2005-01-01

Smith predictor structure stability analysis using Mikhailov stability criterion

Pauline Sourdille
Technological University Dublin

Aidan O'Dwyer
Technological University Dublin, aidan.odwyer@tudublin.ie

Eugene Coyle
Technological University Dublin

Recommended Citation

Sourdille, Pauline and O'Dwyer, Aidan and Coyle, Eugene : Smith predictor structure stability analysis using Mikhailov stability criterion. Proceedings of the 4th Wismarer Automatisierungssymposium, Wismar, Germany, September 22-23.

Smith predictor structure stability analysis using Mikhailov stability criterion

Pauline Sourdille$^{\phi}$, Aidan O'Dwyer* and Eugene Coyle^{+}

School of Control Systems and Electrical Engineering,
Dublin Institute of Technology, Kevin Street
Dublin 8, IRELAND
E-mail: $^{\phi}$ pauline.sourdille@dit.ie
*E-mail: * aidan.odwyer@dit.ie*
E-mail: $^{+}$eugene.coyle@dit.ie

Abstract – **As it well known, stability is an important requirement for control systems. Due to the nature of time-delay processes, common methods to evaluate stability may be difficult to use. In 1938, Mikhailov proved a frequency response criterion, which is sufficient and necessary for the stability of processes described by known n^{th} order constant coefficient linear differential equations. This paper presents the Mikhailov's method and the application of this method to a Smith predictor structure controlled by a PI controller.**
Keywords – **Dead-time compensator, process control, stability analysis.**

I Introduction

As is well known, good control of processes with long time delay may be difficult using PID control. In 1957, O. J. Smith developed the Smith predictor structure to compensate systems with time delay. This structure is a model-based structure, which uses a mathematical model of the process in a minor feedback loop. One of the advantages of the Smith predictor structure is that it can be easily extended from a single input-single output system to a multiple-input multiple-output system. Over the years, many modifications to the Smith predictor structure have been proposed to improve the responses. The starting point of this analysis was the need to evaluate the stability of the first and second modified Smith predictor structures developed by Sourdille and O'Dwyer [1-3].

The Smith predictor is composed of transcendental transfer functions. Therefore, mathematical models for steady-state stability analysis contain transcendental functions and consequently cannot be reduced to state space or polynomial form. As is well known, approximations of such functions by truncated series expansions may distort the stability conditions. Consequently, methods of modal analysis and algebraic stability criteria cannot be used for steady state stability analysis of systems which have a transcendental characteristic equation, i.e., an infinite number of roots. Consequently, frequency domain tests have become increasingly more prevalent in stability analysis. Generally, frequency domain tests are often favoured for their conceptual simplicity and computational ease and because they can be implemented in an efficient manner by plotting graphically an appropriate frequency-dependent measure [4].

In 1938, Mikhailov proved a frequency response criterion, which is sufficient and necessary for the stability of processes described by known n^{th} order constant coefficient linear differential equations. Consequently, stability depends on the shape of the so-called ***Mikhailov hodograph***, that is the curve situated in a complex plane and connected with the locus of the characteristic equation. This criterion belongs to the class of methods applying the principle of argument to various problems in control and stability.

This paper explores the possibility of implementing the Mikhailov stability criterion for a Smith predictor structure controlled by a PI controller. In Part II, the Mikhailov's method is briefly presented. In Part III, this method is applied to a Smith predictor controlled by a PI controller.

II Mikhailov's stability criterion

This section introduces the concept of Mikhailov's method. First, the principle of argument is presented. Then, the Mikhailov's stability criterion is introduced. And finally, the Mikhailov's method is improved for time-delay processes.

a) Principle of argument

As stated before, the Mikhailov criterion is obtained by a simple application of the principle of argument. [5] defines the principle of argument as being:

If a function f is meromorphic on the interior of a rectifiable simple closed curve C, then $\frac{1}{2\pi}\oint_C \frac{f'(z)}{f(z)}dz$ equals the difference between the number of zeros and the number of poles of f counted with multiplicity, where $f'(z)$ is the derivative of $f(z)$.

b) Mikhailov's stability criterion for time-delay processes

A direct application of the argument principle may result in the following extended Mikhailov criterion for time-delay systems:

$p(\lambda)$ is uniformly asymptotically stable if and only if the variation of $\arg p(j\omega)$ is $\frac{n\pi}{2}$ when ω varies from 0 to ∞; i.e., $\arg p(j\omega)\big|_0^\infty = \frac{n\pi}{2}$.

