtrations of free groups.

Lemma 4.29 *A group G of cardinality $\leq \aleph_1$ is $\aleph_1$-free if and only if there exists an $\aleph_1$-filtration of G where each G_μ is free.*

Proof: See [8, IV, Lemma 1.5]. □

Note that the Pontryagin Criterion (see [8, IV, Theorem 2.3]) tells us that every finite subset of an $\aleph_1$-free group G is contained in a pure free subgroup of G. This means that G must be homogeneous of type $\mathbb{Z}$.

Definition 4.30 *A torsion-free group is* coseparable *if it is $\aleph_1$-free and if every subgroup H of G, with G/H finitely generated, contains a summand D of G such that G/D is finitely generated.*

An alternative definition of coseparability can be given in terms of Ext, i.e. a torsion-free group G is coseparable if and only if $\mathrm{Ext}(G, \mathbb{Z})$ is torsion-free and also G is coseparable if and only if G is coseparable and separable (see [8, IV, Theorem 2.13]).

Eklof and Mekler prove the following set of implications for homogeneous torsion-free groups of type $\mathbb{Z}$:

Lemma 4.31 *Free implies hereditarily separable implies coseparable implies separable implies* $\aleph_1$*-free.*

Proof: See [8, p.100]. □

The next proposition shows that every coseparable group is minimal. This gives a fairly large class of examples for separable minimal groups.

Proposition 4.32 *A coseparable group is minimal.*

Proof: Let G be coseparable and let H be of finite index in G, i.e. G/H is finite. Since G is coseparable there exists $D \leq H$ such that $D \sqsubset G$ and G/D is finitely generated, i.e. $G = D \oplus A$ with A finitely generated. Since A is torsion-free we actually have that A is free. Now, the modular law implies that $H = D \oplus (H \cap A)$ and $A/(H \cap A) \cong (H + A)/H = G/H$ is finite. Hence $H \cap A \cong A$ since free groups are minimal and so $H \cong G$. □

Note that Proposition 4.32 gives an alternative proof that Whitehead groups are minimal, since $\mathrm{Ext}(G, \mathbb{Z}) = 0$ obviously implies that $\mathrm{Ext}(G, \mathbb{Z})$

is torsion-free and so every Whitehead group is coseparable.

Corollary 4.33 *A coseparable group is cofinitely hereditarily separable.*

Proof: Coseparable implies minimal implies cofinitely hereditarily separable, by Proposition 4.32 and Corollary 4.22. □

Eklof and Mekler give an example of a group (assuming CH) which is coseparable, and hence cofinitely hereditarily separable, by the previous corollary, but not hereditarily separable (see [8, XII, Corollary 2.12]). Since CH is consistent with ZFC this shows that the statement that every cofinitely hereditarily separable group is hereditarily separable cannot be deduced in ZFC.

The next theorem gives a rather surprising example of a homogeneous separable group of type $\mathbb{Z}$, namely the so-called Baer-Specker group, which is not minimal.

Theorem 4.34 *The Baer-Specker group $P = \prod_{\aleph_0} \mathbb{Z}$ is not minimal.*

Proof: We apply Theorem 4.22 to obtain this result. It is well-known that P is separable and homogeneous of type $\mathbb{Z}$ (see e.g. [10, Theorem 19.2]). We show that, for any prime p, there exists a subgroup of P of index p in P which is not separable. In this way we get that P is not cofinitely hereditarily separable.

Let B be the set of bounded sequences in P, i.e. $B = \{(x_n)_{n\in\mathbb{N}} : sup_n|x_n|$ is finite$\}$. First we show that B is a basic subgroup of P:

(i) By [11, Corollary 97.4] B is free of rank $2^{\aleph_0}$, i.e. $B = \bigoplus_{i\in I} g_i\mathbb{Z}$, say, where $|I| = 2^{\aleph_0}$;

(ii) $B \leq_* P$ since if $(x_n)_n \in B$ and $(x_n)_n = m(y_n)_n$, then obviously $(y_n)_n \in B$ also;

(iii) P/B is divisible since if $(x_n)_n + B \in P/B$ and $m \in \mathbb{Z}$, then, for each $i \in \mathbb{N}$, there exist $y_i, r_i \in \mathbb{Z}$, with $0 \leq r_i < m$, such that $x_i = my_i + r_i$ and so $(x_1, x_2, \ldots) - m(y_1, y_2, \ldots) = (r_1, r_2, \ldots) \in B$.

Therefore B is a basic subgroup of P.

Now, $P = B + pP$, where p be any prime, since P/B is divisible. Consider $A = B_0 + pP$ where $B = B_0 \oplus g_0\mathbb{Z}$ with $B_0 = \bigoplus_{i\in I'} g_i\mathbb{Z}$ $(I = I' \cup \{0\})$. We claim that A is a finite index subgroup of P which is not separable. We have $P/A = (B + pP)/(B_0 + pP)$ and so if $\pi + A \in P/A$, then $\pi + A = ng_0 + A$ for some $n \in \mathbb{Z}$. There exist $k, r \in \mathbb{Z}$, with $0 \leq r < p$, such that $n = kp + r$, so $\pi + A = r(g_0 + A)$. Hence $|P/A| \leq p$. On the other hand, if $r_1g_0 - r_2g_0 \in A$ for $0 \leq r_1, r_2 < p, r_1 \neq r_2$, then $(r_1 - r_2)g_0 \in A$, so $(r_1 - r_2)g_0 = b_0 + px$ for some $b_0 \in B_0$ and $x \in P$.

