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Self and On-selfadjoint Sturm-Liouville Operators with Exponentially Decaying Potentials

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Abstract

In this thesis we investigate the properties of a class of linear differential operators known as Sturm-Liouville operators. Sturm-Liouville operators arise from differential equations of the form

$$-y''+q(x)y=\lambda y,$$

where q is known as the potential and λ is a spectral parameter. This differential equation has been of great importance in mathematics and physics alike. We allow the potential to be real or complex valued and assume that q satisfies

$$|q(x)|\leq ce^{-ax}, \quad x\geq 0,$$

where $a>0$ and $c>0$.

We first give a brief account of the mathematical background we will be using and then proceed to give a summary of Sturm-Liouville theory on the half-line, following Weyl, Titchmarsh, Sims and Naimark.

In chapter three, we investigate the properties of a series derived initially by Eastham, identify zero-free regions for the Jost solution of the Sturm-Liouville equation and conclude by highlighting the significance of our results in terms of the eigenvalues, resonances and spectral singularities of the Sturm-Liouville operator. We also give several examples to support the validity of our results.

The last chapter is dedicated to the study of a series derived by Harris and Gilbert. We show that this series can be used to complement the results obtained in the previous chapter and, finally, we use the property of this series to investigate the points of spectral concentration associated with the Sturm-Liouville operator.

Declaration

I certify that this thesis which I now submit for examination for the award of PhD is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

This thesis was prepared according to the regulations for postgraduate studies by research of the Dublin Institute of Technology and has not been submitted in whole or part for an award in any other Institute or University.

The work reported on in this thesis conforms to the principles and requirements of the Institute's guidelines for ethics in research.

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Signature (Candidate): 22/06/2005

Date:

Bien que l'on ait du coeur a l'ouvrage, l'art est long et le temps est court.

Charles Baudelaire

Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.

Albert Einstein

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Introduction

Our aim is to classify (as far as possible) Sturm-Liouville operators, selfadjoint and nonselfadjoint, with potentials satisfying $|q(x)| \leq ce^{-ax}$, $x \geq 0$, in terms of their eigenvalues, resonances and spectral singularities (and, for selfadjoint operators, also in terms of their points of spectral concentration). Intuitively the idea is that, if $|q(x)| \leq ce^{-ax}$, $x \geq 0$, for some $a > 0$ and $c > 0$, then the operator associated with the Sturm-Liouville equation $-y'' + q(x)y$ should share most of the properties of the selfadjoint operator associated with $q \equiv 0$, whether q is real-valued or not. Our results are mainly based on the study of series associated with the Sturm-Liouville differential equation. The theories of self- and nonself-adjoint Sturm-Liouville operators are not necessarily compatible, for example there is a definition of the absolutely continuous spectrum that applies only to purely nonselfadjoint Sturm-Liouville operators and not to selfadjoint operators. However, for the class of potentials considered, the two theories do have some features in common, as we propose to illustrate. The main new results in this thesis are contained in chapters 3 and 4.

In the first chapter, we give a summary of the mathematical background needed for the following chapters and a summary of the classical Sturm-Liouville theory, concluding with Sim's extension of Weyl's limit-point, limit-circle theory. In the second chapter, we give an overview of the theory of Sturm-Liouville operators with real or complex valued exponentially decaying potentials, i.e. potentials satisfying $q(x) = O(e^{-ax})$ as $x \to +\infty$ and conclude with a version of Naimark's expansion theorem for nonselfadjoint Sturm-Liouville operators. We also introduce the Jost solution and the Jost function and describe the relationship between the latter and the Titchmarsh-Weyl function. Since we are investigating the consequences of changes in the potential q, we restrict ourself to classical, real boundary conditions at 0.

The third chapter is mainly based on the study of some series representing the Jost solution associated with the Sturm-Liouville equation with exponentially decaying potential. These series were derived by Eastham [12, 11]. We examine the convergence of the series for potentials satisfying $|q(x)| \leq ce^{-ax}$, $x \geq 0$, and then then use the properties of the series to provide bounds for the sets of eigenvalues, of spectral singularities and of resonances. The use of the Jost solution allows us to view eigenvalues, resonances and spectral singularities as a single mathematical object, naimly as arising from the zeros of the Jost function. We also show that the series is extremely well-suited for the study of the Jost solution associated with potentials of the form $q(x) = ce^{-ax}$, $x \geq 0$, since we can derive from these series a closed form for the Jost solution. We conclude this chapter by giving a few examples derived from the Bessel and hypergeometric equations.

In the fourth chapter we use some results and ideas due to Harris and Gilbert [17], in particular a series associated with the Titchmarsh-Weyl function and the Riccati equation $v' = -\lambda + q - v^2$ (which in turn is related to the Sturm-Liouville equation), to derive bounds on the sets of eigenvalues, of spectral singularities and of resonances. These results are based on the fact that, via its relationship with the Jost solution, the Titchmarsh-Weyl function can be analytically extended. The results in this chapter are complementary to the ones obtained in the fourth chapter, since we obtain tighter bounds but require (in general) different and more restrictive conditions on q. We show in particular that a relatively simple expansion theorem holds for a class of nonselfadjoint operators. We conclude by using the series to investigate the spectral concentration phenomenon for a class of selfadjoint operators associated with real-valued potentials satisfying $|q(x)| \leq ce^{-ax}$, $x \geq 0$. We show in particular that some operators have no points of spectral concentration at all. We illustrate our results on spectral concentration with some graphs, using Maple. Most of our bounds on the set of points of spectral concentration (but not all) are tighter than the ones given in [17], but we put more restrictive conditions on the potential q.

Most of the results in chapters 3 and 4 are given in terms of conditions on a and c only, for $|q(x)| \leq ce^{-ax}$, $x \geq 0$.

Chapter 1

MATHEMATICAL BACKGROUND

We now give a relatively brief account of the mathematical tools we will be using later.

1.1 REAL AND COMPLEX ANALYSIS

Most of the results are well-known and can be found in [34], so we do not find it necessary to give references for their proof. In this section we recall some results concerning complex analysis (including analytic extensions and complex square roots), series and integration.

1.1.1 Notation and some General Results

By a *region* Ω in the complex plane $\mathbb{C}$ we mean a nonempty connected open subset of $\mathbb{C}$.

Let $r > 0$ and $a \in \mathbb{C}$. We denote by $D(a, r)$ the open ball of centre a and radius r, ie

$$D(a,r) = \{z \in \mathbb{C} : \ |z - a| < r\},$$

by $\overline{D}(a, r)$ the corresponding closed ball and $D'(a, r)$ the deleted open ball of centre a and radius r:

$$D'(a,r) = \{z \in \mathbb{C} : \ 0 < |z - a| < r\}.$$

A function f defined on a region Ω is said to be *holomorphic* on Ω if, for all $z \in \Omega$, $f(z)$ is differentiable. We say $f \in H(\Omega)$.

A function f defined on a region Ω is said to be *analytic* if for each $z_0 \in \Omega$ there exists $r > 0$ such that $D(z_0, r) \in \Omega$ and f can be represented by a power series

$$f(z) = \sum_{n \geq 1} c_n (z - z_0)^n$$

that converges for $z \in D(z_0, r)$.

It is a well-known result that f is analytic on Ω if and only if $f \in H(\Omega)$.

The following theorem says that the set of zeros of a function $f \in H(\Omega)$ can only have limit points on the boundary of Ω.

Theorem 1.1.1. *Let $f \in H(\Omega)$ and set*

$$Z(f) = \{a \in \Omega : \ f(a) = 0\}.$$

The either $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω. In the latter case for each $a \in Z(f)$ there exists $m \in \mathbb{N}$, $r > 0$ such that

$$f(z) = (z-a)^m g(z), \quad z \in \Omega,$$

where $g \in H(\Omega)$, $D(a,r) \subset \Omega$ and g does not vanish inside $D(a,r)$. In this case a is said to be a zero of order m.

We will see later that the eigenvalues of a class of operators are given by the zeros of some function analytic in $\mathbb{C} \setminus \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$. The above theorem implies then that the set of eigenvalues can only have accumulation points on the positive semi-axis.

Let $a \in \Omega$. We say that $f \in H(\Omega \setminus \{a\})$ has a *pole* of order m at a if there exists $g \in H(\Omega)$ such that

$$f(z) = g(z) + \sum_{k=1}^{k=m} \frac{c_k}{(z-a)^k}.$$

$\mathrm{Res}(f,a) = c_1$ is said to be the *residue* of f at the pole a.

If f has a countable number of poles in Ω, we say that f is meromorphic in Ω.

A *path* in $\mathbb{C}$ is a piecewise continuously differentiable function $\gamma[a,b] \subset \mathbb{R} \to \mathbb{C}$. A *closed simple path* γ is a path γ such that its range γ^* is closed in $\mathbb{C}$ and such that

$$\mathrm{Ind}_\gamma(z) = 0 \text{ if } z \in \Omega_0 \text{ and } \mathrm{Ind}_\gamma(z) = 1 \text{ if } z \in \Omega_1,$$

where Ω_0 is the unbounded component of $\mathbb{C} \setminus \gamma^*$, $\Omega_1 = \mathbb{C} \setminus \{\Omega_0 \cup \gamma^*\}$ and

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2i\pi} \int_\gamma \frac{dw}{w-z}, \quad z \in \Omega_0 \cup \Omega_1.$$

We give the following version of the residue theorem:

Theorem 1.1.2. *Let f be meromorphic in Ω and let γ be a closed simple path in Ω such that the interior of γ^* contains the poles $z_1, \ldots, z_n$ of f. Then*

$$f(z) = \frac{1}{2i\pi}\int_\gamma \frac{f(w)}{w-z}dw + \frac{1}{2i\pi}\sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

This theorem allows us, in chapter 2, to sketch the proof of Naimark's expansion theorem for a class of nonselfadjoint operators.

1.1.2 Square Root and Analytic Continuation

We define the notions of complex square root and analytic extension, notions essential to the understanding of eigenvalues, resonances and spectral singularities. For the results in this section, we refer to [19, Chapter 10]

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$. We then have $\lambda = re^{i\theta}$, with $r > 0$ and $\theta \in (0, 2\pi)$.

If we want to find a complex number z such that $z^2 = \lambda$, we can choose it to be

$$z = r^{1/2}e^{i\theta/2},$$

in which case, since $\theta/2 \in (0, \pi)$, $\Im z = r^{1/2}\sin(\theta/2) > 0$.

We can also make the following choice for z:

$$z = -r^{1/2}e^{i\theta/2} = r^{1/2}e^{i(\theta+2\pi)/2},$$

in which case $\Im z < 0$.

The former choice is called the *first analytic determination of the square root* and the latter choice *the second analytic determination of the square root*. The first determination induces a bijective mapping v from $\mathbb{C}\setminus\mathbb{R}^+$ into $\{z \in \mathbb{C} : \ \Im z > 0\}$:

$$\begin{array}{rcl} v : \mathbb{C}\setminus\mathbb{R}^+ & \to & \{z \in \mathbb{C} : \ \Im z > 0\} \\ \lambda & \mapsto & z = \sqrt{\lambda} \end{array}$$

The choice $z = \sqrt{\lambda}$, $\Im z > 0$, determines uniquely the analytic determination. The positive semi axis in the z-plane corresponds to the "upper edge" of the cut $[0, +\infty)$ in the λ-plane and the negative semi axis in the z-plane corresponds to the "lower edge" of the cut $[0, +\infty)$ in the λ-plane (see also [32]).

We will continue this discussion about the complex square root after we have defined the notion of analytic continuation.

Let D_i, $i = 1, 2$, be two open circular discs and let $f_i \in H(D_i)$, $i = 1, 2$.

The functions f_1 and f_2 are said to be *analytic continuations* of each other if $D_1 \cap D_2 \neq \emptyset$ and if $f_1(z) = f_2(z)$ for $z \in D_1 \cap D_2$. In such cases we can define an analytic function f on $D = D_1 \cup D_2$ such that $f(z) = f_i(z)$ for $z \in D_i$, $i = 1, 2$ and f is said to be the analytic continuation of f_i, $i = 1, 2$ on D. Note that, if the analytic continuation exists, it is unique.

Now, let us consider for example the function $g(\lambda) = 1/\left(1 - 2i\sqrt{\lambda}\right)$, $\Im\sqrt{\lambda} > 0$, which can be rewritten as $g(z^2) = 1/(1 - 2iz)$, $z = \sqrt{\lambda}$, $\Im z > 0$. The function $z \mapsto 1/(1 - 2iz)$ is initially defined on the upper half z-plane $\{z \in \mathbb{C} : \Im z > 0\}$ but it is readily seen that it can be analytically extended to the half z-plane $\{z \in \mathbb{C} : \Im z > -1/2\}$ and even to $\mathbb{C} \setminus \{-i/2\}$. Of course, since the lower half z-plane $\{z \in \mathbb{C} : \Im z < 0\}$ has no image in the λ-plane under v^{-1}, the analytic extension of the function $z \mapsto 1/(1 - 2iz)$ is not an analytic extension of the function g itself. The analytically extended function can however provide information on the function g itself, since, for example, the existence of a pole at $z = -i/2$ affects the modulus of $g(\lambda)$.

For $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$, we have

$$\lambda = re^{i\theta} \text{ and } z = \sqrt{\lambda} = r^{1/2}e^{i\theta/2}, \quad \theta \in (0, 2\pi),$$

so that $\Im\sqrt{\lambda} > 0$ and

$$\overline{\lambda} = re^{i(2\pi-\theta)} \text{ and } \sqrt{\overline{\lambda}} = r^{1/2}e^{i(\pi-\theta/2)}, \quad 2\pi - \theta \in (0, 2\pi).$$

We then end up with $\sqrt{\overline{\lambda}} = -r^{1/2}e^{-i\theta/2}$ and $\Im\sqrt{\overline{\lambda}} > 0$

We have therefore

$$\sqrt{\overline{\lambda}} = \overline{-\sqrt{\lambda}}.$$

Note that we cannot take $\overline{\lambda} = re^{-i\theta}$ since we would have $Arg(\overline{\lambda}) \in (-2\pi, 0)$ and would have to choose the second determination of the square root. Intuitively, we have avoided the cut $\{\lambda \geq 0\}$ and rotated anticlockwise in the λ-plane.

Let us consider an example that will be of interest later.

Let $z = \sqrt{\lambda}$, $\Im z > 0$, let $f(z) = \cos(\alpha) + iz\sin(\alpha)$, with $\alpha \in [0, \pi)$. For $\Im z < 0$, define $g(z) = f(-z)$. For $\Im\lambda = 0$ the considerations above imply that

$$g(z) = f(-z) = \cos(\alpha) - iz\sin(\alpha) = \overline{f(z)}$$

and that

$$\lim_{\epsilon \to 0^+} f\left(\sqrt{\lambda + i\epsilon}\right) = f(-\sqrt{\lambda}), \quad \lambda > 0.$$

These kinds of considerations are implicit in, for example, Kodaira's expansion theorem for a class of selfadjoint Sturm-Liouville operators (see [23]) and in Naimark's expansion theorem for a class of nonselfadjoint Sturm-Liouville operators (see [32], appendix II and also chapter two of the present thesis).

1.1.3 Series and Integration

We begin by recalling a few classical results on series.

Let $\{a_n\}_{n\geq 1}$ be a sequence of real or complex numbers. The series $\sum_{n\geq 1} a_n$ is said to *converge* if the sequence $S_n = \sum_{k=1}^n a_k$ converges. S_n is also called the partial sum of the series.

$\sum_{n\geq 1} a_n$ *converges absolutely* if $\sum_{n\geq 1} |a_n|$ converges. We say

$$\sum_{n\geq 1} |a_n| < +\infty.$$

If a series is absolutely convergent, then we can modify the order of the terms without changing the value of the series.

Let

$$a_n = (-1)^n b_n,$$

where $b_n > 0$ and b_n decreases to 0 as n increases. Then the series $\sum_{n\geq 1} a_n$ converges but is not necessarily absolutely convergent: the series is said to be *semi-convergent* or *alternated*.

For alternated series we have

$$|R_n| = \left|\sum_{k=n}^{+\infty} a_k\right| \leq |a_n|.$$

Let $\{f_n(x)\}_{n\geq 1}$ be a sequence of functions of a real variable, which are real or complex valued. Suppose that for each x the series $\sum_{n\geq 1} f_n(x)$ converges.

$\sum_{n\geq 1} f_n(x)$ is said to be *uniformly convergent* if the bound on the partial sum $S_n(x)$ does not depend on x.

Let $\sum_{n\geq 1} a_n$ be an absolutely convergent series. Then if

$$\sum_{n\geq 1} |f_n(x)| \leq \sum_{n\geq 1} |a_n|$$

for all x, $\sum_{n\geq 1} f_n(x)$ converges uniformly (these series are sometimes said to be normally convergent).

If $\{f_n\}$ is a sequence of continuous functions and if $\sum_{n\geq 1} f_n(x)$ converges uniformly, then

$$f(x) = \sum_{n\geq 1} f_n(x)$$

is continuous.

If, moreover, $f_n \in C^1$ and $\sum_{n\geq 1} f_n'$ converges absolutely and uniformly, then $f \in C^1$ and

$$f'(x) = \sum_{n\geq 1} f_n'(x).$$

We find it convenient to recall here two classical examples:

(I) Let $S_n(x) = 1 + x + x^2 + \cdots + x^n$. Then

$$S_n(x) - xS_n(x) = 1 - x^{n+1},$$

so that

$$S_n(x) = \frac{1 - x^{n+1}}{1 - x}.$$

For $|x| < 1$ we therefore have

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \quad \text{and} \sum_{n\geq k} x^n = x^k \sum_{n\geq 0} x^n = x^k \frac{1}{1-x},$$

where the series above converge absolutely and uniformly for $|x| < 1$.

We also have

$$|\sum_{k=0}^{n} x^n| \leq \sum_{k\geq 0} |x^n| = \frac{1}{1-|x|}, \quad |x| < 1.$$

These relations will be especially useful in the last chapter.

(II) We also recall that

$$e^z = 1 + \sum_{n\geq 1} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

and that the series on the right hand side converges absolutely and uniformly.

We now gives a few results concerning Lebesgue and Riemann integrals.

Let us consider Riemann integrals first. Let $[a, b]$ be a interval on the real line, $a < b \leq +\infty$, and let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$. If f_n converges uniformly to a function f, then f is Riemann integrable on $[a, b]$ and

$$\int_a^b \lim_{n\to+\infty} f_n(x)dx = \int_a^b f(x)dx = \lim_{n\to+\infty} \int_a^b f_n(x)dx.$$

For Lebesgue integrals, we recall the Lebesgue Dominated Convergence Theorem:

Theorem 1.1.3. *Let X be a measurable subset of $\mathbb{R}$ and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions such that* *(i)* $f_n(x) \to f(x)$ *as* $x \to +\infty$ *a.e. for* $x \in X$,

(ii) there exists $g \in \mathcal{L}(\mathbb{R}^+)$ *such that* $|f_n(x)| \leq g(x)$ *for all* n *and a.e. for* $x \in X$.

Then $f \in \mathcal{L}(\mathbb{R}^+)$ *and*

$$\int_X \lim_{n\to+\infty} f_n(x)dx = \int_X f(x)dx = \lim_{n\to+\infty} \int_X f_n(x)dx.$$

The result below will also be used regularly in the following chapters:

If $g \in \mathcal{L}(\mathbb{R}^+)$ and if

$$f(x) = \int_x^{+\infty} g(t)dt,$$

then f is absolutely continuous and $f'(x) = -g(x)$ *a.e.*

1.2 LINEAR OPERATORS

In this section, we give an overview of the theoretical setting in which the analysis of the following chapters take place with, of course, particular references to the case considered, i.e Sturm-Liouville operators in $\mathcal{L}^2(\mathbb{R}^+)$.

1.2.1 Hilbert Spaces

An *Inner Product Space* on $\mathbb{C}$ is a vector space H together with an inner product $< .,. >$, i.e. a sesquilinear function $< .,. > H \times H \to \mathbb{C}$ such that
(i) For all $x \in H$, $< x, x > \geq 0$ and $< x, x >= 0$ if and only if $x = 0$.
(ii) For all $x, y \in H$, $< x, y >= \overline{< y, x >}$

$\|x\| =< x, x >^{1/2}$ is a norm on H

A *Hilbert space* H is a complete inner product space.

On any Hilbert space H, the *Schwarz inequality* holds:

$$| < x, y > | \leq \|x\| \|y\| \quad \text{for all } x, y \in H.$$

In a Hilbert space H, we define the orthogonal complement of a subset M of H to be

$$M^\perp = \{x \in H : \ < x, m >= 0 \text{ for all } m \in M\}$$

and write $H = M \oplus N$ if M and N are Hilbert subspaces of H, if $N+M = H$ and if $M^\perp = N$, where $N + M = \{n + m : \ n \in N, \ m \in M\}$.

We give two examples:

(i) $\mathcal{L}^2(\mathbb{R}^+) = \left\{f : \ \int_{\mathbb{R}^+} |f(x)|^2 dx < +\infty\right\}$ together with the inner product

$$< f, g >= \int_{\mathbb{R}^+} f(x)\overline{g(x)} dx$$

is a Hilbert space.

In $\mathcal{L}^2(\mathbb{R}^+)$, the Schwarz inequality takes the form

$$\left| \int_{\mathbb{R}^+} f(x)\overline{g(x)} dx \right|^2 \leq \left(\int_{\mathbb{R}^+} |f(x)|^2 dx \right) \left(\int_{\mathbb{R}^+} |g(x)|^2 dx \right).$$

In particular, if $f, g \in \mathcal{L}^2(\mathbb{R}^+)$ then $fg \in \mathcal{L}(\mathbb{R}^+)$.