The stability of time-delay processes can be evaluated by applying the Mikhailov criterion on the hodograph of the corresponding quasi-polynomial. However, the application of the classical Mikhailov criterion can become difficult, even when a computer is used [4, 6-7]. In fact, a visual conclusion on stability with respect to the constructed part of the hodograph is not practically realisable, since, along with an infinite spiral, the time-delay generates loops whose number is infinite. This point is illustrated in the next part using a Smith predictor structure.

c) Improved Mikhailov's criterion

[4] presents a series of improved Mikhailov's methods and explains that the method that seems the most suitable for the authors's application is the improvements developed by [8-10]. For this method, the following characteristic quasi-polynomial is defined:

$$\delta(s) = d(s) + e^{-s\tau_1}n_1(s) + e^{-s\tau_2}n_2(s) + ... + e^{-s\tau_m}n_m(s)$$

Where $d(s)$, $n_i(s)$ for $i = 1,2,...,m$ are polynomials with real coefficients. The following conditions for the characteristic equation must be respected:

- $\deg\left[n_i(s)\right] < n$ for $i = 1,2,...,m$ with $\deg\left[d(s)\right] = n$
- $0 < \tau_1 < \tau_2 < ... < \tau_m$

Then instead of $\delta(s)$, the following quasi-polynomial can be considered:

$$\delta^*(s) = e^{s\tau_m}\delta(s)$$
$$\delta^*(s) = e^{s\tau_m}d(s) + e^{-s(\tau_m-\tau_1)}n_1(s) + e^{-s(\tau_m-\tau_2)}n_2(s) + ... + n_m(s)$$

The authors explain that $\delta^*(s)$ is stable if and only if:

1. $\delta_r(\omega)$ and $\delta_i(\omega)$ have only simple real roots and these interlace.
2. $\delta_i'(\omega_o)\delta_r(\omega_o) - \delta_i(\omega_o)\delta_r'(\omega_o) > 0$ (1) for some ω_o in (-∞,∞), where $\delta_r'(\omega)$ and $\delta_i'(\omega_o)$ denote the first derivative with respect to ω of $\delta_r(\omega)$ and $\delta_i(\omega_o)$, respectively.

Write $\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$, where $\delta_r(\omega)$ and $\delta_i(\omega)$ represent respectively the real and imaginary parts. Similar analysis can be found in [11-12].

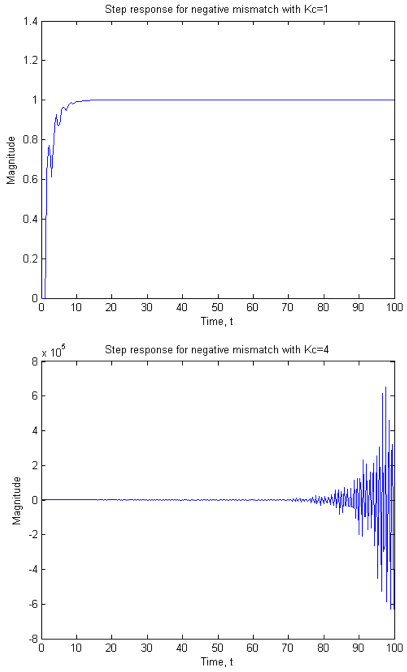
III Implementation of Mikhailov's method for Smith predictor