Therefore $(r_1 - r_2)g_0 - b_0 = px$ and hence $(r_1 - r_2)g_0 - b_0 = p(sg_0 + b'_0)$, for some $s \in \mathbb{Z}$ and $b'_0 \in B_0$, since $B \leq_* P$, and so $r_1 - r_2$ is divisible by p, contradicting $r_1 \neq r_2$. Therefore $|P/A| = p$.

It remains to show that A is not separable. Let $H = p\langle g_0 \rangle \leq A$. H is pure in A as the following arguments show. If $npg_0 = m(b_0 + px)$ where $m \neq 0$, $b_0 = (z_n)_n \in B_0$ and $x = (x_n)_n$, then $mpx = npg_0 - mb_0 \in B$, so $x \in B$. Let $x = rg_0 + b'_0$ where $r \in \mathbb{Z}$ and $b'_0 \in B_0$. Therefore $npg_0 = mb_0 + mrpg_0 + mpb'_0$. Hence $npg_0 = mrpg_0 = m(rpg_0)$ and so H is pure in A.

However, H is not a summand of A, since if $A = H \oplus K$, then any $a \in A$ has the unique form $a = npg_0 + k$,where $k \in K$, and so defining $\phi : A \longrightarrow \mathbb{Z}$ by $(a)\phi = n$, we get that $(pg_0)\phi = 1$. Now, if $p : P \longrightarrow A$ is given by multiplication by p, then $p\phi : P \longrightarrow \mathbb{Z}$ is in P^* and so there exists $m \in \mathbb{N}$ such that $p\phi \restriction \prod_{i>m} e_i\mathbb{Z} = 0$. Therefore $(pg_0)\phi = (pg_0^1, pg_0^2, \ldots)\phi = (pg_0^1 e_1 + \ldots + pg_0^m e_m + (0, 0, \ldots, 0, pg_0^{m+1}, \ldots))\phi = p(g_0^1 e_1 + g_0^2 e_2 + \ldots + g_0^m e_m)\phi$ is divisible by p, a contradiction. We conclude that A has a finite rank pure subgroup, namely H, which is not a summand and hence A is not separable by I, Lemma 3.8. □

Since P is separable and homogeneous of type $\mathbb{Z}$ it must be $\aleph_1$-free and so Theorem 4.34 shows that $\aleph_1$-freeness is not sufficient to guarantee minimality. Recall the stronger concept of strong $\aleph_1$-freeness.

Definition 4.35 *A torsion-free group G is* strongly $\aleph_1$-free *if G is $\aleph_1$-free and every countable subset of G is contained in a countably generated subgroup H of G such that G/H is $\aleph_1$-free.*

Strongly $\aleph_1$-free groups of cardinality $\leq \aleph_1$ can be characterised in a similar way to the $\aleph_1$-free case.

Lemma 4.36 *A group G of cardinality $\leq \aleph_1$ is strongly $\aleph_1$-free if and only if there exists an $\aleph_1$-filtration $G = \bigcup_{\mu<\aleph_1} G_\mu$ such that, for $\mu < \nu$, $G_{\nu+1}$ and $G_{\nu+1}/G_{\mu+1}$ are free.*

Proof: See [8, IV, Proposition 1.11]. □

However, even strong $\aleph_1$-freeness is not sufficient to ensure minimality, as the following illustrates:

Dugas and Göbel (see [6, Corollary 3.4]) have shown (assuming $2^{\aleph_0} < 2^{\aleph_1}$) that there exists a strongly $\aleph_1$-free group G of cardinality $\aleph_1$ such that $\mathrm{End}(G) \cong \mathbb{Z}$. G has a basic subgroup B (see [7, Theorem 10]) and if the rank of B is countable, then $B \leq G_{\mu+1}$ for some μ and so $G/G_{\mu+1}$ is a homomorphic image of G/B, which is a a contradiction, since G/B

is divisible and $G/G_{\mu+1}$ is reduced. Therefore B has uncountable rank. For each prime p we have $G/pG = (B+pG)/pG \cong B/(B \cap pG) \cong B/pB$ so $r_p(G) = r_p(B) = \aleph_1$ and hence $|\text{F.I.}(G)| > \aleph_1$, by III, Theorem 2.2. Now, if G is minimal then $|\text{F.I.}(G)| \leq |\text{End}(G)| = \aleph_0$, by III, Proposition 2.3, again a contradiction. Since the weak diamond condition $(2^{\aleph_0} < 2^{\aleph_1})$ is consistent with ZFC, we can conclude that the minimality of a general strongly $\aleph_1$-free group cannot be deduced in ZFC.