(ii) $l_2(\mathbb{C}) = \left\{c = (c_n)_{n \in \mathbb{N}}, \ c_n \in \mathbb{C} : \ \sum_{n \geq 1} |c_n|^2 < +\infty\right\}$ together with the inner product

$$< a, b >= \sum_{n \geq 1} a_n \overline{b_n}$$

is a Hilbert space.

1.2.2 Linear Operators

A *linear operator* L in a Hilbert space H is a linear mapping $L : D(L) \to H$, where $D(L) \subseteq H$ is the domain of L.

If L is *bounded*, ie if there exists $B > 0$ such that $\|L(x)\| \leq B\|x\|$ for all $x \in H$, we set

$$\|L\| = \sup_{\|x\|=1} \|L(x)\| = \inf\{B > 0 : \ \|L(x)\| \leq B\|x\| \text{ for all } x \in H\}.$$

A linear operator is bounded if and only if it is continuous.

A bounded linear operator P on H is said to be a *projection* if $P^2 = P$. If we set $M = PH$ and $N = (I - P)H$, then we have the direct sum $H = M + N$. M and N are not necessarily orthogonal.

We define the angle θ between M and N via the relations

$$\cos(\theta) = \sup\{| < x, y > | : \ x \in M, \ y \in N, \ \|x\| = \|y\| = 1\}, \quad 0 \leq \theta \leq \pi/2.$$

We say that L is an *unbounded densely defined linear operator* if L is an unbounded linear operator and if there exists a dense subspace D of H such that L is defined on D.

A *bounded linear functional* f on H is a bounded linear operator $f : H \to \mathbb{C}$. The *adjoint* of H is defined to be

$$H^* = \{f : \ f \text{ is a bounded linear functional on } H\}.$$

The Riesz-Fisher theorem says in essence that, if H is a Hilbert space, $H = H^*$, which is one of the most interesting features of Hilbert spaces. It allows us to identify a Hilbert space with its adjoint.

If L is a linear operator between two Banach spaces (ie complete normed spaces) X and Y, then we can define its *adjoint operator* L^*, which is a linear operator between Y^* and X^*. In a Hilbert space H, the adjoint of a linear operator L is still a linear operator from H to H.

In a Hilbert space H, we can define the *adjoint* of a linear operator L with domain $D \subseteq H$ to be the operator L^* with domain D^* such that

$$< L(x), y > = < x, L^*(y) > \quad \text{for all } x \in D, \ y \in D^*.$$

L is said to be *selfadjoint* if $D = D^*$ and if

$$< L(x), y >=< x, L(y) > \qquad \text{for all } x, y \in D.$$

L is said to be *symmetric* if

$$< L(x), y >=< x, L(y) > \qquad \text{for all } x, y \in D.$$

A bounded linear operator is symmetric if and only if it is selfadjoint. For unbounded linear operators the notions of symmetry and selfadjointness are not equivalent.

Consider the linear differential equation

$$\tau y = -y'' + q(x)y \qquad x \geq 0,$$

where $q \in \mathcal{L}(\mathbb{R}^+)$.

The operator L_α defined by $Ly = \tau y$ for $y \in D_\alpha$ with

$$D_\alpha = \{y \in \mathcal{L}^2(\mathbb{R}^+) : y, y' \in AC_{loc}(\mathbb{R}^+), \tau y \in \mathcal{L}^2(\mathbb{R}^+), W_0(y, \phi) = 0\},$$

where ϕ is the solution to $\tau y = \lambda y$ satisfying $\phi(0) = -\sin(\alpha)$, $\phi'(0) = \cos(\alpha)$ and where $\alpha \in [0, \pi)$, is an unbounded densely defined linear operator on $\mathcal{L}^2(\mathbb{R}^+)$. If q is real-valued, then L_α is selfadjoint and if q is complex-valued, L_α is nonselfadjoint.

L_α is the operator associated with the *Sturm-Liouville problem* $\tau y = \lambda y$ with *boundary condition* at 0

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0.$$

The adjoint of L_α is the linear operator defined by $L_\alpha^* y = \overline{\tau} y$ for $y \in D_\alpha^*$, where $\overline{\tau} y = -y'' + \overline{q} y$ and

$$D_\alpha^* = \left\{ f : \ \overline{f} \in D_\alpha \right\}.$$

If $\Im q \equiv 0$, it is not hard to prove that L_α is symmetric: using integration by parts twice, it can be shown that, for $f, g \in D_\alpha$

$$\int_{\mathbb{R}^+} (L_\alpha f)\overline{g} dx = \int_{\mathbb{R}^+} (-f'' + qf)\overline{g} dx = \int_{\mathbb{R}^+} f \left(\overline{-g'' + qg}\right) dx = \int_{\mathbb{R}^+} f \left(\overline{L_\alpha g}\right) dx.$$

1.2.3 Spectrum of a Linear Operator

The notion of the inverse of an operator plays an important role in spectral theory. It allows us, for example, to define the spectrum of the operator and, for Sturm-Liouville operators, to obtain expansion theorems using the method of contour integration.

An operator $L : D(L) \to R(L)$, where $R(L)$ is the range of L, is said to be *invertible* if there exists an operator $L^{-1} : R(L) \to D(L)$ such that $LL^{-1}(h) = h$ for all $h \in D(L)$ and such that $L^{-1}L(h) = h$ for all $h \in R(L)$.

We say that an operator L on a Hilbert space H is *boundedly invertible* if L^{-1} exists and is bounded.

If $\lambda \in \mathbb{C}$ and if $(L - \lambda I)$ is boundedly invertible (I being the identity operator on H), $R_\lambda = (L - \lambda I)^{-1}$ is called the *resolvent operator* associated with $(L - \lambda I)$.

If L is unbounded and densely defined, it is possible for $(L - \lambda I)^{-1}$ to exist without being bounded and to be unbounded and densely defined.

If q is real-valued we will se later that, provided the limit point case at $+\infty$ applies (see §1.3.2), there exists an $\mathcal{L}^2(\mathbb{R}^+)$-solution ψ_α of $\tau y = \lambda y$ for $\Im\lambda > 0$ called the Weyl solution and, for $\Im\lambda > 0$, $f \in \mathcal{L}^2(\mathbb{R}^+)$, we have

$$R_\lambda f(x) = \psi_\alpha(x, \lambda) \int_0^x \phi(t, \lambda) f(t) dt + \phi(x, \lambda) \int_x^{+\infty} \psi_\alpha(t, \lambda) f(t) dt.$$

R_λ is an *integral operator*, i.e. an operator that integrates $\mathcal{L}^2(\mathbb{R}^+)$-functions. Note that, since $\psi_\alpha, f \in \mathcal{L}^2(\mathbb{R}^+)$, $\psi_\alpha f \in \mathcal{L}(\mathbb{R}^+)$ and the last term on the right hand side is well defined.

The function

$$K(x, t, \lambda) = \begin{cases} \psi_\alpha(x, \lambda)\phi(t, \lambda), & \text{for } 0 \le t < x \\ \psi_\alpha(t, \lambda)\phi(x, \lambda), & \text{for } x \le t < +\infty \end{cases}$$

is called the *kernel* of the integral operator R_λ.

If q is complex-valued and $q \in \mathcal{L}(\mathbb{R}^+)$, we can still define R_λ but not necessarily for all λ with $\Im\lambda > 0$.

We will see later that if $q \in \mathcal{L}(\mathbb{R}^+)$, R_λ becomes an unbounded densely defined linear operator as λ approaches the real axis $\mathbb{R}^+$ normally from above, whether q is real-valued or not.

We now introduce a notion that is especially of interest to us, the notion of spectral analysis, i.e. the study of the spectra of operators.

If L is a linear operator on a Hilbert space we define the *resolvent set* of L to be

$$\varrho(L) = \{\lambda \in \mathbb{C} : \ (L - \lambda I) \text{ is boundedly invertible}\}.$$

The resolvent set is therefore the set of complex numbers λ such that R_λ exists.

The *spectrum* of L is the complement in $\mathbb{C}$ of the resolvent set:

$$\sigma(L) = \mathbb{C} \setminus \varrho(L).$$

$\lambda \in \mathbb{C}$ is said to be an *eigenvalue* for L if there exists $y \in H$, $y \neq 0$, such that $L(y) = \lambda y$. In this case y is an *eigenvector* associated with the eigenvalue λ. If λ is an eigenvalue, then $(L - \lambda I)$ is not invertible, so that the eigenvalues of L belong to the spectrum $\sigma(L)$. The eigenvalues of a symmetric operator are real and the spectrum of a selfadjoint operator is contained in $\mathbb{R}$. This is not necessarily the case if the operator is nonselfadjoint.

If the operator L is selfadjoint and its spectrum represents the total energy of a physical system, an eigenvalue λ of L is a *bound state* of the energy. If L is nonselfadjoint, the possible interpretations of the eigenvalues are not as clear.

If L is a matrix on $H = \mathbb{C}^n$, then the spectrum of L and the set of its eigenvalues coincide. This is not necessarily true if the dimension of H is not finite. We therefore need to split the spectrum into different parts. One of the possible decompositions is as follows:
The *essential spectrum* of L is defined to be

$$\sigma_e(L) = \sigma(L) \setminus \sigma_f(L),$$

where

$$\sigma_f(L) = \{\lambda \in \sigma(L) : \ \lambda \text{ is an isolated eigenvalue of finite multiplicity } \}$$

To each of these subsets of the spectrum it is possible to associate a subspace of H, namely to $\sigma_e(L)$ we can associate H_e and to $\sigma_f(L)$ we can associate H_f.

If L is selfadjoint, we have $H = H_f \oplus H_e$ and the spectrum of the restriction of L to the subspace H_f (resp. H_e) is $\sigma_f(L)$ (resp. $\sigma_e(L)$). We will say more about such decompositions later.

Since it is possible for R_λ to exist without being bounded, we can also define

$$\sigma_c(L) = closure\{\lambda \in \sigma(L) : \ R_\lambda \text{ is unbounded and densely defined } \}$$

where $\sigma_c(L)$ is known as the *continuous* spectrum of L. Again, we can associate a subspace H_c of H with $\sigma_c(L)$; however, $\sigma_c(L)$ may contain embedded eigenvalues (see for example [8]).

We will see that, if $q \in \mathcal{L}(\mathbb{R}^+)$, the set of eigenvalues of the Sturm-Liouville operator L_α is at most countable and that $\sigma_c(L) = \mathbb{R}^+$, whether q is real or complex valued.

If $q \equiv 0$ and $\alpha \in (0, \pi/2)$, it is known that L_α has a unique eigenvalue $\lambda_\alpha = -\cot(\alpha)$.

1.3 SINGULAR STURM-LIOUVILLE OPERATORS

1.3.1 Notation and Preliminary Results

We are concerned with the study of properties of the operator L_α associated with the differential expression

$$\tau y = -y'' + q(x)y, \quad x \in \mathbb{R}^+, \tag{1.1}$$

where q is a locally integrable complex-valued function of a real variable, and the boundary condition

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0, \qquad \alpha \in [0, \pi). \tag{1.2}$$

A *Sturm-Liouville* problem consists of the differential equation

$$\tau y = \lambda y, \tag{1.3}$$

where λ is a complex parameter, together with the boundary condition (1.2) and, effectively, with a boundary condition at $+\infty$.

Unless stated otherwise, we will use the above notation throughout the following chapters. We will later impose more stringent conditions on the potential q.

Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be two solutions of (1.3) satisfying, respectively

$$\left.\begin{aligned} \phi(0,\lambda) &= -\sin(\alpha), & \theta(0,\lambda) &= \cos(\alpha) \\ \phi'(0,\lambda) &= \cos(\alpha), & \theta'(0,\lambda) &= \sin(\alpha) \end{aligned}\right\}. \tag{1.4}$$

The existence of such solutions can be proved using the method of successive approximations, whether q is real-valued or complex-valued (see [40] and [38]).

Note that, so far as the initial conditions satisfied by θ and ϕ are concerned, we follow Chaudhuri and Everitt [8] rather than Titchmarsh [40].

From (1.4), it is readily seen that ϕ satisfies (1.2) and that a function f satisfies (1.2) if and only if

$$W_0(f(x), \phi(x)) = 0,$$

where

$$W_X(f(x), \phi(x)) = f(X)\phi'(X) - f'(X)\phi(X)$$

is the Wronskian of f and ϕ evaluated at $x = X$.

Let f and g be two solutions of (1.3) satisfying the corresponding equation with λ and λ' respectively. We give two versions of Green's equality, omitting the arguments:

$$(\lambda' - \lambda) \int_0^b fg dx = \int_0^b (f(\tau g) - (\tau f)g)\, dx = -W_b(f,g) + W_0(f,g), \quad (1.5)$$

$$\left(\lambda - \overline{\lambda'}\right) \int_0^b f\overline{g} dx = -\int_0^b (f\overline{\tau g} - (\tau f)\overline{g})\, dx$$
$$= W_b\,(f,\overline{g}) - W_0\,(f,\overline{g}) + 2i \int_0^b \Im(q) f\overline{g} dx. \quad (1.6)$$

We prove the latter.

$$-\int_0^b (f(\overline{\tau g}) - (\tau f)\overline{g})\, dx = \int_0^b \left(-f''(\overline{g}) + f(\overline{g''})\right) dx + \int_0^b f\overline{g}\,(q - \overline{q})\, dx$$

and since

$$\frac{d}{dx} \{W_x\,(f,\overline{g})\} = -f''\overline{g} + f\overline{g''}$$

we get

$$-\int_0^b (f(\overline{\tau g}) - (\tau f)\overline{g})\, dx = W_b\,(f,\overline{g}) - W_0\,(f,\overline{g}) + \int_0^b 2i\Im(q) f\overline{g} dx$$

from which (1.6) follows.

If f and g satisfy (1.3) with $\lambda = \lambda'$ then, by (1.5), $W_b(f,g) - W_0(f,g) = 0$ so that $W_b(f(x,\lambda), g(x,\lambda))$ is a constant independent of b.

If q is real valued and if λ is real, then

$$\tau\phi(x,\lambda) - \overline{\tau\phi(x,\lambda)} = \lambda\left(\phi(x,\lambda) - \overline{\phi(x,\lambda)}\right)$$

and, on the other hand,

$$\tau\phi(x,\lambda)-\overline{\tau\phi(x,\lambda)}=-\left(\phi''(x,\lambda)-\overline{\phi''(x,\lambda)}\right)+q(x)\left(\phi(x,\lambda)-\overline{\phi(x,\lambda)}\right)$$

so that $\Im(\phi(x,\lambda))$ is a solution of (1.3) with $\lambda\in\mathbb{R}$. In view of (1.4),

$$\Im(\phi(0,\lambda))=0 \quad \text{and} \quad \Im(\phi'(0,\lambda))=0$$

and by the uniqueness of the solution, we see that $\Im(\phi(x,\lambda))\equiv 0$. $\phi(x,\lambda)$ is therefore real for real λ and $\Im q\equiv 0$. Similarly, $\theta(x,\lambda)$ is real for real λ and real q.

We also have, for non-real λ and real q:

$$\overline{\lambda}\phi\left(x,\overline{\lambda}\right)=\tau\phi\left(x,\overline{\lambda}\right)=-\phi''\left(x,\overline{\lambda}\right)+q(x)\phi\left(x,\overline{\lambda}\right),$$

$$\overline{\lambda\phi(x,\lambda)}=\overline{\tau\phi(x,\lambda)}=-\overline{\phi''(x,\lambda)}+q(x)\overline{\phi(x,\lambda)}=\tau\overline{\phi(x,\lambda)},$$

so that $\overline{\phi(x,\lambda)}-\phi\left(x,\overline{\lambda}\right)$ is a solution of (1.3).

Since the boundary condition at 0 is real and independent of λ,

$$\overline{\phi(0,\lambda)}-\phi\left(0,\overline{\lambda}\right)=\overline{\phi'(0,\lambda)}-\phi'\left(0,\overline{\lambda}\right)=0$$

and the uniqueness of the solution implies that $\overline{\phi(x,\lambda)}-\phi\left(x,\overline{\lambda}\right)\equiv 0$, from which we get

$$\overline{\phi(x,\lambda)}=\phi\left(x,\overline{\lambda}\right).$$

Similarly,

$$\overline{\theta(x,\lambda)}=\theta\left(x,\overline{\lambda}\right).$$

Note that the results above concerning the behaviour of θ and ϕ for real and complex λ are not, of course, necessarily true when q is complex-valued or when α in (1.2) is allowed to be complex.

1.3.2 The Classical Theory

The results below can be found in [40], with some modifications due to a different choice of initial conditions for ϕ and θ.

We suppose here that q is real-valued.

Since, by (1.4), θ and ϕ are linearly independent solutions of (1.3) the general solution of (1.3) is of the form $a\theta + b\phi$, $a,\ b \in \mathbb{C}$.
We can therefore define the solution of (1.3) satisfying

$$y(b)\cos(\beta) + y'(b)\sin(\beta) = 0$$

(β being real) to be

$$\psi_b(x,\lambda) = \theta(x,\lambda) + m_b\phi(x,\lambda). \tag{1.7}$$

From (1.7), $\psi_b(x,\lambda)$ satisfies the boundary condition above at b if and only if

$$m_b = -\frac{\theta(b,\lambda)\cot(\beta) + \theta'(b,\lambda)}{\phi(b,\lambda)\cot(\beta) + \phi'(b,\lambda)}.$$

Replacing $\cot(\beta)$ by the complex variable w, we get

$$m_b(w,\lambda) = -\frac{\theta(b,\lambda)w + \theta'(b,\lambda)}{\phi(b,\lambda)w + \phi'(b,\lambda)}.$$

Note that $m_b(w,\lambda)$ is a linear fractional transformation, non-degenerate since $W_0(\theta,\phi) = 1$, which maps the real axis $w \in \mathbb{R}$ to a line or a circle.

The inverse transformation is given by (omitting the arguments)

$$w = -\frac{\theta' + m_b\phi'}{\theta + m_b\phi},$$

so that w is real if and only if $w - \overline{w} = 0$, i.e. if and only if

$$|m_b|^2 W_b(\phi,\overline{\phi}) + m_b W_b(\phi,\overline{\theta}) + \overline{m_b} W_b(\theta,\overline{\phi}) + W_b(\theta,\overline{\theta}) = 0. \tag{1.8}$$

By (1.6) with $\Im q \equiv 0$, $\lambda' = \overline{\lambda}$, $f = \phi$ and $\theta = \overline{\phi}$, we have

$$W_b(\phi,\overline{\phi}) = 2i(\Im\lambda)\int_0^b |\phi|^2 dx, \tag{1.9}$$

so that the coefficient of $|m_b|^2$ in (1.8) is non-zero for $\Im\lambda \neq 0$ and, for each b and each λ such that $\Im\lambda \neq 0$, m_b maps the real line to a circle C_b.

The image under m_b of the point $w = -\phi'(b)/\phi(b)$ is ∞ so that the centre of C_b is the image under m_b of the conjugate $-\overline{\phi'}(b)/\overline{\phi}(b)$.

We have therefore for the radius r_b of C_b:

$$r_b = \left|m_b(0,\lambda) - m_b(-\overline{\phi'}(b)/\overline{\phi}(b),\lambda)\right| = \left|\frac{W_b(\theta,\phi)}{W_b(\phi,\overline{\phi})}\right| = \frac{1}{2(\Im\lambda)\int_0^b |\phi|^2 dx}. \tag{1.10}$$

On the other hand

$$\Im\left(\frac{-\phi'(b)}{\phi(b)}\right) = \frac{1}{2i}\frac{W_b(\phi,\overline{\phi})}{|\phi(b)|^2}$$

hence, by (1.9),

$$\Im\left(\frac{-\phi'(b)}{\phi(b)}\right) = (\Im\lambda)\frac{\int_0^b |\phi|^2 dx}{|\phi(b)|^2} > 0$$

for $\Im\lambda > 0$, so that the upper half plane w-plane is mapped by m_b to the exterior of the circle C_b.

It follows that m_b is inside C_b if $\Im\lambda < 0$ which, according to (1.8), is equivalent to

$$\frac{1}{2i}\left(|m_b|^2 W_b(\phi,\overline{\phi}) + m_b W_b(\phi,\overline{\theta}) + \overline{m_b} W_b(\theta,\overline{\phi}) + W_b(\theta,\overline{\theta})\right) < 0,$$

i.e. m_b is inside C_b if

$$\frac{1}{2i} W_b\left(\psi_b, \overline{\psi_b}\right) < 0.$$

By (1.7) and (1.4), $W_0\left(\psi_b,\overline{\psi_b}\right) = \overline{m_b} - m_b = -2i\Im(m_b)$ so that the above, together with (1.6), yields

$$m_b \text{ is inside } C_b \iff \int_0^b |\psi_b|^2 dx < \frac{\Im m_b}{\Im\lambda}. \tag{1.11}$$

The circles C_b are nested: for $0 < b' < b$, we get $r_b < r_{b'}$ so that

$$\int_0^{b'} |\psi_{b'}|^2 dx < \int_0^b |\psi_b|^2 dx < \frac{\Im m_b}{\Im \lambda}$$

and m_b is inside $C_{b'}$.

The circles C_b tend therefore to a limit-point or to a limit-circle as $b \to +\infty$. Let $m(\lambda)$ denote either the limit-point or a point on the limit-circle.

It follows from (1.11) that

$$\int_{\mathbb{R}^+} |\theta + m\phi|^2 dx \leq \frac{\Im m}{\Im \lambda}, \tag{1.12}$$

so that for $\Im\lambda \neq 0$ (1.3) has an $\mathcal{L}^2(\mathbb{R}^+)$-solution

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda).$$

If C_b tends to a limit point then $r_b \to 0$ as $b \to +\infty$ and by (1.10),

$$\int_0^b |\phi|^2 dx \to +\infty \quad \text{as} \quad b \to +\infty$$

so that ψ is, up to a constant multiple, the unique $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3). The function ψ is known as the Weyl solution [40].