In this section, the Mikhailov's method is analysed for a Smith predictor structure controlled by PI controller. In order to evaluate the reliability of this method, it is necessary to have a reference response, which is presented in the first section. Then, the Mikhailov's hodographs for the Smith predictor structure are plotted. Finally, the results obtained using an improved Mikhailov's method are given. The process and model transfer functions are given by equation $G_p = \frac{K_p}{T_p s + 1}$ (2) and $G_m = \frac{K_m}{T_m s + 1}$ (3), respectively.

a) Step responses

As it is difficult to represent time-delay systems due to their infinite nature, the Nyquist method for determining stability cannot be used directly, for example. So, to obtain some cross-reference on the operation of the Mikhailov's method, closed-loop step responses are determined. In order to obtain a general analysis, the process and model transfer functions are different, i.e., $G_p = \frac{1.6}{0.5s + 1}$ (4) and $G_m = \frac{2}{0.7s + 1}$ (5). It can be noticed that the difference between the process and model transfer functions is negative. For this study, the process time-delay is $\tau_p = 1.2$ and the process model time-delay is fixed at $\tau_m = 1.4$. As the authors' concern is to evaluate the efficiency, accuracy and reliability of the method and not to obtain the best response possible, the controller time constant, T_I, is chosen to be equal to one. The proportional gain of the controller is selected by trial and error to obtain two situations: one which gives stable responses and a second which gives unstable responses.

Figures 1: Step responses of a Smith predictor



b) Mikhailov's hodographs

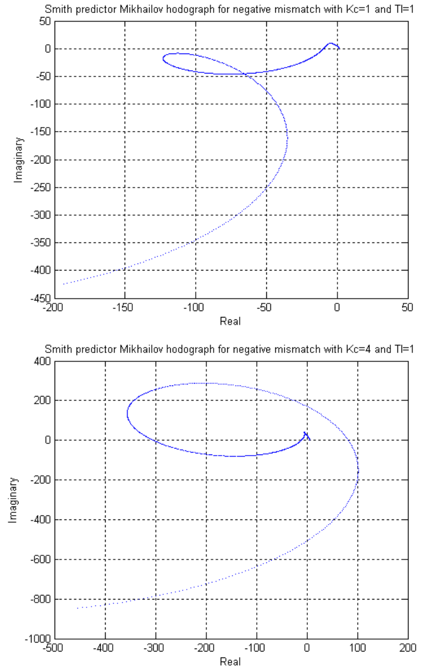
Using Mikhailov's criterion, it is known that $G(s)$ is uniformly asymptotically stable if and only if the variation of $\arg G(j\omega)$ is $\frac{n\pi}{2}$ when ω varies from 0 to ∞; i.e., $\arg G(j\omega)\big|_0^\infty = \frac{n\pi}{2}$. The closed loop transfer function of a Smith predictor controlled by a proportional controller can be calculated using equations (2) and (3), from which the following characteristic equation may be determined:

$$
\begin{aligned}
G_{charact} &= T_m T_p T_I s^3 + (T_p T_I + T_m T_I + K_c K_m T_p T_I)s^2 \\
&+ (T_I + K_c K_m T_I + K_c K_m T_p)s \\
&+ K_c K_m - \left(K_c K_m T_I T_p s^2 + K_c K_m T_I s + K_c K_m T_p s + K_c K_m\right)e^{-s\tau_m} \\
&+ \left(K_c K_p T_I T_m s^2 + K_c K_p T_I s + K_c K_p T_m s + K_c K_p\right)e^{-s\tau_p}
\end{aligned}
\quad (6)
$$

From the above characteristic equation, $n = 3$, so the stability condition becomes $\arg G(j\omega)\big|_0^\infty = \frac{3\pi}{2}$.

The following figures show the Mikhailov's hodographs using the PI controller values defined in the step response trial and error procedure and the process and model parameters detailed previously. The plots represent the imaginary part of equation (6) plotted versus its real part. From these plots, it is difficult to conclude if the closed loop system is stable or not. In fact, a visual conclusion on stability with respect to the constructed part of the hodograph is not practically realisable, since along with an infinite spiral, the time-delay generates loops whose number is infinite [4, 6-7].