Returning to completely decomposable groups we will now apply what we have established about separable minimal groups to the completely decomposable case. For the remainder of this section let G be a completely decomposable group, $G = \bigoplus_{i \in I} G_i$ with $r(G_i) = 1$ for all $i \in I$. Now, if A is a summand of G, then both A and its complementary summand B are completely decomposable, by I, Lemma 3.6. Since $G = A \oplus B$, I, Lemma 3.3 implies that each rank 1 canonical summand of A or B is isomorphic to some G_i. Hence, appealing to Theorem 4.17, all finite rank summands of G are minimal if and only if $r_p(G_i)r_p(G_j) = 0$ for all p and for all G_i, G_j of incomparable type where $i, j \in I$. So in the case of completely decomposable groups we get the following theorem.

Theorem 4.37 *Let* $G = \bigoplus_{i \in I} G_i$ *be a completely decomposable group. Then the following are equivalent:*

(i) *G is minimal;*

(ii) *G is weakly minimal and $r_p(G_i)r_p(G_j) = 0$ for all p and for all G_i, G_j of incomparable type where $i, j \in I$;*

(iii) *If H is a finite index subgroup of G, then there exists a separable group C of finite index in G such that $C \leq H$ and $r_p(G_i)r_p(G_j) = 0$ for all p and for all G_i, G_j of incomparable type where $i, j \in I$;*

(iv) *G is cofinitely hereditarily separable and $r_p(G_i)r_p(G_j) = 0$ for all p and for all G_i, G_j of incomparable type where $i, j \in I$.*

Proof: Theorem 4.20 with what has been said above. □

Corollary 4.38 *Let* $G = \bigoplus_{i \in I} G_i$ *be a completely decomposable group of finite p-rank for all p. Then G is minimal if and only if $r_p(G_i)r_p(Gj) = 0$ for all p and for all G_i, G_j of incomparable type where $i, j \in I$. In particular, any completely decomposable group whose critical typeset is an infinite chain is minimal if it has finite p-rank for all p.*

Proof: G is weakly minimal by Theorem 3.3. Now Theorem 4.37 completes the proof. □

Corollary 4.39 *Suppose a completely decomposable group G can be written in the form $G = \bigoplus_{i \in I} G_i = X_1 \oplus X_2 \oplus \cdots \oplus X_n$, for some $n \in \mathbb{N}$, where each X_i is minimal. Then G is minimal if and only if $r_p(G_i)r_p(G_j) = 0$ for all incomparable types t_i, t_j, where $i, j \in I$, and for all p.*

Proof: G is weakly minimal by Theorem 3.4. Again Theorem 4.37 gives the result. □

Note that if $G = \bigoplus_{i \in I} G_i$ has the property that $r_p(G_i)r_p(G_j) = 0$ for all incomparable pairs G_i, G_j and for all p then:

(i) If $r_p(G)$ is finite for all p then G is minimal by Corollary 4.38;

(ii) If $r_p(G)$ is infinite for some p then the set of critical types of $G, T_{cr}(G)$, must contain an infinite chain for, if not, then $r_p(G_i)r_p(G_j) = 1$ for some incomparable pair G_i, G_j.

Hence if $T_{cr}(G)$ does not contain an infinite chain then $r_p(G)$ is finite for all p and so G is minimal.

To conclude this chapter, we combine all the information we have established on minimal completely decomposable groups.

Theorem 4.40 *Let $G = \bigoplus_{i \in I} G_i$ be a completely decomposable group such* *that the critical typeset of G contains only a finite number of infinite chains, each of whose corresponding groups is homogeneous or has finite p-rank or has an inversely well-ordered critical typeset. Then G is minimal if and only if $r_p(G_i)r_p(G_j) = 0$ for all incomparable pairs G_i, G_j and for all p.*

Proof: If G is minimal then the condition follows from Corollary 4.8 and Theorem 4.11. Conversely, if $G = C_1 \oplus C_2 \oplus \cdots \oplus C_n \oplus G'$ where each $T_{cr}(C_i)$ is a chain of one of the given kinds for $i = 1, \ldots n$ and G' does not contain an infinite chain, then each C_i is minimal, by Corollaries 4.7, 4.9 and 4.38, and the previous note tells us that G' is minimal. Now Corollary 4.39 implies that G is minimal. □

Some Concluding Remarks

Although considerable progress has been made in this work in characterising quasi-minimal and minimal abelian groups, some outstanding questions remain unanswered. In the quasi-minimal case it would be interesting to know if we can dispense with the assumption of GCH in the characterisation of the reduced torsion-free purely quasi-minimal groups (see II, Theorem 3.12). For the directly quasi-minimal groups it seems that the direct quasi-minimality or otherwise of the group described by Corner (see II, 5) needs to be decided as a prelude to the complete characterisation of the countable decomposable torsion-free directly quasi-minimal groups.

Turning to minimal groups, chief among the open problems is the question of characterising arbitrary rank completely decomposable groups in terms of the types of their canonical summands, a characterisation that has been achieved in the finite rank case (see V, Theorem 4.17).

Finally, more information on non-splitting mixed minimal groups, particularly in the torsion-free rank 1 case, would be desirable. However, it is not clear how such information might be obtained.

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