If on the other hand C_b tends to a limit-circle, then r_b tends to a constant as $b \to +\infty$ and by (1.10), $\phi \in \mathcal{L}^2(\mathbb{R}^+)$.
In this case all the solutions of (1.3) are in $\mathcal{L}^2(\mathbb{R}^+)$. The following theorem asserts that the limit-point, limit-circle classification above does not depend on the choice of λ.

Theorem 1.3.1. *If there is, up to a constant multiple, only one $\mathcal{L}^2(\mathbb{R}^+)$-solution of $\tau y = \lambda y$ for some λ with $\Im\lambda \neq 0$, namely*

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda),$$

then $\psi(x,\lambda)$ *is, up to a constant multiple, the unique* $\mathcal{L}^2(\mathbb{R}^+)$*-solution of* $\tau y = \lambda y$ *for all* λ *with* $\Im\lambda \neq 0$.

If all the solutions of $\tau y = \lambda y$ *are in* $\mathcal{L}^2(\mathbb{R}^+)$ *for some* $\lambda \in \mathbb{C}$, *then all the solutions of* $\tau y = \lambda y$ *are in* $\mathcal{L}^2(\mathbb{R}^+)$ *for* all $\lambda \in \mathbb{C}$.

Proof: see [9]

It follows from (1.12) that $\Im m(\lambda) > 0$ for $\Im\lambda > 0$. It can also be proved that (see [40]) $m_b(\lambda)$ converges boundedly to $m(\lambda) \in \{\lambda \in \mathbb{C} : \Im\lambda \neq 0\}$, so that $m(\lambda)$ is analytic in the upper half λ-plane, i.e. $m(\lambda)$ is a Nevanlinna function and the limit

$$\lim_{\Im\lambda \downarrow 0+} m(\lambda)$$

exists and is finite Lebesgue almost everywhere.

It can also be shown that if $m(\lambda)$ has a pole on the real axis, this pole must be simple (see [40]).

If the limit-point case at $+\infty$ applies, the operator L_α defined by $L_\alpha f = \tau f$ for $f \in D_\alpha$, with

$$D_\alpha = \{y \in \mathcal{L}^2(\mathbb{R}^+) : y, y' \in AC_{loc}(\mathbb{R}^+), \tau y \in \mathcal{L}^2(\mathbb{R}^+), W_0(y,\phi) = 0\} \quad (1.13)$$

is selfadjoint (see [9]).
The eigenvalues of L_α are the poles of $m_\alpha(\lambda) = m(\lambda)$ (see [9]).

We now give a brief account of the relationship between the Titchmarsh-Weyl function and the spectral function. Consider the Sturm-Liouville problem on $[0,b]$, i.e. the differential equation

$$-y'' + q(x)y = \lambda y, \quad x \in [0,b],$$

where q is locally integrable, with the boundary conditions

$$\begin{aligned} y(0) \quad \cos(\alpha) + y'(0)\sin(\alpha) &= 0, \\ y(b) \quad \cos(\beta) + y'(b)\sin(\beta) &= 0. \end{aligned}$$

The spectrum of the associated operator on $\mathcal{L}^2[0,b]$ consists only of eigenvalues $\lambda_{n,b}$, $n \geq 0$, and the corresponding eigenfunctions are $\phi(x, \lambda_{n,b})$ (see [25]). Let

$$\alpha_{n,b}^2 = \frac{1}{\int_0^b |\phi(x, \lambda_{n,b})|^2 dx}$$

and let

$$\phi_{n,b}(x) = \frac{1}{\alpha_{n,b}} \phi(x, \lambda_{n,b}).$$

For $f \in \mathcal{L}^2[0,b]$, we have the following Parseval equality (see [25]):

$$\int_0^b f^2(x)dx = \sum_{n=0}^{+\infty} \frac{1}{\alpha_{n,b}^2} \left(\int_0^b f(x)\phi_{n,b}(x)dx \right)^2.$$

Let $\rho_b(\lambda)$ be the monotonically increasing step function

$$\rho_b(\lambda) = \begin{cases} -\displaystyle\sum_{\lambda < \lambda_{n,b} \leq 0} \frac{1}{\alpha_{n,b}^2}, & \lambda \leq 0 \\ \displaystyle\sum_{0 < \lambda_{n,b} \leq \lambda} \frac{1}{\alpha_{n,b}^2}, & \lambda > 0. \end{cases}$$

The Parseval equality can then be rewritten as follows:

$$\int_0^b f^2(x)dx = \int_{-\infty}^{+\infty} F^2(\lambda) d\rho_b(\lambda),$$

where

$$F(\lambda) = \int_0^b f(x)\phi(x, \lambda)dx.$$

We can select a sequence $\{b_k\}$ for which $\rho_b(\lambda)$ converges to a monotonically increasing function $\rho(\lambda)$ as $b_k \to +\infty$. Provided $q \in \mathcal{L}(\mathbb{R}^+)$ the function $\rho(\lambda)$ is a step function for $\lambda \leq 0$ and is continuous for $\lambda > 0$. Moreover, still assuming $q \in \mathcal{L}(\mathbb{R}^+)$, we have for $f \in \mathcal{L}^2(\mathbb{R}^+)$

$$f(x) = \int_{-\infty}^{+\infty} F(\lambda)\phi(x, \lambda) d\rho(\lambda),$$

where

$$F(\lambda) = \lim_{n \to +\infty} \int_0^n f(x)\phi(x,\lambda)dx,$$

the convergence being in $\mathcal{L}^2(\mathbb{R}^+)$ (see [25] and [40]). The Titchmarsh-Weyl function and the spectral function are related as follows (see [40]):

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\delta \to 0+} \int_0^\lambda \Im(m(\mu + i\delta))d\mu. \tag{1.14}$$

1.3.3 Sims' Limit-point Limit-circle Theory

We suppose here that q is complex-valued, that $\Im q \leq 0$ and that α in (1.2) is complex with $\Im\alpha \leq 0$.

Following [38], we shall prove that for $\Im\lambda > 0$ there is an $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3), that the limit-point limit-circle classification also applies here although limit-point does not necessarily imply the uniqueness of the $\mathcal{L}^2(\mathbb{R}^+)$-solution and that there is an m-function, analytic in the upper-half λ-plane. We then give McLeod's extension of Sims' results on the the m-function. We highlight only those parts of the theory that are different from the classical case.

The solution

$$\psi_b(x,\lambda) = \theta(x,\lambda) + m_b\phi(x,\lambda)$$

of (1.3) satisfies the boundary condition

$$wy(b) + y'(b) = 0$$

at $x = b$ if

$$m_b(w,\lambda) = -\frac{\theta(b,\lambda)w + \theta'(b,\lambda)}{\phi(b,\lambda)w + \phi'(b,\lambda)}.$$

The latter equality defines a non-degenerate linear fractional transformation, the inverse of which is given by

$$w = -\frac{\theta' + m_b\phi'}{\theta + m_b\phi}$$

and $m_b(w,\lambda) = \infty$ corresponds to $w = -\phi'(b)/\phi(b)$.

Now

$$\Im\left(-\frac{\phi'(b)}{\phi(b)}\right) = \frac{1}{2i}\frac{W_b\left(\phi,\overline{\phi}\right)}{|\phi(b)|^2}$$

and, by (1.6), we have

$$W_b\left(\phi,\overline{\phi}\right) = W_0\left(\phi,\overline{\phi}\right) + 2i\int_0^b \Im(\lambda - q)|\phi|^2 dx.$$

According to (1.4),

$$W_0\left(\phi,\overline{\phi}\right) = -\sin(\alpha)\overline{\cos(\alpha)} + \cos(\alpha)\overline{\sin(\alpha)},$$

$$= 2i\Im\left(\cos(\alpha)\overline{\sin(\alpha)}\right),$$

$$= 2i\Im\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\frac{e^{-i\overline{\alpha}}-e^{+i\overline{\alpha}}}{-2i}\right)$$

from which

$$W_0\left(\phi,\overline{\phi}\right) = i\sinh\left(2(\Im\alpha)\right),$$

so that

$$\Im\left(-\frac{\phi'(b)}{\phi(b)}\right) = |\phi|^{-2}\left(\frac{1}{2}\sinh\left(2(\Im\alpha)\right) + \int_0^b \Im(\lambda - q)|\phi|^2 dx\right).$$

It follows that m_b maps the upper half plane outside a circle C_b of equation $w - \overline{w} = 0$ and the lower half plane inside C_b.

A point m_b is inside or on C_b if

$$\frac{1}{2i}\left(w - \overline{w}\right) \leq 0,$$

i.e. if

$$\frac{1}{2i}W_b\left(\psi_b,\overline{\psi_b}\right) \leq 0.$$

Now, according to (1.6),

$$W_b\left(\psi_b,\overline{\psi_b}\right) = W_0\left(\psi_b,\overline{\psi_b}\right) + 2i\int_0^b \Im(\lambda - q)|\psi_b|^2 dx$$

and by (1.4)

$$W_0\left(\psi_b, \overline{\psi_b}\right) = 2i\Im\left(\cos(\alpha)\overline{\sin(\alpha)}\right)(1+|m_b|^2) - 2i\Im(m_b)\left(|\cos(\alpha)|^2 + |\sin(\alpha)|^2\right),$$

so that

$$W_0\left(\psi_b, \overline{\psi_b}\right) = i\sinh(2\Im(\alpha))(1+|m_b|^2) - 2i\Im(m_b)\cosh\left(2\Im(\alpha)\right).$$

A point m_b is therefore inside or on C_b if

$$\int_0^b \Im(\lambda - q)|\psi_b|^2 dx \leq -\frac{1}{2}\sinh(2\Im(\alpha))(1+|m_b|^2) + \Im(m_b)\cosh\left(2\Im(\alpha)\right). \tag{1.15}$$

Note that (1.15) is consistent with (1.11) (with $\Im\alpha = 0$ and $\Im q = 0$).

From (1.15), as $b \to +\infty$, we get

$$\int_0^{+\infty} \Im(\lambda - q)|\psi|^2 dx \leq -\frac{1}{2}\sinh(2\Im(\alpha))(1+|m|^2) + \Im(m)\cosh\left(2\Im(\alpha)\right), \tag{1.16}$$

where

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

and where $m(\lambda)$ is the limit point or any point on the limit-circle. The function $m(\lambda)$, defined for $\Im\lambda > 0$, is analytic in the upper half-plane $\Im\lambda > 0$ (see [38]).

From (1.16) with $\Im\alpha = 0$ it follows that, for $\Im\lambda > 0$, $\Im m(\lambda) > 0$ so that $m(\lambda)$ is Nevanlinna. Note that this is not true if $\Im\alpha \neq 0$.

There are three distinct possibilities:

Case I: $\psi(x,\lambda)$ is, up to a constant multiple, the only $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3). This solution has the property

$$\int_0^{+\infty} \Im(\lambda - q)|\psi|^2 dx < +\infty.$$

This is a limit-point case.

Case II: all the solutions of (1.3) are in $\mathcal{L}^2(\mathbb{R}^+)$ but only constant multiples of $\psi(x,\lambda)$ are square integrable in $\mathbb{R}^+$ with weight $\Im(\lambda - q)$. This is also a limit-point case.

Case III: all the solutions of (1.3) are square integrable in $\mathbb{R}^+$ with weight $\Im(\lambda - q)$. This is a limit-circle case.

The second case does not have an equivalent in the classical theory, but its existence was justified in [38].

The following theorem asserts that the classification does not depend on λ:

Theorem 1.3.2. *If*

$$\int_{\mathbb{R}^+} |\phi(x,\lambda)|^2 dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^+} |\theta(x,\lambda)|^2 dx < +\infty$$

holds for some λ with $\Im\lambda > 0$, then the same holds for all λ with $\Im\lambda > 0$.

If

$$\int_{\mathbb{R}^+} \Im(\lambda - q(x))|\phi(x,\lambda)|^2 dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^+} \Im(\lambda - q(x))|\theta(x,\lambda)|^2 dx < +\infty$$

holds for some λ with $\Im\lambda > 0$, then the same holds for all λ with $\Im\lambda > 0$.

Proof: see [38]

The following theorem, giving a useful generalisation of Sims' results, is due to McLeod (see [28]). In his concise style, McLeod proved that Sim's results hold in a more general setting, removing some of the conditions on α and q.

McLeod first noticed that, in Sims's case I, $m(\lambda)$ is meromorphic for $\Im\lambda > 0$ without any restriction on α.
Suppose that α is allowed to take any values, that $\psi(x,\lambda)$ is the corresponding $\mathcal{L}^2(\mathbb{R}^+)$-solution and that $\Psi(x,\lambda) = \Theta(x,\lambda) + M(\lambda)\Phi(x,\lambda)$ is the $\mathcal{L}^2(\mathbb{R}^+)$-solution corresponding to the solutions Θ and Φ of (1.3) satisfying (1.4) with $\alpha = 0$. Then since there is, up to a constant multiple, only one $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3) we must have

$$\frac{\sin(\alpha) + m\cos(\alpha)}{\cos(\alpha) - m\sin(\alpha)} = M,$$

According to Sims, $M(\lambda)$ is analytic for $\Im\lambda > 0$, from the equality above we see that $m(\lambda)$ is meromorphic for $\Im\lambda > 0$.

McLeod further noticed that if $\Im q$ is not supposed to be of one sign but if $\Im q(x)$ tends to a finite limit as $x \to +\infty$, say $\Im q(x) \to 0$, then for any λ such that $\Im\lambda \geq \delta > 0$ there exists $a > 0$ such that $\Im\lambda > (\Im q) + \delta/2$ holds if $x \geq a$; so that Sims' results apply and there is an $\mathcal{L}^2(a,+\infty)$ of (1.3). This solution can obviously be extended to be in $\mathcal{L}^2(\mathbb{R}^+)$.

This is the basis of the proof of the theorem below.

Theorem 1.3.3. *We do not suppose that $\Im q$ is of one sign.*

(i) If

$$L = \limsup_{x\to+\infty} \Im q(x) \quad \text{and} \quad l = \liminf_{x\to+\infty} \Im q(x),$$

then there is an $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3) for all λ with $\Im\lambda > L$ or $\Im\lambda < l$ and the corresponding function $m(\lambda)$ is meromorphic on the half planes $\Im\lambda > L$ and $\Im\lambda < l$.

(ii) If

$$\lim_{x \to +\infty} \Im q(x) = \pm\infty$$

then there is an $\mathcal{L}^2(\mathbb{R}^+)$*-solution of* (1.3) *for all* λ *and the corresponding function* $m(\lambda)$ *is meromorphic on the whole* λ*-plane.*

Proof: see [28]

In particular, if $\Im q \to 0$, then there is an $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3) for $\Im\lambda \neq 0$ and the corresponding m-function is meromorphic on the upper and lower half λ-plane.

Chapter 2

EXPONENTIALLY DECAYING POTENTIALS

In this chapter, we present the basic theory of nonselfadjoint Sturm-Liouville operators with exponentially decaying potentials, as investigated by Naimark ([30] and [32], appendix II). We note that this theory also applies to self-adoint Sturm-Liouville operators, although of course the results in this case have been obtained previously by, for example, Titchmarsh ([40] and [41]) and Kodaira ([23]).

We show in particular that, when attempts have been made to adapt it to the nonselfadjoint case, Weyl's limit-point limit circle theory has not yielded satisfactory results or, at least, the results that have been obtained so far are not as spectacular as the ones obtained by Weyl for selfadjoint Sturm-Liouville operators. Another approach was needed and, in the case where $q \in \mathcal{L}(\mathbb{R}^+)$, this leads to the investigation of the properties of a function called the Jost function.

2.1 THE JOST SOLUTION

2.1.1 Definition and Preliminary Results

Let $z = \sqrt{\lambda}$, $\Im z > 0$.

Theorem 2.1.1. *Suppose that q is complex-valued and that $q \in \mathcal{L}(\mathbb{R}^+)$. Then (1.3) has a solution $\chi(x, z)$ satisfying*

$$\chi(x,z) = e^{izx}(1 + o(1)), \qquad \frac{d}{dx}\chi(x,z) = e^{izx}(iz + o(1)) \quad \text{as } x \to +\infty,$$

uniformly in the domain $\Im z \geq 0$, $|z| > \delta$ for every $\delta > 0$.

Moreover, for $\Im z \geq 0$,

$$\chi(x,z) = e^{izx}(1 + O(1/z)), \qquad \frac{d}{dx}\chi(x,z) = e^{izx}(iz + O(1)) \quad \text{as } |z| \to +\infty,$$

uniformly on $x \in \mathbb{R}^+$.

For every $x \in \mathbb{R}^+$, $\chi(x, z)$ is continuous function of z for $\Im z \geq 0$, $z \neq 0$ and is analytic on $\Im z > 0$.

Proof: see [32] (theorem 1, appendix II) and [15]. □

The solution $\chi(., z)$ of (1.3) is known as the Jost solution. This is the solution of (1.3) which is asymptotically close to the $\mathcal{L}^2(\mathbb{R}^+)$-solution e^{izx} of $-y'' = \lambda y$ as $x \to +\infty$ for $\Im z > 0$. The existence of the Jost solution depends on the fact that $q \in \mathcal{L}(\mathbb{R}^+)$.

Taking ϕ to be the solution of (1.3) defined in (1.4), we set

$$W_0(\chi, \phi) = \chi_\alpha(z) = \chi(0, z)\cos(\alpha) + \frac{d}{dx}\chi(x, z)_{|x=0}\sin(\alpha), \qquad (2.1)$$

where $\chi(z) = \chi(0, z) = \chi_0(z)$.

The function $\chi(z)$ is known as the Jost function. As we will see in the next section, the Jost function can be used to determine the properties of the spectrum of L_0. Note that the zeros of $\chi_\alpha(z)$ are the points in the z-domain at which the Jost solution satisfies the boundary condition (1.2), so that the zeros of the Jost function are the points at which the Jost solution satisfies a Dirichlet boundary condition at 0.

With the above notation,

$$\frac{d}{dx}\chi(x, z)_{|x=0} = \chi_{\pi/2}(z).$$

The following theorem identifies a solution of (1.3) that is large at ∞.

Theorem 2.1.2. *For every $\delta > 0$ there exists a solution $\chi_1(x, z)$ of (1.3), analytic as a function of z in the domain $\Im z \geq 0$, $|z| \geq \delta$, satisfying, for each $\delta > 0$,*

$$\chi_1(x, z) = e^{-izx}(1 + o(1)), \qquad \chi_1'(x, z) = e^{-izx}(-iz + o(1)) \quad \textit{as } x \to +\infty$$

uniformly in the domain $|z| > \delta$, $\Im z \geq \delta$.

Proof: see [32] (theorem 3, appendix II). ☐

If we suppose in addition that, for some $a > 0$,

$$q(x) = O(e^{-ax}) \qquad \text{as } x \to +\infty \tag{2.2}$$

then, for every $x \in \mathbb{R}^+$, the solution χ of (1.3) can be analytically extended as a function of z to the half plane $\Im z > -a/2$ (see [32], §3).

It follows from theorems 2.1.1 and 2.1.2 that

$$W_X(\chi(x,z), \chi_1(x,z)) = -2iz + o(1) \qquad \text{as } X \to +\infty$$

uniformly on $\Im z \geq 0$, $|z| > \delta$ and since the Wronskian does not depend on X, we have

$$W(\chi(x,z), \chi_1(x,z)) = -2iz \qquad \Im z \geq 0, \quad |z| > \delta. \tag{2.3}$$

Similarly, for $\Im z = 0$, $|z| \geq \delta$,

$$W(\chi(x,z), \chi(x,-z)) = -2iz. \tag{2.4}$$

Note that, if (2.2) holds, $\chi(x,z)$ is defined for $\Im z > -a/2$ and (2.4) holds for $z \in \{z \in \mathbb{C} : \ |\Im z| < a/2\}$.

It will be necessary later to express various solutions of (1.3) as linear combinations of two other linearly independent solutions of (1.3). To this end we prove the following, using the notation of (2.1):

$$\phi(x,z^2) = \frac{1}{2iz}(\chi_{1,\alpha}(z)\chi(x,z) - \chi_\alpha(z)\chi_1(x,z)) \qquad \Im z > 0, |z| \geq \delta, \tag{2.5}$$

$$\phi(x,z^2) = \frac{1}{2iz}(\chi_\alpha(-z)\chi(x,z) - \chi_\alpha(z)\chi(x,-z)) \qquad \Im z = 0, |z| \geq \delta, \tag{2.6}$$

Note that if (2.2) holds, then (2.6) is valid for $|\Im z| < a/2$.

$$\chi(x,z) = \chi_\alpha(z)\theta(x,z^2) + \omega_\alpha(z)\phi(x,z^2) \qquad \Im z \geq 0, |z| \geq \delta, \tag{2.7}$$

with

$$\omega_\alpha(z) = -\chi(z)\sin(\alpha) + \chi_{\pi/2}(z)\cos(\alpha)) = -W_0(\chi, \theta). \qquad (2.8)$$

We prove (2.5), the proofs of (2.6), (2.7) are similar.
By (2.3), χ and χ_1 are linearly independent solutions of (1.3) so that ϕ can be expressed as a linear combination of them, say

$$\phi = a\chi + b\chi_1.$$

Now,

$$W_0(\chi, \phi) = b\chi_1'\chi - b\chi_1\chi' = bW_0(\chi, \chi_1)$$

and hence by (2.3),

$$W_0(\chi, \phi) = -2izb$$

so that $b = -W_0(\chi, \phi)/2iz = -\chi_\alpha(z)/2iz$ by (2.1) and (1.4).