Figures 2: Imaginary part versus real part



In conclusion, improved Mikhailov methods have to be used.

c) Improved methods

In the following section, the stability for a Smith predictor structure controlled by a proportional is studied using the improved Mikhailov method developed by [8-10].

First, the initial conditions have to be checked for the characteristic equation (6) which are respected. Secondly, the characteristic equation is multiplied by $e^{\tau_m s}$, which permits to obtain the following real and imaginary parts with $\tau = \tau_m - \tau_p$.

$$
\begin{aligned}
\mathrm{Re}[G^*_{charact}] &= \left(-T_I\left(T_p + T_m + K_c K_m T_p\right)\omega^2 + K_c K_m\right)\cos\omega\tau_m \\
&+ \left(T_m T_p T_I \omega^2 - \left(T_I + K_c K_m T_I + K_c K_m T_p\right)\right)\omega\sin\omega\tau_m \\
&+ K_c K_m\left(T_I T_p \omega^2 - 1\right) + K_c K_p\left(-T_I T_m \omega^2 + 1\right)\cos\omega\tau \\
&- \left(K_c K_p T_I + K_c K_p T_m\right)\omega\sin\omega\tau
\end{aligned}
\quad (7)
$$

$$
\begin{aligned}
\mathrm{Im}[G^*_{charact}] &= \left(-T_m T_p T_I \omega^2 + T_I + K_c K_m T_I + K_c K_m T_p\right)\omega\cos\omega\tau_m \\
&+ \left(-T_I\left(T_p + T_m + K_c K_m T_p\right)\omega^2 + K_c K_m\right)\sin\omega\tau_m \\
&- \left(K_c K_m T_I + K_c K_m T_p\right)\omega + \left(K_c K_p T_I + K_c K_p T_m\right)\omega\cos\omega\tau \\
&+ K_c K_p\left(-T_I T_m \omega^2 + 1\right)\sin\omega\tau
\end{aligned}
\quad (8)
$$

Now, the first stability condition expressed by equation (1) can be checked. For simplicity $\omega_0 = 0$, as [8-10] define $\omega_0 \in (-\infty, +\infty)$. Using equations (4) and (5) with $K_c = 1$ and $T_I = 1$,

$Re[G^*_{charact}]*Im[G^*_{charact}]' - Re[G^*_{charact}]'*Im[G^*_{charact}] = 10.944$ and with $K_c = 4$ and $T_I = 1$, $Re[G^*_{charact}]*Im[G^*_{charact}]' - Re[G^*_{charact}]'*Im[G^*_{charact}] = 155.904$. This permits to conclude that the first stability condition is respected for both proportional controller values when the process and model time-delay are equal or different.

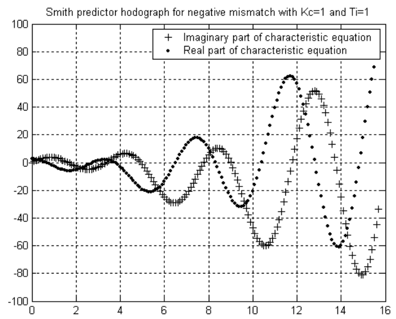
This stability condition also permits to calculate limits for the proportional controller, K_c. In fact, if $Re[G^*_{charact}]*Im[G^*_{charact}]' - Re[G^*_{charact}]'*Im[G^*_{charact}] = 0$,

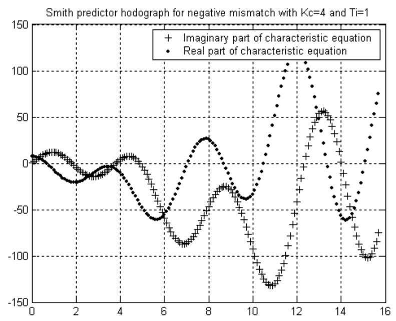
$$K_p K_c \left[K_c \left(K_m \tau_m + K_p T_I + K_p T_m + K_p \tau \right) + K_p T_I \right] = 0$$

By plotting this expression, it can be seen that $Re[G^*_{charact}]*Im[G^*_{charact}]' - Re[G^*_{charact}]'*Im[G^*_{charact}] > 0$ when $K_c < -\dfrac{T_I}{K_m \tau_m + K_p T_I + K_p T_m + K_p \tau}$ or $K_c > 0$.