Similarly, $a = W_0(\chi_1, \phi)/2iz$.

Note that, since $\chi \in \mathcal{L}^2(\mathbb{R}^+)$ and $\chi_1 \notin \mathcal{L}^2(\mathbb{R}^+)$ are linearly independent solutions of (1.3), Sims' case 1 applies at ∞.

2.1.2 Zeros of the Jost Function

Suppose that $\lambda \in \mathbb{C} \setminus [0, +\infty)$, so that $\Im z > 0$.
Any solution of (1.3) satisfying (1.2) is a constant multiple of ϕ and, recalling (2.5),

$$\phi(x, z^2) = \frac{1}{2iz}\left(\chi_{1,\alpha}(z)\chi(x, z) - \chi_\alpha(z)\chi_1(x, z)\right).$$

Since $\chi \in \mathcal{L}^2(\mathbb{R}^+)$ and $\chi_1 \notin \mathcal{L}^2(\mathbb{R}^+)$, $\phi \in \mathcal{L}^2(\mathbb{R}^+)$ if and only if $\chi_\alpha(z) = 0$, so that $\lambda \neq 0$ is an eigenvalue if and only if

$$\lambda = z^2, \qquad \Im z > 0 \qquad \text{and} \quad \chi_\alpha(z) = 0.$$

Now, if q is real valued

$$\chi(x, z) = \overline{\chi}(x, -z)$$

so that, by (2.4),

$$W_0(\chi,\overline{\chi}) = \chi(z)\overline{\chi_{\pi/2}}(z) - \chi_{\pi/2}(z)\overline{\chi}(z) = -2iz, \quad \Im z = 0$$

and $\chi(z) \neq 0$, $\chi_\alpha(z) \neq 0$ for $\Im z = 0,\ z \neq 0$.
If we do not assume that q is real-valued and that α is real, nothing guaranties that the function $\chi_\alpha(z)$ does not have zeros on the real axis $\Im z = 0$. If $\chi_\alpha(z)$ has such a zero z, then for $\lambda = z^2 > 0$ the Jost solution of (1.3) satisfies (1.2), although this solution does not belong to $\mathcal{L}^2(\mathbb{R}^+)$.

We have therefore identified a phenomenon which for the case $q \in \mathcal{L}(\mathbb{R}^+)$ is related only to nonselfadjoint Sturm-Liouville operators.

Definition 2.1.3. $\lambda = z^2$ is said to be a *spectral singularity* for L_α if $z \neq 0$

is a solution of $\chi_\alpha(z) = 0$ with $\Im z = 0$.

If we suppose that (2.2) holds, the set of spectral singularities is finite (see [32] §3, 2., appendix II).

In order to clarify the concept of spectral singularity, we provide an example (we refer to the next chapter for further examples).

Example:
We suppose that $q \equiv 0$ and that $\alpha \in \mathbb{C}$.

In this case the Jost solution associated with (1.3) is $\chi_\alpha(x,z) = e^{izx}$ and we have

$$\chi_\alpha(z) = \cos(\alpha) + iz\sin(\alpha).$$

We recall that the eigenvalues of L_α arise from the zeros of $\chi_\alpha(z)$ which satisfy $\Im z > 0$ and the spectral singularities from the zeros of $\chi_\alpha(z)$ satisfying $\Im z = 0$, $z \neq 0$. Solving $\chi_\alpha(z) = 0$ we obtain

$$z_\alpha = i\frac{\cos(\alpha)}{\sin(\alpha)}.$$

Since

$$\cos(\alpha) = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}), \quad \sin(\alpha) = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$$

we get

$$z_\alpha = -\frac{(e^{i\alpha}+e^{-i\alpha})(e^{-i\overline{\alpha}}-e^{i\overline{\alpha}})}{|e^{i\alpha}-e^{-i\alpha}|^2}$$

and finally

$$z_\alpha = -\frac{\sinh(2\Im(\alpha))}{\cosh(2\Im(\alpha))-\cos(2\Re(\alpha))} + i\frac{\sin(2\Re(\alpha))}{\cosh(2\Im(\alpha))-\cos(2\Re(\alpha))}.$$

Hence z_α gives rise to an eigenvalue if $\sin(2\Re(\alpha)) > 0$ and to a spectral singularity if $\sin(2\Re(\alpha)) = 0$.

Also, if we can extend $\chi_\alpha(z)$ analytically to some region of the lower-half z-plane, it is possible for $\chi_\alpha(z)$ to have zeros with $\Im z < 0$ and these zeros are called resonances.

Definition 2.1.4. Suppose that $q \in \mathcal{L}(\mathbb{R}^+)$. If $\chi_\alpha(z)$ can be analytically extended to some region of the lower-half z-plane $\Im z < 0$, the zeros of $\chi_\alpha(z)$ such that $\Im z < 0$ are called *resonances* for the operator L_α. If $q \in \mathcal{L}(\mathbb{R}^+)$ is real valued, a resonance situated on the semi axis $-i(0,+\infty)$ is said to be an *anti-bound state*.

Example:
Let $q \equiv 0$ and suppose $\alpha \in (0,\pi/2) \cup (\pi/2,\pi)$.

In this case

$$\chi_\alpha(z) = \cos(\alpha) + iz\sin(\alpha)$$

and $\chi_\alpha(z)$ can be extended analytically to the lower-half z-plane.

$$z_\alpha = i\cot(\alpha)$$

is therefore a resonance for $\Im z_\alpha < 0$, ie for $\alpha \in (\pi/2,\pi)$ and gives rise to an eigenvalue for $\alpha \in (0,\pi/2)$.

2.1.3 Jost Function and Titchmarsh-Weyl Function

We suppose in the first instance that $q \in \mathcal{L}(\mathbb{R}^+)$ is real valued. Since the eigenvalues and more generally the spectrum of the operator L_α have traditionally been studied using the properties of the Titchmarsh-Weyl function $m_\alpha(\lambda)$, we discuss the relationship between the Jost function and the Titchmarsh-Weyl function. Note that we make use in chapter 4 of the relationship between these two functions.

Suppose for the sake of simplicity that $\alpha = 0$. Let $\lambda \in \mathbb{C}$ be such that $\Im\lambda > 0$ so that $\Im z > 0$, $\Re z > 0$. Since there is, up to a constant multiple, a unique $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3) for $\Im\lambda > 0$ we have, in the notations of sections 1.3.2 and 2.1.1,

$$\frac{\psi'(x,z^2)}{\psi(x,z^2)} = \frac{\chi'(x,z)}{\chi(x,z)}, \quad \Im z > 0, \quad \Re z > 0.$$

On the other hand $\psi(x,\lambda) = \theta(x,\lambda) + m_0(\lambda)\phi(x,\lambda)$ so that, according to (1.4),

$$\psi(0,\lambda) = 1 \quad \text{and} \quad \psi'(0,\lambda) = m_0(\lambda).$$

We have therefore

$$m_0(z^2) = \frac{\chi'(0,z)}{\chi(0,z)}, \quad \Im z > 0, \quad \Re z > 0,$$

or, in the notation of (2.1),

$$m_0(z^2) = \frac{\chi_{\pi/2}(z)}{\chi(z)}, \quad \Im z > 0, \quad \Re z > 0.$$

Note that $\chi_{\pi/2}(z)$ and $\chi(z)$ are analytic for $\Im z > 0$, so that the equality above provides an analytic extension of $m_0(\lambda)$ into $\mathbb{C}\backslash\{[0,+\infty) \cup \{\lambda = z^2 : \ \chi(z) = 0\}\}$. Since $\chi(z)$ is analytic for $\Im z > 0$, the set $\{z : \ \chi(z) = 0\}$ is at most countable, so that the extended Titchmarsh-Weyl function $m_0(\lambda)$ is meromorphic on $\mathbb{C}\backslash[0,+\infty)$ and the non-zero poles of $m_0(\lambda)$ must satisfy $\lambda = z^2$, $\chi(z) = 0$, $\Im z > 0$.

Suppose now that $\alpha \in (0,\pi)$. There is still a unique $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3) for $\Im\lambda > 0$ so that

$$m_0(\lambda) = \frac{\psi_\alpha'(0,\lambda)}{\psi_\alpha(0,\lambda)}, \quad \Im\lambda > 0,$$

with $\psi_\alpha(x,\lambda) = \theta(x,\lambda) + m_\alpha(\lambda)\phi(x,\lambda)$, so that according to (1.4)

$$m_0(\lambda) = \frac{\sin(\alpha) + m_\alpha(\lambda)\cos(\alpha)}{\cos(\alpha) - m_\alpha(\lambda)\sin(\alpha)}, \quad \Im\lambda > 0.$$

Solving the above for $m_\alpha(\lambda)$ yields

$$m_\alpha(\lambda) = \frac{m_0(\lambda)\cos(\alpha) - \sin(\alpha)}{m_0\sin(\alpha) + \cos(\alpha)}, \quad \Im\lambda > 0.$$

It follows that

$$m_\alpha(z^2) = \frac{\chi_{\pi/2}(z)\cos(\alpha) - \chi(z)\sin(\alpha)}{\chi_{\pi/2}(z)\sin(\alpha) + \chi(z)\cos(\alpha)}, \quad \Im z > 0, \quad \Re z > 0,$$

so that, according to (2.1) and (2.8),

$$m_\alpha(z^2) = \frac{\omega_\alpha(z)}{\chi_\alpha(z)}, \quad \Im z > 0, \quad \Re z > 0. \tag{2.9}$$

Proceeding as above, we can now extend $m_\alpha(\lambda)$ to a function meromorphic on $\mathbb{C} \setminus [0, +\infty)$ and the non-zero poles of m_α are the squares of the zeros of $\chi_\alpha(z)$ satisfying $\Im z > 0$.
Note that, in terms of the Weyl solution ψ_α,

$$\psi_\alpha(x,\lambda) = \frac{1}{\chi_\alpha(z)}\chi(x,z). \tag{2.10}$$

If we suppose now that q is complex valued and that $\Im q \leq 0$, then we can still construct a function $m_\alpha(\lambda)$ analytic for $\Im\lambda > 0$ (see section 1.3.3). Provided we also assume that $q \in \mathcal{L}(\mathbb{R}^+)$, we can extend $m_\alpha(\lambda)$ to a function meromorphic on $\mathbb{C} \setminus [0, +\infty)$ using the arguments decribed above.
If $\Im q$ is not necessarily of one sign and if $\limsup \Im q(x)$ or $\liminf \Im q(x)$ is finite as $x \to +\infty$ then, according to theorem 1.3.3, we can still construct a m-function $m_\alpha(\lambda)$, associated with an $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3), meromorphic in some region of the λ-plane. This m-function can also be extended to a function meromorphic on $\mathbb{C} \setminus [0, +\infty)$ via the Jost function.

If $q \in \mathcal{L}(\mathbb{R}^+)$ is real-valued, then $\chi(z) = \overline{\chi(-z)}$ for $\Im z = 0$ so that, for

$\lambda > 0$,

$$\begin{aligned}\Im m_0(\lambda) &= \frac{1}{2i}\left(\frac{\chi_{\pi/2}(z)}{\chi(z)} - \frac{\overline{\chi_{\pi/2}(z)}}{\overline{\chi(z)}}\right), \\ &= \frac{1}{2i}\frac{W_0(\overline{\chi(z)}, \chi(z))}{\chi(z)\overline{\chi(z)}},\end{aligned}$$

so, from (2.4),

$$\Im m(\lambda) = \frac{z}{|\chi(z)|^2}, \quad \lambda > 0.$$

The equality above and (1.14) yield

$$\rho_0'(\lambda) = \frac{1}{\pi}\frac{\sqrt{\lambda}}{\left|\chi\left(\sqrt{\lambda}\right)\right|^2}, \quad \lambda > 0.$$

2.2 NAIMARK'S EXPANSION THEOREM

In [32] Naimark obtained an expansion theorem for a class of nonselfadjoint Sturm-Liouville operators under strong conditions on the potential q, the proof relying heavily on the fact that $q(x)$ is exponentially decaying as $x \to +\infty$.

In [32] (appendix II) it is assumed, for the sake of simplicity, that $\alpha = 0$ in (1.2). We suppose here that $\alpha \in [0, \pi)$ and only highlight those results which, as a consequence, take a slightly different form.

We suppose thoughout this section that $q \in \mathcal{L}(\mathbb{R}^+)$, and unless otherwise stated is complex-valued.

2.2.1 Eigenvalues

The following theorem summarizes the properties of the operator L_α associated with (1.3) and (1.2).

Theorem 2.2.1. *The operator L_α defined by $L_\alpha y = \tau y$ for $y \in D_\alpha$ with*

$$D_\alpha = \{y \in \mathcal{L}^2(\mathbb{R}^+) : y, y' \in AC_{loc}(\mathbb{R}^+), \tau y \in \mathcal{L}^2(\mathbb{R}^+), W_0(y, \phi) = 0\}$$

is a closed, densely defined linear operator on $\mathcal{L}^2(\mathbb{R}^+)$. Moreover, for any $y \in D_\alpha$

$$y(x),\ y'(x) \to 0 \qquad as\ x \to +\infty.$$

The adjoint operator $L^\star_\alpha$ is defined by $L^\star_\alpha y = \tau^\star y$ for $y \in \overline{D_\alpha}$, with

$$\overline{D_\alpha} = \{f \in \mathcal{L}^2(\mathbb{R}^+) : \overline{f} \in D_\alpha\}$$

and $\tau^\star y = -y'' + \overline{q}y$.

Proof: see [30] (theorems 3.4.2, 3.4.3 and lemma 3.4).

Theorem 2.2.2. *The operator L_α has no eigenvalue in the interval $(0, +\infty)$. The set of eigenvalues of L_α is bounded, no more than countable and its limit points can only lie on the half-axis $\lambda \geq 0$.*

If we suppose also that (2.2) holds, then the number of eigenvalues of L_α is finite and 0 is not an eigenvalue.

Proof: Suppose that $\lambda > 0$, so that $\Im z = 0$, $z \neq 0$. Any solution y of (1.3) can be expressed in the form

$$y = a\chi(x, z) + b\chi(x, -z), \quad \Im z = 0, \quad z \neq 0.$$

From theorem 2.1.1 we have

$$y = ae^{izx} + be^{-izx} + o(1) \qquad \text{as } x \to +\infty$$

so that, for X and Y large enough,

$$\left(\int_Y^X |y|^2 dx\right)^{1/2} \geq \left(\int_Y^X |ae^{izx} + be^{-izx}|^2 dx\right)^{1/2} - \left(\int_Y^X o(1) dx\right)^{1/2},$$

from which it follows that provided a and b are not both equal to 0, $y \notin \mathcal{L}^2(\mathbb{R}^+)$. Hence (1.3) has no non-trivial $\mathcal{L}^2(\mathbb{R}^+)$-solution for $\lambda > 0$ and, in particular, L_α has no eigenvalue for $\lambda > 0$.

Suppose that $\lambda \in \mathbb{C} \setminus [0, +\infty)$, so that $\Im z > 0$.
Any solution of (1.3) satisfying (1.2) is a constant multiple of ϕ and, recalling (2.5),

$$\phi(x, z^2) = \frac{1}{2iz}\left(\chi_{1,\alpha}(z)\chi(x,z) - \chi_\alpha(z)\chi_1(x,z)\right).$$

Since $\chi \in \mathcal{L}^2(\mathbb{R}^+)$ and $\chi_1 \notin \mathcal{L}^2(\mathbb{R}^+)$, $\phi \in \mathcal{L}^2(\mathbb{R}^+)$ if and only if $\chi_\alpha(z) = 0$.

Now according to theorems 2.1.1 and 2.1.2, as $|z| \to +\infty$,

$$\chi_\alpha(z) = \cos(\alpha) + iz\sin(\alpha) + O(1/z),$$

so that $\chi_\alpha(z) \neq 0$ for $|z|$ large enough; and the set of eigenvalues is bounded. The function $\chi_\alpha(z)$ being analytic for $\Im z > 0$, the set of solutions of $\chi_\alpha(z) = 0$ is countable and its limit points can only lie on the real axis.
In terms of the λ plane, it means that the accumulation points of the set of eigenvalues of L_α can only be situated on the semi axis $\lambda \geq 0$.

For the last assertion, we refer to [30] (theorem 3.7.2). □

2.2.2 The Continuous Spectrum

The results presented here hold for $q \in \mathcal{L}(\mathbb{R}^+)$ but can be strengthened if we assume that q satisfies (2.2).
We now proceed to examine the essential spectrum, following Naimark (see [30] and [32]).

Theorem 2.2.3. *Let $\lambda \in \rho(L_\alpha)$ and let $R_\lambda = (L_\alpha - \lambda)^{-1}$. Then, for every $\lambda_0 > 0$,*

$$\lim_{\lambda \to \lambda_0} \|R_\lambda\| = +\infty$$

and the operator R_{λ_0} is densely defined in $\mathcal{L}^2(\mathbb{R}^+)$, so that λ_0 is in the continuous spectrum of L_α.

If we suppose that (2.2) holds then every point $\lambda_0 \geq 0$ belongs to the continous spectrum of L_α.

Proof. Let $\lambda \in \rho(L_\alpha)$. We set

$$K(x,t,z^2) = \begin{cases} \dfrac{1}{\chi_\alpha(z)}\chi(x,z)\phi(t,z^2), & \text{for } 0 \leq t < x \\ \dfrac{1}{\chi_\alpha(z)}\chi(t,z)\phi(x,z^2), & \text{for } x \leq t < +\infty \end{cases} \tag{2.11}$$

Let $f \in \mathcal{L}^2(\mathbb{R}^+)$ and set

$$y(x,z^2) = \int_{\mathbb{R}^+} K(x,t,z^2) f(t)dt.$$

It can be proved that (see [32], §3, theorem 1) for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|y\| \leq \frac{C_\delta}{|\chi_\alpha(z)|\Im z}\|f\| \qquad \Im z > 0, \quad |z| \geq \delta,$$

so that the integral operator R_λ with kernel $K(x,t,z^2)$ is bounded in $\mathcal{L}^2(\mathbb{R}^+)$.

Differentiating twice, we obtain (omitting the arguments)

$$\frac{d^2}{dx^2}y = \frac{\chi''}{\chi_\alpha}\int_0^x \phi f\,dt + \frac{\phi''}{\chi_\alpha}\int_x^{+\infty} \chi f\,dt + \frac{f}{\chi_\alpha}\phi\chi' - \frac{f}{\chi_\alpha}\phi'\chi$$

and since χ, ϕ satisfy (1.3),

$$\frac{d^2}{dx^2}y = (\lambda - q)\left\{\frac{\chi}{\chi_\alpha}\int_0^x \phi f\,dt + \frac{\phi}{\chi_\alpha}\int_x^{+\infty}\chi f\,dt\right\} + \frac{f}{\chi_\alpha(z)}W_x(\phi, \chi).$$

Since the Wronskian on the right hand side does not depend on x, we obtain

$$\frac{d^2}{dx^2}y = (\lambda - q)y - \frac{f}{\chi_\alpha(z)}W_0(\chi, \phi)$$

and, by virtue of (2.1),

$$\frac{d^2}{dx^2}y = (\lambda - q)y - f,$$

so that y satisfies $(L_\alpha - \lambda)y = f$. It is readily seen that y satisfies the boundary condition (1.2). Hence

$$(L_\alpha - \lambda)R_\lambda f = f \qquad \text{for all } f \in \mathcal{L}^2(\mathbb{R}^+).$$

It can also be proved that (see [40])

$$R_\lambda(L_\alpha - \lambda)y = y$$

for all

$$y \in D_\alpha = \{y \in \mathcal{L}^2(\mathbb{R}^+) : y, y' \in AC_{loc}(\mathbb{R}^+), \tau y \in \mathcal{L}^2(\mathbb{R}^+), W_0(y, \phi) = 0\},$$

so that

$$R_\lambda = (L_\alpha - \lambda)^{-1}.$$

We set

$$f_b(x) = \begin{cases} \overline{\phi}(x, z^2), & \text{for } 0 \le x < b \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_b \in \mathcal{L}^2(\mathbb{R}^+)$, $\|f_b\| > 0$ and

$$(R_\lambda f_b)(x) = \|f_b\|^2 \frac{\chi(x, z)}{\chi_\alpha(z)} \qquad \text{for } x > b.$$

On the other hand

$$\|R_\lambda f_b\|^2 \ge \int_b^{+\infty} |R_\lambda f_b(x)|^2 dx$$

so that

$$\|R_\lambda f_b\|^2 \geq \frac{\|f_b\|^4}{|\chi_\alpha|^2} \int_b^{+\infty} |\chi(x,z)|^2 dx.$$

According to theorem 2.1.1 we can choose b so large that

$$|\chi(x,z)| > \frac{1}{2} e^{-x\Im z} \qquad \text{for } x > b,$$

from which we obtain

$$\|R_\lambda f_b\| \geq \frac{\|f_b\|^2}{2\sqrt{2\Im z}|\chi_\alpha(z)|} e^{-b\Im z}$$

and the first part of the theorem follows.

Let $\lambda_0 > 0$. We are left to prove that R_{λ_0} is densely defined. Since

$$Domain(R_{\lambda_0}) = Range(L_\alpha - \lambda_0),$$

the domain of R_{λ_0} is dense in $\mathcal{L}^2(\mathbb{R}^+)$ if $(Range(L_\alpha - \lambda_0))^\perp = \{0\}$. On the other hand

$$(Range(L_\alpha - \lambda_0))^\perp = \{f : L^* f = \lambda_0 f\},$$

so that proving that R_{λ_0} is densely defined amounts to showing that λ_0 is not an eigenvalue for the adjoint operator L_α^*, which is readily seen from theorems 2.2.2 and 2.2.1. □

The expansion theorem below was derived in [30] and [32] in a more general setting, but we limit ourselves to sketching the proof in the case when L_α has no spectral singularity.

2.2.3 Expansion Theorem

We suppose here that (2.2) holds and that L_α has no spectral singularities. In this case, we will see that the expansion of $\mathcal{L}^2(\mathbb{R}^+)$-functions in eigenfunctions and generalised eigenfunctions of L_α is remarkably similar to the one obtain by Kodaira in [23].

We recall that, since (2.2) holds, the number of eigenvalues of L_α is finite.

We can therefore list the poles of $K(x, t, \lambda)$ as follows: $\lambda_1, \ldots, \lambda_n$ with multiplicity $m_1, \ldots, m_n$ respectively.

Let $R > 0$ and let $\epsilon > 0$. Let $\Gamma_{R,\epsilon}$ denote the closed contour in the λ-plane consisting of
- the arc $C_{R,\epsilon}$ of the circle $|\lambda| = R$ which does not include the points λ such that $|\Im\lambda| < \epsilon$, $\Re\lambda > 0$,
- the arc $c_\epsilon = \{\lambda : |\lambda| = \epsilon, \Re\lambda \leq 0\}$,
- the segment lines $l_{R,\epsilon,+}$ and $l_{R,\epsilon,-}$ of equations $\lambda = a \pm i\epsilon$, $0 \leq a \leq R$.

The orientation of $\Gamma_{R,\epsilon}$ is shown on the figure (2.12).

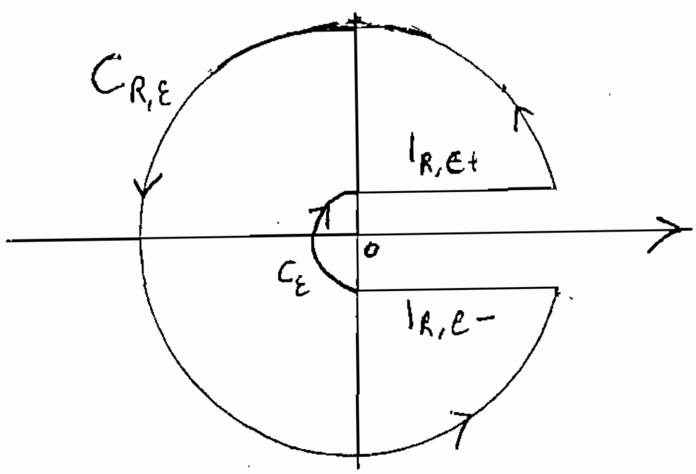


Figure 2.12 Contour $\Gamma_{R,\epsilon}$

Let $\lambda_0 \in \rho(L_\alpha)$. We choose R and ϵ so that the points λ_0, $\lambda_1, \ldots, \lambda_n$ are inside $\Gamma_{R,\epsilon}$ and we set

$$I_{R,\epsilon} = \frac{1}{2i\pi} \int_{\Gamma_{R,\epsilon}} \frac{K(x, t, \lambda)}{\lambda - \lambda_0} d\lambda.$$

The residue theorem yields

$$I_{R,\epsilon} = K(x, t, \lambda_0) + \sum_{k=1}^{k=n} \text{Res}\left(\frac{K(x, t, \lambda_0)}{\lambda - \lambda_0}, \lambda = \lambda_k \right). \qquad (2.13)$$

It follows from (2.7) that we can rewrite the kernel K as follows:

$$K(x,t,\lambda) = \frac{\omega_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})}\phi(x,\lambda)\phi(t,\lambda) + v(x,t,\lambda),$$

where $v(x,t,\lambda)$ is an entire function of λ.

Setting for $k = 1 \dots n$

$$M_k(\lambda) = -\frac{(\lambda-\lambda_k)^{m_k}}{(m_k-1)!}\frac{\omega_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})},$$

we obtain for (2.13)

$$K(x,t,\lambda_0) = I_{R,\epsilon} + \sum_{k=1}^{k=n}\left(\frac{d}{d\lambda}\right)^{m_k-1}\left\{M_k(\lambda)\frac{\phi(x,\lambda)\phi(t,\lambda)}{\lambda-\lambda_0}\right\}.$$

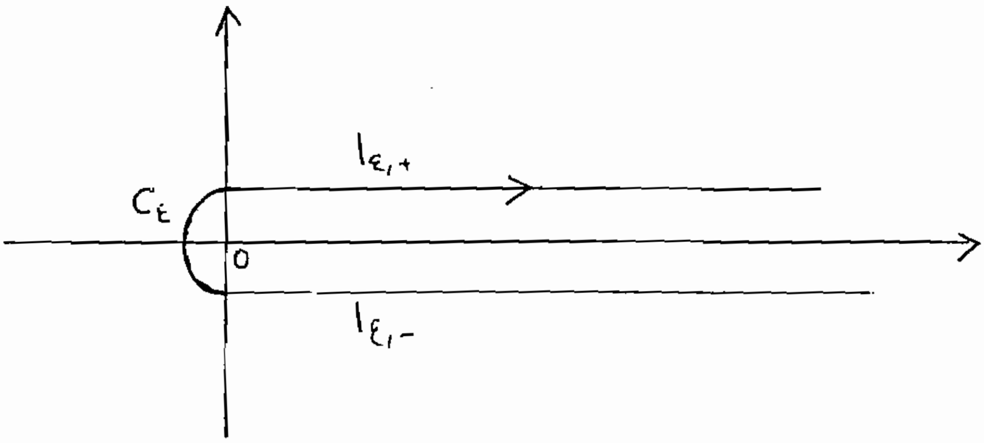
It can be shown that, as $R \to +\infty$,

$$\int_{\mathbf{C}_{R,\epsilon}} \frac{K(x,t,\lambda)}{\lambda-\lambda_0}d\lambda \to 0$$

(see [32] equation 60, §3, 4., appendix II and [30] equation 5.2.7). We then obtain for (2.13)

$$\lim_{R\to+\infty} I_{R,\epsilon} = \frac{1}{2i\pi}\int_{\Gamma_\epsilon}\frac{K(x,t,\lambda)}{\lambda-\lambda_0}d\lambda,$$

where Γ_ϵ, the contour consisting of the arc c_ϵ and of the lines $l_{\epsilon,\pm}$ of equation $\lambda = a \pm i\epsilon$, $a \geq 0$, is shown in figure (2.14)



Contour Γ_ϵ

Figure 2.14

Letting $\epsilon \longrightarrow 0$, we end up with

$$\lim_{R\to+\infty,\epsilon\to 0} I_{R,\epsilon} = \frac{1}{2i\pi}\int_{\mathbb{R}+}\left(\frac{\omega_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})} - \frac{\omega_\alpha(-\sqrt{\lambda})}{\chi_\alpha(-\sqrt{\lambda})}\right)\frac{\phi(x,\lambda)\phi(t,\lambda)}{\lambda-\lambda_0}d\lambda,$$

which shows that

$$K(x,t,\lambda_0) = \frac{1}{2i\pi}\int_{\mathbb{R}+}\left(\frac{\omega_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})} - \frac{\omega_\alpha(-\sqrt{\lambda})}{\chi_\alpha(-\sqrt{\lambda})}\right)\frac{\phi(x,\lambda)\phi(t,\lambda)}{\lambda-\lambda_0}d\lambda +$$

$$\sum_{k=1}^{k=n}\left(\frac{d}{d\lambda}\right)^{m_k-1}\left\{M_k(\lambda)\frac{\phi(x,\lambda)\phi(t,\lambda)}{\lambda-\lambda_0}\right\}.$$

On the other hand, we have

$$\frac{\omega_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})} - \frac{\omega_\alpha(-\sqrt{\lambda})}{\chi_\alpha(-\sqrt{\lambda})} = \frac{\omega_\alpha(\sqrt{\lambda})\chi_\alpha(-\sqrt{\lambda}) - \omega_\alpha(-\sqrt{\lambda})\chi_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})\chi_\alpha(-\sqrt{\lambda})}$$

and according to (2.4), (2.7),

$$\frac{\omega_\alpha(\sqrt{\lambda})}{\chi_\alpha(\sqrt{\lambda})} - \frac{\omega_\alpha(-\sqrt{\lambda})}{\chi_\alpha(-\sqrt{\lambda})} = \frac{2i\sqrt{\lambda}}{\chi_\alpha(\sqrt{\lambda})\chi_\alpha(-\sqrt{\lambda})}$$

so that the above can be rewritten as follows:

$$K(x,t,\lambda_0) = \frac{1}{\pi}\int_{\mathbb{R}+}\left(\frac{\sqrt{\lambda}}{\chi_\alpha(\sqrt{\lambda})\chi_\alpha(-\sqrt{\lambda})}\right)\frac{\phi(x,\lambda)\phi(t,\lambda)}{\lambda-\lambda_0}d\lambda +$$

$$\sum_{k=1}^{k=n}\left(\frac{d}{d\lambda}\right)^{m_k-1}\left\{M_k(\lambda)\frac{\phi(x,\lambda)\phi(t,\lambda)}{\lambda-\lambda_0}\right\}.$$

For a function $f \in \mathcal{L}^2(\mathbb{R}^+)$, we set

$$g(\lambda) = \int_{\mathbb{R}^+} f(x)\phi(x,\lambda)dx.$$

For a justification of the convergence of the integral on the right-hand side in the space $\mathcal{L}^2(\mathbb{R}^+)$ relative to the measure $\sqrt{\lambda}d\lambda$, see [32] (appendix II, §4, 6., lemma 2).

The following theorem can be deduced from the expansion of the kernel of the resolvant (see [32], appendix II, §4, 9., A and [30] theorem 5.4).

Theorem 2.2.4. *For every function $f \in \mathcal{L}^2(\mathbb{R}^+)$, the following expansion formula holds:*

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{\mathbb{R}^+}\left(\frac{\sqrt{\lambda}}{\chi_\alpha(\sqrt{\lambda})\chi_\alpha(-\sqrt{\lambda})}\right)\phi(x,\lambda)g(\lambda)d\lambda + \\ &+ \sum_{k=1}^{k=n}\left(\frac{d}{d\lambda}\right)^{m_k-1}\{M_k(\lambda)\phi(x,\lambda)g(\lambda)\}, \end{aligned}$$

where the integral on the right-hand side converges absolutely and uniformly for $x \in \mathbb{R}^+$.

Note that in the case $\Im q \equiv 0$ and $\alpha \in [0,\pi)$, theorem 2.2.4 reduces to the expansion theorem obtain by Kodaira in [23].

There is also an expansion theorem if we do not suppose that there is no spectral singularity (see [32], appendix II, §5, 8., theorem 3) although the form of the expansion is rather different and involves the regularized value of a divergent integral.

Chapter 3

APPLICATION TO SPECTRAL ANALYSIS

In this chapter, we suppose that $q \in \mathcal{L}(\mathbb{R}^+)$, derive a series for the Jost solution following the methods of [12], examine some of the properties of the associated Sturm-Liouville operator using this series and give two examples related to the Bessel equation and to the hypergeometric equation.

3.1 SERIES REPRESENTATION

The theorem below can be found in ([12], section 3). The proof given below is an extended version of the one outlined in ([12], section 3), following the method given in [11]. We note that we consider here necessary conditions for the convergence of the series associated with the Jost solution, conditions which were not mentioned in [12].

3.1.1 Main Theorem

We begin by recalling the variation of parameters formula for a 2×2 system of first order linear differential equations:

A fundamental matrix for the system

$$Y'(x) = A(x)Y(x),$$

is a matrix $M(x)$ whose columns are linearly independent solutions of the system.

Lemma 3.1.1. *If $W(x, z)$ is a solution of*

$$Y(x,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - M(x,z) \int_x^{+\infty} M^{-1}(t,z) \left(\frac{i}{2z} q(t)\right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} Y(t,z)dt,$$

with

$$M(x,z) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2ixz} \end{pmatrix},$$

then $W(x,t)$ satisfies the following differential equation:

$$Y' = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} + \frac{i}{2z} q \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right\} Y.$$

Proof. Suppose that

$$W(x,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - M(x,z) \int_x^{+\infty} M^{-1}(t,z) \left(\frac{i}{2z} q(t)\right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(t,z)dt.$$

Then

$$\begin{aligned} W'(x,z) &= -M'(x,z) \int_x^{+\infty} M^{-1}(t,z) \left(\frac{i}{2z} q(t)\right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(t,z)dt \\ &+ \frac{i}{2z} q(x) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(x,z). \end{aligned}$$

Since M is a fundamental matrix for the system

$$Y' = \begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} Y,$$

we have

$$M'(x,z) = \begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} M(x,z),$$

so that

$$\begin{aligned} W'(x,z) &= -\begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} M(x,z) \int_x^{+\infty} M^{-1}(t,z) \left(\frac{i}{2z} q(t)\right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(t,z)dt \\ &+ \frac{i}{2z} q(x) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(x,z). \end{aligned}$$

Since

$$-M(x,z) \int_x^{+\infty} M^{-1}(t,z) \left(\frac{i}{2z} q(t)\right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(t,z)dt = W(x,z) - \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

we finally obtain

$$W' = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} + \frac{i}{2z} q \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right\} W,$$

as required. □

Also, we define for a matrix $A(x) = (a_{i,j}(x))_{1 \le i,j \le n}$,

$$|A(x)| = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} |a_{i,j}|.$$

Theorem 3.1.2. *Let* $z = \sqrt{\lambda}$, $\Im(z) > 0$. *Equation* (1.3) *has a solution* $\chi(x,z)$ *satisfying, for each* $\delta > 0$

$$\chi(x,z) = e^{izx}(1 + o(1)) \text{ as } x \to +\infty,$$

uniformly in z for $|z| \geq \delta$ and

$$\chi(x, z) = e^{ixz} \left\{1 + O\left(\frac{1}{z}\right)\right\} \text{ as } |z| \to +\infty,$$

uniformly on $x \geq 0$.

Moreover, this solution can be expressed in the form

$$\chi(x, z) = e^{ixz} \left(1 + \sum_{n \geq 1} r_n(x, z)\right),$$

where the series on the right hand side converges absolutely and uniformly for $x \in \mathbb{R}^+$, $|z| \geq c > \|q\|_1$ and

$$r_0(x, z) = 1, \quad r_n(x, z) = \frac{i}{2z} \int_x^{+\infty} q(t) r_{n-1}(t, z) \left(1 - e^{2iz(t-x)}\right) dt, \quad n \geq 1.$$

For the first derivative we also have

$$\chi'(x, z) = e^{ixz} \left(iz + \sum_{n \geq 1} s_n(x, z)\right),$$

where the series converges absolutely and uniformly for $x \in \mathbb{R}^+$, $|z| \geq c > \|q\|_1$ and

$$s_n(x, z) = -\frac{1}{2} \int_x^{+\infty} q(t) r_{n-1}(t, z)(1 + e^{2iz(t-x)}) dt \quad n \geq 1.$$

Also

$$\chi'(x, z) = e^{izx}(iz + o(1)) \text{ as } x \to +\infty$$

uniformly with respect to z for $|z| \geq \delta$.

Proof. We first transform (1.3) in order to obtain a linear system of the form $W' = (A + R)W$, where A is diagonal and R is integrable.

We then apply lemma 3.1.1 to the system $W' = (A + R)W$ in order to obtain an integral equation. Finally we find a solution of the integral equation by the method of successive approximations.

Let $\chi(x, z)$ be a solution of (1.3) and define

$$W = \frac{1}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi \\ \chi' \end{pmatrix}.$$

Then

$$W' = \frac{-iz}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi \\ \chi' \end{pmatrix} + \frac{1}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi' \\ \chi'' \end{pmatrix}$$

and, since χ is a solution of (1.3),

$$W' = \frac{-iz}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi \\ \chi' \end{pmatrix} + \frac{1}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi' \\ (q - z^2)\chi \end{pmatrix}.$$

Rewriting the last term on the right hand side, we get

$$W' = \frac{-iz}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi \\ \chi' \end{pmatrix} + \frac{1}{2}e^{-ixz}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ q - z^2 & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \chi' \end{pmatrix},$$

so that

$$W' = \frac{1}{2}e^{-ixz}\left\{\begin{pmatrix} 0 & 0 \\ -2iz & 2 \end{pmatrix} + \frac{i}{z}q\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\right\}\begin{pmatrix} \chi \\ \chi' \end{pmatrix}$$

and

$$W' = \frac{1}{2}e^{-ixz}\left\{\begin{pmatrix} 0 & 0 \\ -2iz & 2 \end{pmatrix}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}^{-1} + \frac{i}{z}q\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}^{-1}\right\}\begin{pmatrix} 1 & -i/z \\ 1 & i/z \end{pmatrix}\begin{pmatrix} \chi \\ \chi' \end{pmatrix}$$

The last equality yields

$$W' = \left\{\begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} + \frac{i}{2z}q\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right\}W. \tag{3.1}$$

A fundamental matrix for the system

$$Y'(x,z) = \begin{pmatrix} 0 & 0 \\ 0 & -2iz \end{pmatrix} Y(x,z)$$

is

$$M(x,z) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2ixz} \end{pmatrix},$$

so that according to lemma 3.1.1 a solution W of (3.1) can be represented in the form

$$W(x,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - M(x,z) \int_x^{+\infty} M^{-1}(t,z) \left(\frac{i}{2z} q(t) \right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} W(t,z) dt.$$

The last equality yields

$$W(x,z) = e_1 + \frac{i}{2z} \int_x^{+\infty} q(t) K(t-x,z) W(t,z) dt \tag{3.2}$$

with

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K(s,z) = \begin{pmatrix} 1 & 1 \\ -e^{2isz} & -e^{2isz} \end{pmatrix}.$$

Set

$$Z_0(x,z) = e_1, \quad Z_n(x,z) = e_1 + \frac{i}{2z} \int_x^{+\infty} q(t) K(t-x,z) Z_{n-1}(t,z) dt \quad n \geq 1. \tag{3.3}$$

With $|Z_0(x,z)| = 1$ and

$$|Z_n(x,z)| \leq 1 + \frac{2}{|z|} \int_0^{+\infty} |q(t)||Z_{n-1}(t,z)| dt \quad n \geq 1,$$

it is readily seen, by induction, that each Z_n is bounded.

Moreover it not difficult to see, by induction, that

$$|Z_n(x,z)| \leq \sum_{k=0}^{n-1} \left\{ \frac{2\|q\|_1}{|z|} \right\}^k \leq \sum_{n \geq 0} \left\{ \frac{2\|q\|_1}{|z|} \right\}^n$$

so that $|Z_n|$ is bounded uniformly with respect to n for $|z| \geq d > 2\|q\|_1$.

We also have

$$Z_{n+1}(x,z) = Z_0(x,z) + \sum_{k=0}^{k=n} (Z_{k+1}(x,z) - Z_k(x,z)). \tag{3.4}$$

Now we prove by induction that $|Z_{k+1}(x,z) - Z_k(x,z)| \leq \rho^k$, $k \geq 0$, for some $\rho \in (0,1)$.

$$|Z_1(x,z) - Z_0(x,z)| \leq \frac{1}{2|z|} \int_x^{+\infty} |q(t)||K(t-x,z)Z_0(t,z)|dt$$

and, since

$$|K(t-x,z)Z_0(t,z)| \leq 2,$$

$$|Z_1(x,z) - Z_0(x,z)| \leq \frac{1}{|z|} \int_x^{+\infty} |q(t)|dt.$$

Suppose that

$$|z| > 2\int_0^{+\infty} |q(t)|dt$$

and set

$$\rho = \frac{2\int_0^{+\infty} |q(t)|dt}{|z|}.$$

Suppose that $|Z_k(x,z) - Z_{k-1}(x,z)| \leq \rho^{k-1}$. Then, by (3.3),

$$|Z_{k+1}(x,z) - Z_k(x,z)| \leq \frac{1}{2|z|} \int_x^{+\infty} |q(t)||K(t-x,z)(Z_k(t,z) - Z_{k-1}(t,z)|dt,$$

$$|Z_{k+1}(x,z) - Z_k(x,z)| \leq \frac{4}{2|z|} \int_x^{+\infty} |q(t)||(Z_k(t,z) - Z_{k-1}(t,z)|dt,$$

and finally

$$|Z_{k+1}(x,z) - Z_k(x,z)| \leq 2\frac{\rho^{k-1}}{|z|} \int_x^{+\infty} |q(t)|dt \leq \rho^k,$$

so that

$$\sum_{k=1}^{+\infty} (Z_{k+1}(x,z) - Z_k(x,z))$$

converges absolutely and uniformly for $x \geq 0$ and $|z| \geq d > 2\|q\|_1$.