The second stability condition states that the roots of the imaginary and real parts should be simple and real roots and interlace. This condition can be evaluated graphically by plotting the imaginary and real parts versus the frequency [8-10, 12].

Figures 3: Roots interlacing of the imaginary and real parts





From these figures, it can be noticed that the roots interlace when $K_c = 1$ and they do not interlace when $K_c = 4$. This corresponds to step response results, i.e., the closed-loop system is stable for $K_c = 1$ and unstable for $K_c = 4$., It has also been noticed that the roots interlace for $\tau_m = \tau_p$, which again corresponds to the step response results found.

IV Conclusion

In conclusion, the Mikhailov's method improved using [8-10]'s method gives the same conclusions as the step response when the graphical method is used. So, it can be concluded that this method is suitable for the analysis of the stability of the Smith predictor controlled by a PI controller. Using [8-10]'s stability criteria, limits for the proportional controller have been defined. As future work, the interlacing of the roots will be studied analytically. A necessary condition is that the roots of the imaginary and real parts are real and simple, which can be proven using Pontryagin's theorem [13], for example.

REFERENCES

[1] Sourdille, P. and O'Dwyer, A. (2003a). "An outline and further development of Smith predictor based methods for compensation of processes with time delay". *Irish Signals and Systems Conference*, Limerick, Ireland ,pp. 338-343

[2] Sourdille, P. and O'Dwyer, A. (2003b). "A new modified Smith predictor Design". *International Symposium on Information and Communication Technologies Conference,*Trinity College, Dublin, Ireland ,pp.404-409

[3] Sourdille, P. and O'Dwyer, A. (2003c). "New modified smith predictor designs". *International Federation of Automatic Control workshop on Time-delay systems, INRIA*, Rocquencourt, France

[4] Sourdille, P. and O'Dwyer, A. (2005) "A graphical method to evaluate the stability of a Smith predictor". *Submitted at Irish Signals and Systems Conference, 2005*.

[5] PlanetMath (2005). "Argument principle", *www.mathworld.wolfram.com/ArgumentPrinciple.html [accessed February 2005]*

[6] Stroev, V.A., Karasev, E.D. and Legkokonets, P.V. (2002). "Generalisation of steady state stability criterion for power systems", *14th PSCC*, Sevilla, Session 2, paper 3.

[7] Drozd, A.P. and Filer, Z.E. (1999). "Application of Mikhailov Criterion to the study of Stability of Systems with delay", *Journal of Computer and Systems Sciences International*, Vol. 38, 3, pp 339-342

[8] Silva, G.J., Data, A. and Bhattacharyya, S.P. (2000). "Stabilisation of Time Delay systems", *Proceeding of the American Control Conference*, pp 963-970

[9] Silva, G.J., Data, A. and Bhattacharyya, S.P. (2002a). "New results on the synthesis of PID controllers", *IEEE Transaction on automatic control*, Vol. 47, No 2, pp 241-252

[10] Silva, G.J., Data, A. and Bhattacharyya, S.P. (2002b). "PID tuning revisited: guaranteed stability and non-fragility", *Proceeding of the American Control Conference*, pp 5000-5006

[11] Kuang, Y. (1993). "Delay differential equations: with applications in population dynamics", *Academic Press, London.*

[12] Marshall, J. (1979). "Control of Time-Delay Systems", *IEE Control Engineering Series 10. Peter Peregrinus Ltd.*

[13] Pontryagin, L.S. (1958). "On the zeros of some elemenatary transcendental functions", English translation, *American Mathematical society Translation*, Vol. 2, pp 95-110