By (3.4) we now have

$$Z(x,z) = \lim_{n\to+\infty} Z_{n+1}(x,z) = Z_0(x,t) + \sum_{k=1}^{+\infty} (Z_{k+1}(x,z) - Z_k(x,z)).$$

$Z(x,z)$ satisfies (3.2) since, by (3.3) and the Lebesgue dominated convergence theorem,

$$Z(x,z) = \lim_{n\to+\infty} Z_{n+1}(x,z) = \lim_{n\to+\infty} \left(e_1 + \frac{i}{2z}\int_x^{+\infty} q(t)K(t-x,z)Z_n(t,z)dt \right)$$

$$= e_1 + \frac{i}{2z}\int_x^{+\infty} q(t)K(t-x,z) \lim_{n\to+\infty} Z_n(t,z)dt$$

and

$$Z(x,z) = e_1 + \frac{i}{2z}\int_x^{+\infty} q(t)K(t-x,z)Z(t,z)dt.$$

Setting

$$W_0(t,z) = e_1 \text{ and } W_n(x,z) = Z_{n+1}(x,z) - Z_n(x,z) \quad n \geq 1,$$

we obtain a solution

$$W(x,z) = e_1 + \sum_{n=1}^{+\infty} W_n(x,z) \tag{3.5}$$

of (3.2) with

$$W_n(x,z) = \frac{i}{2z}\int_x^{+\infty} q(t)K(t-x,z)W_{n-1}(t,z)dt \quad n \geq 1. \tag{3.6}$$

Setting

$$W_n(x,z) = \begin{pmatrix} u_n(x,z) \\ v_n(x,z) \end{pmatrix},$$

(3.6) becomes

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \frac{i}{2z}\int_x^{+\infty} q(t)\{u_{n-1}(t,z) + v_{n-1}(t,z)\} \begin{pmatrix} 1 \\ -e^{2iz(t-x)} \end{pmatrix} dt$$

and since

$$\begin{pmatrix}\chi(x,z)\\ \chi'(x,z)\end{pmatrix} = \left\{\frac{e^{-ixz}}{2}\begin{pmatrix}1 & -i/z\\ 1 & i/z\end{pmatrix}\right\}^{-1} W(x,z),$$

(3.5) becomes

$$\chi(x,z) = e^{ixz}\left(1 + \sum_{n\geq 1} r_n(x,z)\right) \tag{3.7}$$

$$\chi'(x,z) = e^{ixz}\left(iz + \sum_{n\geq 1} s_n(x,z)\right), \tag{3.8}$$

where, by (3.6), $r_0(x,z) = 1$,

$$r_n(x,z) = u_n(x,z) + v_n(x,z) \Rightarrow r_n(x,z) = \frac{i}{2z}\int_x^{+\infty} q(t) r_{n-1}(t,z)\left(1 - e^{2iz(t-x)}\right) dt \tag{3.9}$$

and

$$s_n(x,z) = iz(u_n(x,z) - v_n(x,z)) \Rightarrow s_n(x,z) = -\frac{1}{2}\int_x^{+\infty} q(t) r_{n-1}(t,z)\left(1 + e^{2iz(t-x)}\right) dt. \tag{3.10}$$

Since $r_0(x,z) = 1$ it is readily seen, by induction, that

$$|r_n(x,z)| \leq \left(\frac{\|q\|_1}{|z|}\right)^n \quad \forall n \geq 0, \tag{3.11}$$

from which we see that the series in (3.7) converges absolutely and uniformly for $x \geq 0$, $|z| \geq c > \|q\|_1$.

Similarly,

$$|s_n(x,z)| \leq \int_x^{+\infty} |r_{n-1}(t,z)||q(t)| dt \leq |z|\left(\frac{\|q\|_1}{|z|}\right)^n$$

from which we see that the series in (3.8) converges absolutely and uniformly for $x \geq 0$, $|z| \geq c > \|q\|_1$.

Also, from (3.7) and (3.9)

$$|\chi(x,z)| \le \left|e^{ixz}\right| \left\{1 + \sum_{n=1}^{+\infty} \left(\frac{1}{|z|}\int_x^{+\infty} |q(t)|dt\right)^n\right\} \quad \forall x \ge 0$$

$$= \left|e^{izx}\right| \left(1 + \frac{\left(\frac{1}{|z|}\int_x^{+\infty} |q(t)|dt\right)}{1 - \left(\frac{1}{|z|}\int_x^{+\infty} |q(t)|dt\right)}\right)$$

for x large enough since $q \in \mathcal{L}(\mathbb{R}^+)$ and finally

$$\chi(x,z) = e^{ixz}(1 + o(1)) \text{ as } x \to +\infty$$

uniformly in $|z| > \delta$ for each $\delta > 0$.

Similarly

$$\chi'(x,z) = e^{izx}(iz + o(1)) \text{ as } x \to +\infty$$

uniformly in $|z| > \delta$ for each $\delta > 0$.

It also follows from (3.7) and (3.11) that

$$\chi(x,z) = e^{ixz}\left\{1 + O\left(\frac{1}{z}\right)\right\} \text{ as } |z| \to +\infty,$$

uniformly on $x \ge 0$. □

Note that the function $\chi(x,z)$ described above is the Jost solution associated with (1.3) and that $\chi(z) = \chi(0,z)$ is the Jost function associated with (1.3).

Since $q \in \mathcal{L}(\mathbb{R}^+)$, we see that (3.9) converges for $\Im z = 0$, so that $\chi(x,.)$ is analytic for $\Im z > 0$ and continuous for $\Im z = 0$, $z \ne 0$.

The following corollary is a straight forward application of theorem 3.1.2.

Corollary 3.1.3. *Suppose that $q \in \mathcal{L}(\mathbb{R}^+)$. Let L_0 be the operator (not necessarily selfadjoint) associated with* (1.3) *and the Dirichlet boundary con-*

dition at 0 *and let* $z^2 = \lambda$*, with* $\Im z > 0$.

Then L_0 *has no eigenvalue* $\lambda = z^2$ *such that* $\Im z > 2\|q\|_1$.

Proof. According to (3.7) and (3.9),

$$|\chi(z)| = \left|1 + \sum_{n\geq 1} r_n(0,z)\right| \geq 1 - \sum_{n\geq 1} |r_n(0,z)|$$

and according to (3.11) $|r_n(0,n)| \leq (\|q\|_1/|z|)^n$ so that for $|z| > \|q\|_1$

$$|\chi(z)| \geq 1 - \sum_{n\geq 1} \left(\frac{\|q\|_1}{|z|}\right)^n = 1 - \frac{\|q\|_1}{|z| - \|q\|_1}.$$

It follows that if

$$\frac{\|q\|_1}{|z| - \|q\|_1} < 1$$

$\chi(z)$ cannot vanish.

Hence $\chi(z) > 0$ for $|z| > 2\|q\|_1$. □

3.1.2 Direct Computation

We derive the Jost solution for (1.3) by direct computation, using (3.7) and (3.9).

Let

$$q(x) = ce^{-ax}$$

in (1.3), where $c \in \mathbb{C}$ and $a \in \mathbb{C}$ is such that $\Re a > 0$.

According to (3.7) and (3.9),

$$r_0(x,z) = 1,$$

$$r_1(x,z) = \frac{ic}{2z} \int_x^{+\infty} e^{-at}(1 - e^{2iz(t-x)})dt$$

so that

$$r_1(x,z) = \frac{ic}{2z}\int_x^{+\infty}(e^{-at} - e^{t(2iz-a)-2izx})dt$$

and

$$r_1(x,z) = \frac{ic}{2z}\left[\frac{e^{-at}}{-a} - \frac{e^{t(2iz-a)-2izx}}{2iz-a}\right]_x^{+\infty}$$

which yields

$$r_1(x,z) = \frac{ic}{2z}\left(\frac{e^{-ax}}{a} + \frac{e^{-ax}}{2iz-a}\right)$$

and finally

$$r_1(x,z) = \frac{ice^{-ax}}{2z}\left(\frac{1}{a} + \frac{1}{2iz-a}\right).$$

Rewriting the above, we obtain

$$r_1(x,z) = \frac{ice^{-ax}}{2z}\frac{2iz}{a(2iz-a)},$$

and

$$r_1(x,z) = i^2ce^{-ax}\frac{1}{a(2iz-a)}.$$

We can then calculate the second term in the series (3.7)

$$r_2(x,z) = \frac{ic}{2z}i^2c\left(\frac{1}{a(2iz-a)}\right)\int_x^{+\infty} e^{-2at}(1-e^{2iz(t-x)})dt,$$

$$r_2(x,z) = \frac{ic}{2z}i^2c\left(\frac{1}{a(2iz-a)}\right)\int_x^{+\infty}(e^{-2at} - e^{t(2iz-2a)-2izx})dt,$$

$$r_2(x,z) = \frac{ic}{2z}i^2c\left(\frac{1}{a(2iz-a)}\right)\left[\frac{e^{-2at}}{-2a} - \frac{e^{t(2iz-2a)-2izx}}{2iz-2a}\right]_x^{+\infty},$$

$$r_2(x,z) = \frac{ic}{2z}i^2ce^{-2ax}\left(\frac{1}{a(2iz-a)}\right)\left(\frac{1}{2a} + \frac{1}{2iz-2a}\right),$$

and finally

$$r_2(x,z) = (ci^2e^{-ax})^2\left(\frac{1}{a(2iz-a)}\right)\left(\frac{1}{2a(2iz-2a)}\right).$$

Suppose that

$$r_p(x,z) = (ci^2 e^{-ax})^p \left(\frac{1}{a(2iz-a)} \cdots \frac{1}{pa(2iz-pa)} \right).$$

Then

$$\begin{aligned}
r_{p+1}(x,z) &= \frac{ic}{2z}(ci^2)^p \left(\frac{1}{a(2iz-a)} \cdots \frac{1}{pa(2iz-pa)} \right) \int_x^{+\infty} e^{-(p+1)at}(1-e^{2iz(t-x)})dt, \\
&= \frac{ic}{2z}(ci^2)^p \left(\frac{1}{a(2iz-a)} \cdots \frac{1}{pa(2iz-pa)} \right) \left[\frac{e^{-(p+1)at}}{-(p+1)a} - \frac{e^{t(2iz-(p+1)a)-2izx}}{2iz-(p+1)a} \right]_x^{+\infty}, \\
&= \frac{ic}{2z}(ci^2)^p \left(\frac{1}{a(2iz-a)} \cdots \frac{1}{pa(2iz-pa)} \right) \left(\frac{1}{(p+1)a} + \frac{1}{2iz-(p+1)a} \right) e^{-(p+1)ax} \\
&= (ci^2 e^{-ax})^{p+1} \left(\frac{1}{a(2iz-a)} \cdots \frac{1}{pa(2iz-pa)} \right) \left(\frac{1}{a(p+1)(2iz-(p+1)a)} \right).
\end{aligned}$$

Hence we obtain

$$r_0(x,z) = 1$$

and

$$r_n(x,z) = (ci^2 e^{-ax})^n \left(\frac{1}{a(2iz-a)} \cdots \frac{1}{na(2iz-na)} \right) \quad n \geq 1.$$

Rewriting the above we get

$$r_n(x,z) = i^{2n}(ca^{-2}e^{-ax})^n \frac{1}{n!} \left(\frac{-1}{(1-2iz/a)} \cdots \frac{-1}{(n-2iz/a)} \right),$$

which yields

$$r_n(x,z) = (ca^{-2}e^{-ax})^n \frac{1}{n!} \left(\frac{1}{(1-2iz/a)} \cdots \frac{1}{(n-2iz/a)} \right) \quad n \geq 1$$

and finally

$$\chi(x,z) = e^{izx} \left\{ 1 + \sum_{n\geq 0} (ca^{-2}e^{-ax})^n \frac{1}{n!} \left(\frac{1}{(1-2iz/a)} \cdots \frac{1}{(n-2iz/a)} \right) \right\}, \tag{3.12}$$

for $z \in \mathbb{C} \setminus \bigcup_{n\geq 1}\{-ina/2\}$. Note that the series above converges absolutely and uniformly for $z \in \mathbb{C}\setminus\bigcup_{n\geq 1}\{-ina/2\}$ and that (3.12) is in fact the Bessel function (see [42])

$$\frac{\Gamma(-2iz+1)}{(ia^{-1}\sqrt{c})^{-iz}} J_{-2iz/a}\left(2ia^{-1}\sqrt{c}e^{-ax/2}\right).$$

3.1.3 Convergence of the Series

We recall that for $q \in \mathcal{L}(\mathbb{R}^+)$, the eigenvalues of L_α are given by $\lambda = z^2$, z being a zero of $\chi_\alpha(z)$ with $\Im z > 0$. Also, if we can extend $\chi_\alpha(z)$ analytically to some region of the lower-half z-plane, it is possible for $\chi_\alpha(z)$ to have zeros with $\Im z < 0$ and these zeros are called resonances.

For the remainder of this section, we do not suppose that q is real valued but put stringent conditions on q, namely we suppose that

$$|q(x)| \leq ce^{-ax}, \qquad x \geq 0, \tag{3.13}$$

for some constants $c > 0$ and $a > 0$.

Let $\delta > 0$ and let

$$\Lambda_{a,\delta} = \{z \in \mathbb{C} : \ \Im z > -a/3, \ |z| > \delta\}.$$

Lemma 3.1.4. *Suppose that* (3.13) *holds and fix* $\delta > 2c/a$. *Then*

$$|r_n(x,z)| \leq \frac{1}{n!}\left(\frac{2c}{|z|a}\right)^n e^{-nax}, \quad x \geq 0, \ \Im z > -a > 3, \ n \geq 1$$

and the series (3.7) *for the Jost solution converges absolutely and uniformly for* $x \geq 0, \ z \in \Lambda_{a,\delta}$.

Proof. According to (3.9) we have

$$r_0(x,z) = 1$$

and, from (3.9) and (3.13),

$$r_1(x,z) \leq \frac{c}{2|z|}\int_x^\infty \left(e^{-at} + e^{-t(a+2\Im z)+2x\Im z}\right) dt,$$

which yields

$$|r_1(x,z)| \leq \frac{c}{a|z|}\left(\frac{a+\Im z}{a+2\Im z}\right) e^{-ax}.$$

The result is therefore true for $n=1$. Suppose that it were true for $1 \leq k \leq n-1$. From (3.9)

$$|r_n(x,z)| \leq \frac{1}{2|z|}\int_x^\infty |q(t)r_{n-1}(t,z)| \left(1+e^{-2(t-x)\Im z}\right) dt,$$

so that, from (3.13) and the induction hypothesis,

$$\begin{aligned} |r_n(x,z)| \quad &\leq \quad \frac{c}{2|z|(n-1)!}\left(\frac{c}{\delta a}\right)^{n-1}\left(\frac{a+\Im z}{a+2\Im z}\right) \ldots \times \\ \ldots \quad &\times \left(\frac{(n-1)a+\Im z}{(n-1)a+2\Im z}\right)\int_x^{+\infty} e^{-nat}(1+e^{-2(t-x)\Im z})dt, \end{aligned}$$

which yields

$$|r_n(x,z)| \leq \frac{1}{n!}\left(\frac{c}{\delta a}\right)^n \left(\frac{a+\Im z}{a+2\Im z}\right)\cdots\left(\frac{(n-1)a+\Im z}{(n-1)a+2\Im z}\right)\left(\frac{1}{na}+\frac{1}{na+2\Im z}\right)e^{-nax}.$$

The lemma is proved when we notice that

$$0 < \frac{na+\Im z}{na+2\Im z} < 2 \quad \text{and} \quad \frac{2c}{|z|a} < \frac{2c}{\delta a} < 1$$

for $\Im z > -a/3$ and $|z| > \delta > 2c/a$. □

We are now in position to identify a region in the z-plane where $\chi(z)$ cannot vanish:

Theorem 3.1.5. *Suppose* (3.13) *holds and fix* $\delta > 2c/a$. *Then, for* $z \in \Lambda_{a,\delta}$,

$$|\chi(z)| \geq 2 - \exp\left(\frac{2c}{\delta a}\right).$$

In particular, if

$$\delta > \frac{2c}{a\ln(2)},$$

then $\chi(z)$ cannot vanish inside the set $\Lambda_{a,\delta}$ and the operator L_0 has

1. *no eigenvalue $\lambda = z^2$ such that $z \in \Lambda_{a,\delta} \cap \{\Im z > 0\}$,*

2. *no spectral singularity $\lambda = z^2$ such that $z \in (-\infty, \delta) \cup (\delta, +\infty)$,*

3. *no resonance inside $\Lambda_{a,\delta} \cap \{\Im z < 0\}$.*

Proof. According to lemma 3.1.4 we have, for $z \in \Lambda_{a,\delta}$,

$$|r_n(x,z)| \leq \frac{1}{n!}\left(\frac{2c}{\delta a}\right)^n e^{-nax}, \quad x \geq 0,$$

so that

$$\left|\sum_{n\geq 1} r_n(x,z)\right| \leq \sum_{n\geq 1} \frac{1}{n!}\left(\frac{2c}{\delta a}\right)^n e^{-nax} = \exp\left(\frac{2c}{\delta a}e^{-ax}\right) - 1.$$

Since

$$|\chi(x,z)| = e^{-x\Im z}\left|1 + \sum_{n\geq 1} r_n(x,z)\right| \geq e^{-x\Im z}\left\{1 - \left|\sum_{n\geq 1} r_n(x,z)\right|\right\},$$

we obtain

$$|\chi(z)| \geq 2 - \exp\left(\frac{2c}{\delta a}\right).$$

In particular, $\chi(z)$ does not vanish if

$$2 - \exp\left(\frac{2c}{\delta a}\right) > 0,$$

i.e. if

$$\delta > \frac{2c}{a\ln(2)},$$

from which 1, 2 and 3 follow. □

We begin our study of the case $\alpha \neq 0$ by giving a criterion for the convergence of the series in (3.8):

Lemma 3.1.6. *Fix $\delta > 2c/a$ and suppose that* (3.13) *holds. Then*

$$|s_n(x,z)| \leq \frac{|z|}{n!}\left(\frac{2c}{|z|a}\right)^n e^{-nax}, \quad x \geq 0,\ \Im z > -a/3,\ n \geq 1,$$

and the series (3.8) *converges absolutely and uniformly for $x \geq 0$, $z \in \Lambda_{a,\delta}$.*

Proof. From (3.10), we get

$$\frac{d}{dx}\chi(x,z) = e^{izx}\left(iz + \sum_{n\geq 1} s_n(x,z)\right)$$

and

$$|s_n(x,z)| \leq \frac{|z|}{2|z|}\int_x^{+\infty} |q(t)r_{n-1}(t,z)|\left(1+e^{-2\Im z(t-x)}\right)dt, \quad n \geq 1.$$

Arguing as in lemma 3.1.4, we obtain the stated result. □

The bounds we obtain for $\alpha \neq 0$ are not as tight as the ones obtained in theorem 3.1.5, which is rather natural as, for $\alpha \neq 0$, it is possible to find resonances far below the real axis or large negative eigenvalues, depending on the value of α. We refer to the first example in the next section for an illustration of this phenomenon.

Theorem 3.1.7. *Suppose that* (3.13) *holds and let δ be such that*

$$\delta > \frac{2c}{a\ln(2)}.$$

Then 1, 2 and 3 of theorem 3.1.5 hold as they stand for the operator $L_{\pi/2}$ and 1, 2 and 3 of theorem 3.1.5 continue to hold for the operator L_α, $\alpha \in (0,\pi/2) \cup (\pi/2,\pi)$, provided we replace δ by $\max\{\delta, \delta_\alpha\}$, where

$$\delta_\alpha = |\cot(\alpha)| \frac{\exp\left(\frac{2c}{\delta a}\right)}{2 - \exp\left(\frac{2c}{\delta a}\right)}.$$

Proof. We first suppose that $\alpha = \pi/2$. According to (3.8) and lemma 3.1.6 we have, for $z \in \Lambda_{a,\delta}$,

$$|\chi_{\pi/2}(z)| \geq |z| - |z| \left\{ \exp\left(\frac{2c}{\delta a}\right) - 1 \right\} = |z| \left\{ 2 - \exp\left(\frac{2c}{\delta a}\right) \right\}. \qquad (3.14)$$

It follows that $\chi_{\pi/2}(z)$ cannot vanish inside $\Lambda_{a,\delta}$ if $\delta > 2c/a\ln(2)$, and the first part of the theorem is proved.
Suppose now that $\alpha \in (0,\pi/2) \cup (\pi/2,\pi)$. From (3.7) and lemma 3.1.4 we get

$$|\chi(z)| \leq 1 + \sum_{n\geq 1} |r_n(0,z)| \leq \exp\left(\frac{2c}{\delta a}\right).$$

On the other hand, according to (2.1),

$$|\chi_\alpha(z)| \geq |\sin(\alpha)\chi_{\pi/2}(z)| - |\cos(\alpha)\chi(z)|$$

so that, with (3.14), we obtain

$$|\chi_\alpha(z)| \geq |z\sin(\alpha)| \left\{ 2 - \exp\left(\frac{2c}{\delta a}\right) \right\} - |\cos(\alpha)| \exp\left(\frac{2c}{\delta a}\right).$$

From the equality above, it is not hard to see that $\chi_\alpha(z) > 0$ for

$$|z| > |\cot(\alpha)| \frac{\exp\left(\frac{2c}{\delta a}\right)}{2 - \exp\left(\frac{2c}{\delta a}\right)},$$

from which the last part of the theorem follows. □

We show next that, as stated in [40], we can derive a closed form for the Jost solution associated with potentials of the type $q(x) = ce^{-ax}$. We then use this closed form to approximate some eigenvalues and resonances.

3.2 EXAMPLE: THE BESSEL EQUATION

We show first that, for $q(x) = ce^{-ax}$, (1.3) is a form of the Bessel equation and the associated Jost solution can therefore be expressed in terms of Bessel functions. We show next that this Jost solution is the one found in section 4.1.2, as expected, and then locate some eigenvalues and resonances using approximations.

3.2.1 Preliminaries

For simplicity's sake, we take $q(x) = -ie^{-x}$.

Let $\lambda \in \mathbb{C}$, let z be such that $\lambda = z^2$ and consider the Bessel function $J_{-2iz}(2i\sqrt{-i}e^{-x/2})$.
We have, with $\nu = -2iz$, (see [42] p 40 and also [40] 4.14, [12] example 2.1)

$$J_\nu\left(2i\sqrt{-i}e^{-x/2}\right) = \sum_{n\geq 0} \frac{(-i)^{n+\nu/2}e^{-x(n+\nu/2)}}{n!\Gamma(\nu+n+1)}. \tag{3.15}$$

Arguing as in [42], we shall prove that the Bessel function above is a solution of

$$-y'' - ie^{-x}y = \lambda y.$$

We need the two following results:

$$\frac{d}{dx}\left(e^{-x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right) = -(-i)^{1/2}e^{-x(1+\nu)/2}J_{\nu-1}\left(2i\sqrt{-i}e^{-x/2}\right) \tag{3.16}$$

and

$$\frac{d}{dx}\left(e^{x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right) = -(-i)^{1/2}e^{x(\nu-1)/2}J_{\nu+1}\left(2i\sqrt{-i}e^{-x/2}\right). \tag{3.17}$$

Note that, since the series in (3.15) is convergent uniformly with respect to x, we can perform the usual operations on the series, namely term by term differentiation and integration.

From (3.15), we get

$$e^{-x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right) = \sum_{n\geq 0} \frac{(-i)^{n+\nu/2}e^{-x(n+\nu)}}{n!\Gamma(\nu+n+1)}$$

so that, with $\Gamma(\nu+n+1)=(\nu+n)\Gamma(\nu+n)$,

$$\frac{d}{dx}\left(e^{-x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)=\sum_{n\geq 0}\frac{-(-i)^{n+\nu/2}e^{-x(n+\nu)}}{n!\Gamma(\nu+n)}.$$

Rewriting the terms in the series on the right hand side yields

$$\frac{d}{dx}\left(e^{-x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)=-(-i)^{1/2}e^{-x(1+\nu)/2}\sum_{n\geq 0}\frac{(-i)^{n+\nu/2-1/2}e^{-x(n+\nu/2-1/2)}}{n!\Gamma(\nu+n)}$$

and the series on the right hand side is $J_{\nu-1}\left(2i\sqrt{-i}e^{-x/2}\right)$, so that (3.16) is proved.

(3.17) is proved in a similar manner: by (3.15) we have

$$e^{x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)=\sum_{n\geq 0}\frac{(-i)^{n+\nu/2}e^{-xn}}{n!\Gamma(\nu+n+1)},$$

so that

$$\frac{d}{dx}\left(e^{x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)=\sum_{n\geq 1}\frac{-(-i)^{n+\nu/2}e^{-xn}}{(n-1)!\Gamma(\nu+n+1)}$$

and

$$\frac{d}{dx}\left(e^{x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)=\sum_{n\geq 0}\frac{-(-i)^{n+\nu/2+1}e^{-x(n+1)}}{n!\Gamma(\nu+n+2)}.$$

Rewriting the terms in the series on the right hand side yields

$$\frac{d}{dx}\left(e^{x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)=-(-i)^{1/2}e^{x(\nu-1)/2}\sum_{n\geq 0}\frac{(-i)^{n+\nu/2+1/2}e^{-x(n+\nu/2+1/2)}}{n!\Gamma(\nu+n+2)}$$

and the series on the right hand side is $J_{\nu+1}\left(2i\sqrt{-i}e^{-x/2}\right)$. (3.17) is therefore proved.

3.2.2 A System of Solutions

We rewrite (3.17) in the form

$$\frac{d}{dx}\left(e^{x(\nu-1)/2}J_{\nu-1}\left(2i\sqrt{-i}e^{-x/2}\right)\right)=-(-i)^{1/2}e^{-x(1-\nu/2)}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right). \tag{3.18}$$

We now proceed to eliminate the terms in $J_{\nu-1}$. From (3.16) we have

$$-(-i)^{1/2}e^{x\nu}\frac{d}{dx}\left(e^{-x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)=e^{x(\nu-1)/2}J_{\nu-1}\left(2i\sqrt{-i}e^{-x/2}\right)$$

which, together with (3.18), yields

$$\frac{d}{dx}\left[-(-i)^{1/2}e^{x\nu}\frac{d}{dx}\left(e^{-x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)\right]=-(-i)^{1/2}e^{-x(1-\nu/2)}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right).$$

Calculating the left hand side explicitly, we obtain

$$\frac{d}{dx}\left[-(-i)^{-1/2}\frac{\nu}{2}e^{x\nu/2}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)-(-i)^{-1/2}e^{x\nu/2}\frac{d}{dx}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right]=$$

$$-(-i)^{1/2}e^{-x(1-\nu/2)}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)$$

and upon performing the last differentiation on the left hand side and multiplying on both sides by $(-i)^{1/2}e^{-x\nu/2}$, we get

$$-\frac{d^2}{dx^2}\left(J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)\right)+\left(\frac{\nu^2}{4}-ie^{-x}\right)J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)=0,$$

so that $J_{-2iz}\left(2i\sqrt{-i}e^{-x/2}\right)$ is a solution of $-y''-ie^{-x}y=\lambda y$.

Note that, from (3.15),

$$J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)=e^{ixz}\sum_{n\geq 0}\frac{(-i)^{n+\nu/2}e^{-xn}}{n!\Gamma(\nu+n+1)},$$

so that $J_{-2iz}\left(2i\sqrt{-i}e^{-x/2}\right)$ is an $\mathcal{L}^2(\mathbb{R}^+)$ solution of $\tau y=z^2y$ for $\Im z>0$.

Also,

$$J_\nu\left(2i\sqrt{-i}e^{-x/2}\right)=e^{ixz}\left(\frac{(-i)^{\nu/2}}{\Gamma(\nu+1)}+\sum_{n\geq 1}\frac{(-i)^{n+\nu/2}e^{-xn}}{n!\Gamma(\nu+n+1)}\right)$$

and since the Jost solution satisfies $\chi(x,z)=e^{ixz}(1+o(1))$ as $x\to+\infty$ the Jost solution is here

$$\chi(x,z)=\frac{\Gamma(\nu+1)}{(-i)^{\nu/2}}J_\nu\left(2i\sqrt{-i}e^{-x/2}\right),$$

that is, the Jost solution is equal to

$$\chi(x,z) = e^{izx}\left(1+\sum_{n\geq 1}\frac{(-i)^n e^{-nx}}{n!(1-2iz)(...)(n-2iz)}\right)$$

since $\Gamma(\nu+n+1) = (\nu+n)(\nu+n-1)(...)(\nu+1)\Gamma(\nu+1)$.

Note that $\chi(x,-z)$ is the solution of $\tau y = z^2 y$ satisfying $\chi(x,-z) = e^{-ixz}(1+o(1))$ as $x \to +\infty$, so that $\chi(.,-z) \in \mathcal{L}^2(\mathbb{R}^+)$ for $\Im z < 0$.

For $\Im z = 0$, making use of the asymptotic behavior of $\chi(x,z)$ and $\chi(x,-z)$, we obtain

$$W\left(\chi(x,z), \chi(x,-z)\right) = -2iz,$$

so that $\chi(.,z)$ and $\chi(.,-z)$ are linearly independent for $\Im z = 0$, $z \neq 0$.

More generally, it can be proved that for $q(x) = ce^{-ax}$, we have

$$\chi(x,z) = \frac{\Gamma(-2iz+1)}{(ia^{-1}\sqrt{c})^{-iz}} J_{-2iz/a}(2ia^{-1}\sqrt{c}e^{-ax/2})$$

so that

$$\chi(z) = 1+\sum_{n\geq 1}\frac{(ca^{-2})^n}{n!}\left(\frac{1}{1-2iz/a}\right)(\dots)\left(\frac{1}{n-2iz/a}\right). \tag{3.19}$$

3.2.3 Applications

If $q(x) = ce^{-ax}$ with $c > 0$ and $a > 0$ then, according to (3.19), we have

$$\chi(z) = 1+\sum_{n\geq 1}\frac{(ca^{-2})^n}{n!}\left(\frac{1}{1-2iz/a}\right)(\dots)\left(\frac{1}{n-2iz/a}\right)$$

and for $z = iy$, $y > -a/2$, $\chi(iz)$ is real-valued since

$$\chi(iy) = 1+\sum_{n\geq 1}\frac{(ca^{-2})^n}{n!}\left(\frac{1}{1+2y/a}\right)(\dots)\left(\frac{1}{n+2y/a}\right)$$

so that, as explained in [12], $\chi(iy) \geq 1$ for $y > -a/2$ and the operator L_0 has no eigenvalue and no resonance $z = iy$ such that $-a/2 < y < 0$. If we wish to make use of (3.19) to locate the eigenvalues and the resonances $z = iy$ such that $-a/2 < y < 0$, we therefore have to consider negative values of c.

Set $q(x) = -e^{-x}$. We prove that $\chi(iy)$ has a unique zero for $-0.5 < y < +\infty$. It is readily seen from (3.19) that

$$\lim_{y \to +\infty} \chi(iy) = 1.$$

We now show that

$$\lim_{y+1/2 \to 0+} \chi(iy) = -\infty.$$

According to (3.19) we have

$$\begin{aligned}
\chi(z) &= 1 + \sum_{n \geq 1} \frac{(-1)^n}{n!} \left(\frac{1}{1+2y}\right) (\dots) \left(\frac{1}{n+2y}\right), \\
&= 1 + \frac{1}{1+2y} \left\{ -1 + \sum_{n \geq 2} \frac{(-1)^n}{n!} \left(\frac{1}{2+2y}\right) (\dots) \left(\frac{1}{n+2y}\right) \right\}, \\
&= 1 + \frac{1}{1+2y} \left\{ -1 + R(y) \right\},
\end{aligned}$$

where

$$R(y) = \sum_{n \geq 2} \frac{(-1)^n}{n!} \left(\frac{1}{2+2y}\right) (\dots) \left(\frac{1}{n+2y}\right).$$

On the other hand,

$$|R(y)| \leq \left| \frac{1}{2(2+2y)} \right| \leq \frac{1}{2}, \quad -1/2 < y < +\infty,$$

so that $-1 + R(iy) < 0$ and

$$\lim_{iy+1/2 \to 0+} \chi(iy) = -\infty,$$

as required. Hence $\chi(iy)$ has at least one zero for $-1/2 < y < +\infty$. Next, we show that χ is strictly increasing as a function of iy for $-1/2 < y < +\infty$. Let $-1/2 < y_1 < y_2 < +\infty$ and set

$$\begin{aligned}
R_2(y_1, y_2) &= \sum_{n \geq 2} \frac{(-1)^n}{n!} \left\{ \left(\frac{1}{1+2y_1}\right) (\dots) \left(\frac{1}{n+2y_1}\right) - \right. \\
&\quad \left. - \left(\frac{1}{1+2y_2}\right) (\dots) \left(\frac{1}{n+2y_2}\right) \right\}.
\end{aligned}$$

From (3.19) we get

$$\begin{aligned}
\chi(iy_1) - \chi(iy_2) &= \sum_{n\geq 1} \frac{(-1)^n}{n!} \left\{ \left(\frac{1}{1+2y_1}\right)(\dots)\left(\frac{1}{n+2y_1}\right) - \right. \\
&- \left. \left(\frac{1}{1+2y_2}\right)(\dots)\left(\frac{1}{n+2y_2}\right)\right\}, \\
&= \frac{-1}{(1+2y_1)} + \frac{1}{1+2y_2} + R_2(y_1, y_2), \\
&= \frac{-2(y_2-y_1)}{(1+2y_1)(1+2y_2)} + R_2(y_1, y_2).
\end{aligned}$$

We also have

$$\begin{aligned}
|R_2(y_1, y_2)| &\leq \frac{1}{2}\left(\frac{1}{(1+2y_1)(1+2y_1)} - \frac{1}{(1+2y_2)(1+2y_2)}\right), \\
&\leq \frac{3(y_2-y_1) + 4(y_2^2) - y_1^2)}{2(1+2y_1)(2+2y_1)(1+2y_2)(2+2y_2)}, \\
&\leq \frac{2(y_2-y_1)}{(1+2y_1)(1+2y_2)} \frac{3+4|y_2+y_1|}{2(2+2y_1)(2+2y_2)}.
\end{aligned}$$

Let $\delta > 0$ and let $y_2 = y_1 + \delta$. The second fraction on the right hand side is strictly smaller than 1 if

$$3 + 4|y_2 + y_1| < 2(2+2y_1)(2+2y_2),$$

i.e. if

$$0 < 5 + 4|2y_1 + \delta| + 8\delta + 16y_1 + 8y_1\delta + 8y_1^2.$$

The polynomial in y_1 on the right hand side is strictly positive for δ small enough, whether $|2y_1 + \delta| = 2y_1 + \delta$ or $|2y_1 + \delta| = -(2y_1 + \delta)$. Hence

$$|R_2(y_1, y_2)| < \frac{2(y_2-y_1)}{(1+2y_1)(1+2y_2)}$$

for $-1/2 < y_1 < y_2 < +\infty$, $y_2 = y_1 + \delta$ and δ small enough. It follows that

$$\chi(iy_1) - \chi(iy_2) < 0 \quad \text{for } -1/2 < y_1 < y_2 < +\infty,$$

as required. Therefore, $\chi(iy)$ has a unique zero for $-1/2 < y < +\infty$. Setting

$$f(iy) = 1 - \frac{1}{1+2y} + \frac{1}{2!(1+2y)(2+2y)} - \frac{1}{3!(1+2y)(2+2y)(3+2y)}$$

we obtain

$$\chi(iy) = f(iy) + R_4(iy),$$

with

$$R_4(iy) = \sum_{n\geq 4} \frac{(-1)^n}{n!}\left(\frac{1}{1+2y}\right)(\ldots)\left(\frac{1}{n+2y}\right).$$

It can be shown that $f(iy)$ has a unique zero iy_0, $y_0 \approx -0.1256$, in the interval $(-1/2, 0]$, that $\chi(-i0.13) < 0$ and that $\chi(-i0.12) > 0$ so that $\chi(iy)$ has a unique zero iy_1 for $-1/2 < y < +\infty$ with

$$y_1 \in [-0.13, -0.12].$$

On the other hand

$$|R_4(iy)| \leq \frac{1}{4!(1+2y)(2+2y)(3+2y)(4+2y)} \leq 0.00351 \quad \text{for } y > -0.15,$$

so that

$$\chi(iy_0) \in (-0.00351,\ 0.00351).$$

We have shown that L_0 has no eigenvalue and a unique antibound state iy_1 such that $-1/2 < y_1 < 0$.

If $q(x) = -3e^{-x}$, we can show using similar methods that that L_0 has a unique eigenvalue $\lambda_1 = (iy_1)^2$, $y_1 \in [0.27, 0.553]$ and no antibound state on the segment line iy, $-1/2 < y < 0$.

Suppose that $y > -1/2$. Then

$$\chi(iy) = 1 + \sum_{n\geq 1} \frac{(-3)^n}{n!}\left(\frac{1}{1+2y}\right)(\ldots)\left(\frac{1}{n+2y}\right),$$

and setting

$$f(iy) = 1 - \frac{3}{1+2y} + \frac{3^2}{2!(1+2y)(2+2y)} - \frac{3^3}{3!(1+2y)(2+2y)(3+2y)}$$

we get

$$\chi(iy) = f(iy) + R_4(iy),$$

with

$$R_4(iy) = \sum_{n\geq 4} \frac{(-3)^n}{n!} \left(\frac{1}{1+2y}\right) (\dots) \left(\frac{1}{n+2y}\right).$$

Since $R_4(iy)$ is an alternating series, we have

$$|R_4(iy)| \leq \frac{(-3)^4}{4!} \left(\frac{1}{1+2y}\right) (\dots) \left(\frac{1}{4+2y}\right)$$

so that, for $y > 0$,

$$|R_4(iy)| \leq \frac{3^4}{(4!)^2} = \frac{9}{64}.$$

On the other hand

$$f(iy) = \frac{8y^2 + 4y - 3}{2(1+2y)(3+2y)} = 4\frac{(y + 1/4 + \sqrt{7}/4)(y + 1/4 - \sqrt{7}/4)}{(1+2y)(3+2y)}.$$

and $f(iy)$ has a unique zero for $y > -1/2$, namely

$$y_0 = \frac{-1+\sqrt{7}}{4}.$$

Taking into account the remainder $R_4(iy)$, we have

$$\chi(iy_0) \in (-0.141,\ 0.141).$$

We now investigate the properties of some operators associated with a potential satisfying (3.13).

3.3 EXAMPLE: THE HYPERGEOMETRIC EQUATION

We note that, in this section, q satisfies (3.13) and not merely (2.2).

3.3.1 Preliminaries

Remark: the constants a and c below do not relate to the constants a and c in (3.13). We consider the hypergeometric equation

$$X(1-X)\frac{d^2Y}{dX^2} + (c-(a+b+1)X)\frac{dY}{dX} - abY = 0, \tag{3.20}$$

a solution of which is (see [1])

$$F(a,b;c;X) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{n!(c)_n} X^n, \tag{3.21}$$

where F is the hypergeometric function ${}_2F_1$ and $(d)_n = d(d+1)(...)(d+n-1)$.

Arguing as in ([40], sections 4.18, 4.19, 4.20), we transform (3.20) into an equation of the form $y'' + (\lambda - q(x))y = 0$ and derive a solution in terms of the hypergeometric function.

In the following sections we produce the Jost solution associated with the differential equation $y'' + (\lambda - q(x))y = 0$ and examine some of its properties.

It is readily seen that (3.21) is a solution of (3.20) if and only if $Y = F(a,b;c;-X)$ is a solution of

$$X(1+X)\frac{d^2Y}{dX^2} + (c+(a+b+1)X)\frac{dY}{dX} + abY = 0. \tag{3.22}$$

Taking

$$Y_1 = X^{\frac{1}{2}c}(1+X)^{\frac{1}{2}(a+b+1-c)}Y$$

we obtain for (3.22) (see [40] section 4.20)

$$\frac{d^2Y_1}{dX^2} + ZY_1 = 0, \tag{3.23}$$

where

$$Z = \frac{1-(1-c)^2}{4X^2} + \frac{1-(c-a-b)^2}{4(1+X)^2} + \frac{(1-c)^2-(a-b)^2+(c-a-b)^2-1}{4X(1+X)}.$$

At this stage however we allow ourselves a small modification in the subsequent change of variable and we put

$$y = X^{-\frac{1}{2}} Y_1, \quad X = e^{2x}.$$

With the chain rule

$$\frac{d^2 Y_1}{dX^2} = \left(\frac{dX}{dx}\right)^{-2} \frac{d^2 Y_1}{dx^2} - \frac{d^2 X}{dx^2} \left(\frac{dX}{dx}\right)^{-3} \frac{dY_1}{dx}$$

we get

$$\frac{d^2 Y_1}{dX^2} = (2X)^{-2} \frac{d^2 Y_1}{dx^2} - \frac{X^{-2}}{2} \frac{dY_1}{dx}. \tag{3.24}$$

Since $Y_1 = e^x y$,

$$\frac{dY_1}{dx} = e^x y + e^x \frac{dy}{dx} \text{ and } \frac{d^2 Y_1}{dx^2} = e^x y + 2e^x \frac{dy}{dx} + e^x \frac{d^2 y}{dx^2}$$

which, together with (3.25) yields

$$\frac{d^2 Y_1}{dX^2} = (2e^{2x})^{-2} \left(e^x y + 2e^x \frac{dy}{dx} + e^x \frac{d^2 y}{dx^2}\right) - \frac{(e^{2x})^{-2}}{2} \left(e^x y + e^x \frac{dy}{dx}\right)$$

and finally

$$\frac{d^2 Y_1}{dX^2} = \frac{1}{4} e^{-3x} \left(\frac{d^2 y}{dx^2} - y\right).$$

In terms of (3.23) we now have

$$\frac{d^2 Y_1}{dX^2} + Z Y_1 = \frac{1}{4} e^{-3x} \left(\frac{d^2 y}{dx^2} - y\right) + Z e^x y = 0$$

or

$$\frac{d^2 y}{dx^2} + (4ZX^2 - 1) y = 0. \tag{3.25}$$

We have

$$\begin{aligned} 4ZX^2 - 1 &= -(1-c)^2 + \frac{X^2 \left(1 - (c-a-b)^2\right)}{(1+X)^2} + \\ &+ \frac{X(1+X)\left((1-c)^2 - (a-b)^2 + (c-a-b)^2 - 1\right)}{(1+X)^2} \end{aligned}$$

which simplifies into

$$4X^2Z - 1 = -(1-c)^2 - \frac{AX}{1+X} - \frac{BX}{(1+X)^2}$$

where

$$A = (a-b)^2 - (1-c)^2 \text{ and } B = 1 - (c-a-b)^2.$$

In order to end up with an equation of the form described in ([40] section 4.19, [12] example 2.2) we choose

$$A = 0, \quad B = 1 - \beta^2, \quad \text{and } z^2 = -(1-c)^2,$$

which leads to

$$z^2 = -(1-c)^2, \quad 1 - \beta^2 = 1 - (c-a-b)^2, \quad (a-b)^2 = (1-c)^2.$$

Due to the squares in the equalities above, we can choose different solutions for a, b and c in terms of z and β. We fix those choices to be

$$c = 1 + iz, \quad a = \frac{1}{2}(1-\beta) + iz \quad \text{and } b = \frac{1}{2}(1-\beta). \tag{3.26}$$

Equation (3.25) now becomes

$$\frac{d^2y}{dx^2} + \left\{ z^2 - \left(\frac{X(1-\beta^2)}{(1+X)^2} \right) \right\} y = 0$$

or

$$\frac{d^2y}{dx^2} + (\lambda - q(x))y = 0 \tag{3.27}$$

with $\lambda = z^2$ and

$$q(x) = \frac{(1-\beta^2)e^{2x}}{(1+e^{2x})^2}.$$

Note that

$$q(x) = \frac{1-\beta^2}{4\cosh^2(x)} = \frac{\gamma}{\cosh^2(x)}. \tag{3.28}$$

Now,

$$Y_1 = X^{\frac{1}{2}c}(1+X)^{\frac{1}{2}(a+b+1-c)}Y = X^{\frac{1}{2}c}(1+X)^bY$$

and $y = X^{-\frac{1}{2}}Y_1$ so that

$$y = X^{\frac{1}{2}(c-1)}(1+X)^b Y \tag{3.29}$$

is a solution of (3.27).

A solution of (3.22) in the neighborhood of the singular point $+\infty$ is (see [1], 15.5.13)

$$(X-1)^{-b} F\left(b, c-a; b-a+1; \frac{1}{1-X}\right),$$

so that a solution of (3.20) in the neighborhood of $+\infty$ is

$$Y = (-1)^{-b}(1+X)^{-b} F\left(b, c-a; b-a+1; \frac{1}{1+X}\right).$$

In terms of (3.26) the above becomes

$$Y = (-1)^{-\frac{1}{2}(1-\beta)}(1+X)^{-\frac{1}{2}(1-\beta)} F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{1+X}\right).$$

The above, together with (3.29) yield

$$y = (-1)^{\frac{1}{2}(1-\beta)} X^{\frac{1}{2}iz} F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{1+X}\right)$$

and, with $X = e^{2x}$, we have

$$y = (-1)^{\frac{1}{2}(1-\beta)} e^{izx} F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{1+e^{2x}}\right) \tag{3.30}$$

as a solution of (3.27).

3.3.2 The Associated Jost Solution

We set

$$\chi(x,z) = e^{izx} F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{1+e^{2x}}\right), \tag{3.31}$$

where, from (3.21)

$$F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{1+e^{2x}}\right) = 1+\sum_{n\geq 1} \frac{(1/2-\beta/2)_n(1/2+\beta/2)_n}{n!(1-iz)_n} \frac{1}{(1+e^{2x})^n}.$$

Note that $\chi(x,z)$, as a function of z, is analytic in $\mathbb{C}$ apart from the simple poles $z=-in$, $n\geq 1$.

According to (3.30), $\chi(x,z)$ is a solution of (3.27). We show that $\chi(x,z)$, $\Im z>0$, is the Jost solution associated with (3.27), that $\chi(x,-z)$ is a $\mathcal{L}^2(\mathbb{R}^+)$ solution of (3.27) for $\Im z<0$ and finally we obtain a compact form for the Jost function $\chi(z)=\chi(0,z)$.

Now, $|1-iz+k|^2=(1+\Im z+k)^2+(\Re z)^2$ so that, since $\Im z>0$, $|1-iz+k|\geq k+1$, $k\geq 1$.

On the other hand, $|1/2\pm\beta/2+k|\leq|1/2\pm\beta/2|+k\leq j+k$ $\forall k\geq 1$, where $j=\inf\{n\in\mathbb{N}: n\geq|1/2+\pm\beta/2|\}$, so that

$$\left|F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{1+e^{2x}}\right)\right| \leq 1+\sum_{n\geq 1}\frac{((n+j)!)^2}{(n!)^2}\frac{1}{(1+e^{2x})^n}$$

$$\leq 1+\sum_{n\geq 1}(n+j)^{2j}\frac{1}{(1+e^{2x})^n}.$$

The series on the right hand side converges for $x\geq 0$ by the ratio test and clearly tends to 0 as $x\to+\infty$, so that $\chi(x,z)=e^{izx}(1+o(1))$ as $x\to+\infty$, and $\chi(x,z)$ is the Jost solution associated with (3.27).

Another linearly independent solution of (3.20) in the neighborhood of $+\infty$ is (see [1], 15.5.14)

$$X^{1-c}(X-1)^{c-a-1}F(a-c+1, 1-b; a-b+1; \frac{1}{1-X})$$

and a solution of (3.22) is therefore

$$Y=(-X)^{1-c}(-X-1)^{c-a-1}F(a-c+1, 1-b; a-b+1; \frac{1}{1+X})$$

and, recalling (3.26),

$$Y = (-1)^{-b} X^{-iz} (-1)^{-b} (X+1)^{-b} F(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1+iz; \frac{1}{1+X}).$$

With $y = X^{\frac{1}{2}(c-1)}(1+X)^{a+b+1-c} Y = X^{iz/2}(1+X)^b Y$, we obtain

$$y = (-1)^{-iz}(-1)^{-b} X^{-iz/2} F(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1+iz; \frac{1}{1+X})$$

and, since $X = e^{2x}$,

$$y = (-1)^{-iz-b} e^{-izx} F(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1+iz; \frac{1}{1+e^{2x}}).$$

A solution of (3.27) is therefore

$$\chi(x,-z) = e^{-izx} F(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1+iz; \frac{1}{1+e^{2x}}). \qquad (3.32)$$

Note that $\chi(x,-z)$ is an $\mathcal{L}^2(\mathbb{R}^+)$ solution of (3.27) for $\Im z < 0$ and that $\chi(x,-z)$ is analytic in $\mathbb{C}$ except for the simple poles $z = in$, $n \geq 1$.

Setting $\chi(z) = \chi(0,z)$, we have

$$\chi(z) = F(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; \frac{1}{2})$$

and according to ([1], 15.1.26),

$$\chi(z) = 2^{iz}\pi^{\frac{1}{2}} \frac{\Gamma(1-iz)}{\Gamma(\frac{1}{2}(\frac{3}{2}-\frac{\beta}{2}-iz))\Gamma(\frac{1}{2}(\frac{3}{2}+\frac{\beta}{2}-iz))} \qquad (3.33)$$

for the determination $\Im z > 0$ of $z = \sqrt{\lambda}$ and

$$\chi(-z) = 2^{iz}\pi^{\frac{1}{2}} \frac{\Gamma(1+iz)}{\Gamma(\frac{1}{2}(\frac{3}{2}-\frac{\beta}{2}+iz))\Gamma(\frac{1}{2}(\frac{3}{2}+\frac{\beta}{2}+iz))} \qquad (3.34)$$

for the determination $\Im z < 0$ of $z = \sqrt{\lambda}$.

We now examine the location and properties of the eigenvalues, spectral singularities and resonances associated with an operator arising from (3.27) with Dirichlet boundary conditions.

The zeros of the Jost solution (3.33) are the poles of the denominator in (3.33). The zeros of the Jost solution are therefore obtained for

$$\frac{1}{2}(\frac{3}{2} \pm \frac{\beta}{2} - iz) = -M, \quad M = 0, 1, \ldots$$

i.e. the zeros of $X(z)$ are

$$z_M = \pm\frac{1}{2}\Im\beta - i\left(\frac{3}{2} \pm \frac{1}{2}\Re\beta + 2M\right), \quad M = 0, 1, \ldots \tag{3.35}$$

$\lambda_M = z_M^2$ is an eigenvalue of the operator L_0 associated with (3.27) with Dirichlet boundary condition $y(0) = 0$ if and only if $\Im(z_M) > 0$.

The resonances of L_0 are given by $\Im(z_M) < 0$ and the possible spectral singularities are obtained from $\Im(z_M) = 0$.

For a positive real number x we denote by $[x]$ the integer part of x, where the integer part of x is here defined to be

$$[x] = \sup\{n \in \mathbb{N} : n \leq x\}.$$

Now, according to (3.35) L_0 has eigenvalue(s) if and only if $\Im(z_M) > 0$, i.e $\iff$

$$\frac{3}{2} + \frac{1}{2}\Re\beta + 2M < 0 \quad \text{or} \quad \frac{3}{2} - \frac{1}{2}\Re\beta + 2M < 0$$

$$\iff M < \frac{1}{4}(-3 - \Re\beta) \quad \text{or} \quad M < \frac{1}{4}(-3 + \Re\beta)$$

and L_0 can only have eigenvalue(s) for $\Re\beta < -3$ or $\Re\beta > 3$ respectively, the eigenvalues being given by $\lambda_M = z_m^2$, z_M as in (3.35) and M such that

$$M \in \left[0, \left[\frac{1}{4}(-3 - \Re\beta)\right]\right) \text{ if } \Re\beta < -3 \tag{3.36}$$

and

$$M \in \left[0, \left[\frac{1}{4}(-3 + \Re\beta)\right]\right) \text{ if } \Re\beta > 3. \tag{3.37}$$

If $\Im(z_M) = 0$, then L_0 has a spectral singularity at $\lambda_s = z_M^2$. This happens only if, by (3.35),

$$\Re\beta = -4M - 3 \quad \text{or} \quad \Re\beta = 4M + 3 \quad \text{for some integer } M \ ,$$

i.e. L_0 has a spectral singularity at λ_s if there exists $M \in \mathbb{N}$ such that

$$M = \left[\frac{1}{4}(-3 - \Re\beta)\right] \quad \text{or} \quad M = \left[\frac{1}{4}(-3 + \Re\beta)\right]. \tag{3.38}$$

According to (3.27), L_0 is selfadjoint for $\gamma = 4(1 - \beta^2) \in \mathbb{R}$, so that L_0 is selfadjoint if $\Im\beta = 0$ or $\Re\beta = 0$ and nonselfadjoint otherwise.

If L_0 is selfadjoint, i.e. if $\Im\beta = 0$ or $\Re\beta = 0$, then it is readily seen from (3.35) that L_0 has no spectral singularity.

Note also that the number of resonances of L_0 is in any case infinite.
From (3.35) it is readily seen that they are placed symmetrically with respect to the semi axis $-i\mathbb{R}^+$ if and only if $\Re\beta = 0$ and situated on the semi axis $-i\mathbb{R}^+$ if and only if $\Im\beta = 0$.
Those two situations correspond to the selfadjoint case.

In order to get a clear picture, let us consider the case $\beta = -11 + 2i$, i.e $\gamma = 4(-116 + 44i)$. Then, in accordance with (3.36)

$$M = 0 \Longrightarrow z_0 = 1 + 4i \Longrightarrow \lambda_0 = z_0^2 = -15 + 8i \quad \text{is an eigenvalue ,}$$

$$M = 1 \Longrightarrow z_1 = 1 + 2i \Longrightarrow \lambda_1 = z_1^2 = -3 + 4i \quad \text{is an eigenvalue}$$

and in accordance with (3.38)

$$M = 2 \Longrightarrow z_2 = 1 \Longrightarrow \lambda_s = z_2^2 = 1 \quad \text{is a spectral singularity ,}$$

while the remaining zeros of $\chi(z)$ are the resonances

$$z_M = 1 + i(4 - 2M), \quad M \geq 3, \quad z_N = -1 - i(2N + 7), \quad N \geq 0.$$

3.3.3 The Case $\alpha = \pi/2$

In this case the eigenvalues of the operator L_α associated with (3.27) and the boundary condition (1.2) are $\lambda_M = z_M^2$, where z_M are the zeros of $\chi_\alpha(z)$ satisfying $\Im(z_M) > 0$.

Arguing as in ([12], example 2.2), we derive a convenient expression for $\chi'(z)$, then obtain the zeros of $\chi_\alpha(z)$ and examine their location and significance in terms of the spectral properties of L_α.

First we prove a simple relation:

$$x\frac{d}{dx}F(a,b;c;x) = (c-1)\left(F(a,b;c-1;x) - F(a,b;c;x)\right). \qquad (3.39)$$

We begin with the right hand side:

$$F(a,b;c-1;x) - F(a,b;c;x) = \sum_{n\geq 1}\left(\frac{a_n b_n}{n!(c-1)_n} - \frac{a_n b_n}{n! c_n}\right)x^n$$

and

$$F(a,b;c-1;x) - F(a,b;c;x) = \sum_{n\geq 1}\left(\frac{a_n b_n(c+n-1)}{n! c_n(c-1)} - \frac{a_n b_n}{n! c_n}\right)x^n,$$

since

$$(c-1)_n = (c-1)(c-1+1)...(c-1+n-1) = (c-1)\frac{c_n}{c+n-1}.$$

We then obtain

$$F(a,b;c-1;x) - F(a,b;c;x) = \sum_{n\geq 1}\left(\frac{a_n b_n(c+n-1) - a_n b_n(c-1)}{n! c_n(c-1)}\right)x^n,$$

$$F(a,b;c-1;x) - F(a,b;c;x) = \frac{x}{c-1}\sum_{n\geq 1}\left(\frac{a_n b_n}{(n-1)! c_n}\right)x^{n-1}$$

and finally

$$F(a,b;c-1;x) - F(a,b;c;x) = \frac{x}{c-1}\frac{d}{dx}F(a,b;c;x),$$

as required.

Let

$$V = \frac{1}{1 + e^{2x}}.$$

Then

$$\frac{dV}{dx} = -2\frac{e^{2x}}{(1+e^{2x})^2} = -2V\frac{e^{2x}}{(1+e^{2x})}$$

so that

$$\frac{dV}{dx} = -2V(1-V). \tag{3.40}$$

We have, from (3.31)

$$\frac{d}{dx}\chi(x,z) = ize^{izx}F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; V\right) +$$

$$+e^{izx}\frac{d}{dx}\left\{F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; V\right)\right\}$$

and, from (3.40),

$$\chi'(x,z) = ize^{izx}F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; V\right) -$$

$$-2V(1-V)e^{izx}\frac{dF}{dV}\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; V\right).$$

From (3.39) we now obtain

$$\chi'(x,z) = ize^{izx}(2V-1)F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); 1-iz; V\right) +$$

$$+2ize^{izx}(1-V)F\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); -iz; V\right).$$

For $x = 0$, $V = 1/2$ so that

$$\chi'(z) = \chi'(0,z) = izF\left(\frac{1}{2}(1-\beta), \frac{1}{2}(1+\beta); -iz; \frac{1}{2}\right).$$

According to ([1], 15.1.26), the above expression takes the form

$$\chi'(z) = iz2^{1+iz}\sqrt{\pi}\frac{\Gamma(-iz)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{2}-iz\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}+\frac{\beta}{2}-iz\right)\right)},$$

or

$$\chi'(z) = -2^{1+iz}\sqrt{\pi}\frac{(-iz)\Gamma(-iz)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{2}-iz\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}+\frac{\beta}{2}-iz\right)\right)},$$

which yields

$$\chi'(z) = -2^{1+iz}\sqrt{\pi}\frac{\Gamma(1-iz)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{2}-iz\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}+\frac{\beta}{2}-iz\right)\right)}, \tag{3.41}$$

since $(-iz)\Gamma(-iz) = \Gamma(1-iz)$.

Note that, as explained in ([12], example 2.2) and employing the method detailed in the previous section, (3.41) gives the spectral information related to the operator $L_{\pi/2}$ for the Neumann boundary condition $y'(0) = 0$.

Chapter 4

RICCATI'S EQUATION

This chapter is dedicated to the study of a series associated with the Riccati equation, an equation in turn related to (1.3). We then apply the results obtained to investigate the eigenvalues, resonances and spectral singularities of the Sturm-Liouville operators and, when q is real-valued, the points of spectral concentrations of the said operators.

4.1 HARRIS-GILBERT SERIES

In this section, we study the series initially investigated by Harris ([20]) and Gilbert and Harris ([17], [18]). The asymptotic behaviour of the series was initially used to obtain the asymptotic behaviour of $\rho_\alpha''(\lambda)$, $\lambda > 0$ in order to get a bound on the set of points of spectral concentration.

We use here some of their ideas and methods for the study of the series, although we place ourselves in a more general setting (we do not suppose initially that q is real-valued), but we suppose that q is not only exponentially decaying but satisfies (3.13).

4.1.1 Derivation of the Series

We recall that, for $q \in \mathcal{L}(\mathbb{R}^+)$, there exists (up to a constant multiple) for $\Im z > 0$ a unique $\mathcal{L}^2(\mathbb{R}^+)$-solution of (1.3) and this solution must be a scalar multiple of the Jost solution $\chi(x, z)$, so that the ratio

$$m(x,z) = \frac{\chi'(x,z)}{\chi(x,z)}$$

is unique in that respect (the dash representing derivation with respect to the real variable) and analytic as a function of z, apart from poles at the zeros of $\chi(x, z)$.

It is readily seen that, if $y(x, z^2)$ satisfies (1.3), then $v = y'/y$ satisfies the Riccati equation

$$v' = -z^2 + q - v^2 \tag{4.1}$$

As explained in [17], a solution $v(x, z)$ of (4.1) satisfying $v(x, z) - iz \to 0$ as $x \to +\infty$ and $v(x, z) - iz \in \mathcal{L}(\mathbb{R}^+)$ is unique.

We therefore seek a solution $v(x, z)$ of (4.1) of the form

$$v(x,z) = iz + \sum_{n\geq 1} v_n(x,z). \tag{4.2}$$

The condition (3.13) is quite different from the conditions on q in [17], but we found the conditions in [17] cleverly chosen and well-suited for the case they considered and therefore difficult to improve on without making specific assumptions on a bound for q.

Theorem 4.1.1. *If q satisfies (3.13), i.e. if $|q(x)| \leq ce^{-ax}$, $x \geq 0$, with $c > 0$ and $a > c + 1$ then the terms of the series*

$$v(x,z) = iz + \sum_{n\geq 1} v_n(x,z),$$

with

$$v_1(x,z) = -e^{-2izx}\int_x^{+\infty} e^{2izt} q(t)dt, \tag{4.3}$$

$$\begin{aligned} v_2(x,z) &= e^{-2izx}\int_x^{+\infty} e^{2izt} v_1^2(t,z)dt, \\ v_n(x,z) &= e^{-2izx}\int_x^{+\infty} e^{2izt}\left\{ v_{n-1}^2(t,z) + 2v_{n-1}(t,z)\sum_{k=1}^{n-2} v_k(t,z)\right\} dt, \quad n \geq 3 \end{aligned}$$

satisfy

$$|v_n(x,z)| \leq \left(\frac{c}{a}\right)^n e^{-nax}, \qquad n \geq 1,$$

so that the series converges absolutely and uniformly for $x \geq 0$, $\Im z \geq 0$, $z \neq 0$.

Moreover, $v(x,z)$ *is a solution of the Riccati equation* (4.1) *satisfying*

$$v(x,z) - iz \to 0 \quad \text{as } x \to +\infty \qquad \text{and } v(x,z) - iz \in \mathcal{L}(\mathbb{R}^+,\ dx),$$

so that $v(x,z)$ *can be identified with the the quotient*

$$m(x,z) = \frac{\chi'(x,z)}{\chi(x,z)}$$

for $x \geq 0$, $\Im z \geq 0$, $z \neq 0$.

Proof. We begin as in [17]: substitution of (4.2) into (4.1) and rearrangment gives (omitting the arguments)

$$v_1' + 2izv_1 + v_2' + 2izv_2 + \sum_{n\geq 3}(v_n' + 2izv_n) = q - v_1^2 - \sum_{n\geq 3}\left\{ v_{n-1}^2 + 2v_{n-1}\sum_{k=1}^{n-2} v_m \right\}.$$

The terms in the equality above are chosen so that

$$v_1' + 2izv_1 = q, \tag{4.4}$$

$$\begin{aligned} v_2' + 2izv_2 &= -v_1^2, \\ v_n' + 2izv_n &= -\left\{ v_{n-1}^2 + 2v_{n-1} \sum_{k=1}^{n-2} v_k \right\}, \quad n \geq 3. \end{aligned}$$

We formally choose

$$v_1(x,z) = -e^{-2izx} \int_x^{+\infty} e^{2izt} q(t)dt,$$

$$v_2(x,z) = e^{-2izx} \int_x^{+\infty} e^{2izt} v_1^2(x,t)dt,$$

and

$$v_n(x,z) = e^{-2izx} \int_x^{+\infty} e^{2izt} \left\{ v_{n-1}^2(t,z) + 2v_{n-1}(t,z) \sum_{k=1}^{n-2} v_k(t,z) \right\} dt \quad n \geq 3.$$

We of course need to justify the convergence of the integrals above, as well as the convergence of the series itself.

We note that, for $\Im z \geq 0$,

$$u(x,z) = e^{-2izx} \int_x^{+\infty} e^{2izt} w(t,z)dt$$

satisfies

$$u' + 2izu = -w$$

if $w \in \mathcal{L}(\mathbb{R}^+)$. On the other hand

$$\begin{aligned} |u(x,z)| &\leq |e^{-2izx}| \int_x^{+\infty} |e^{2izt} w(t,z)|dt, \\ &\leq e^{2x\Im z} \int_x^{+\infty} e^{-2t\Im z} |w(t,z)|dt \end{aligned}$$

and finally

$$|u(x,z)| \leq \int_x^{+\infty} |w(t,z)|dt,$$

since $e^{-2t\Im z} \leq e^{-2x\Im z}$ for $t \geq x$.

Suppose that (3.13) holds with $a > c+1$. We then have

$$|v_1(x,z)| \leq c\int_x^{+\infty} e^{-at}dt = \frac{c}{a}e^{-ax}$$

and

$$|v_2(x,z)| \leq \left(\frac{c}{a}\right)^2 \int_x^{+\infty} e^{-2at}dt = \frac{c^2}{2a^3}e^{-2ax}.$$

Also,

$$v_3(x,z) = e^{-2izx}\int_x^{+\infty} e^{2izt}(v_2^2(x,z) + 2v_2(x,z)v_1(x,z))dt,$$

$$\begin{aligned}
|v_3(x,z)| &\leq \int_x^{+\infty} (|v_2(t,z)|^2 + 2|v_2(t,z)v_1(t,z)|)dt, \\
&\leq \int_x^{+\infty} \left\{\frac{c^4}{2^2a^6}e^{-4at} + \frac{c^3}{a^4}e^{-3at}\right\} dt, \\
&\leq \left(\frac{c}{a}\right)^3 \int_x^{+\infty} \left\{\frac{c}{2^2a^3}e^{-4at} + \frac{1}{a}e^{-3at}\right\} dt, \\
&\leq \left(\frac{c}{a}\right)^3 \int_x^{+\infty} \left\{\frac{c}{2^2a^3}e^{-3at} + \frac{1}{a}e^{-3at}\right\} dt
\end{aligned}$$

and

$$|v_3(x,z)| \leq \left(\frac{c}{a}\right)^3 \frac{e^{-3ax}}{3a}\left\{\frac{c}{2^2a^3} + \frac{1}{a}\right\}.$$

The condition $a > c+1$ implies that

$$\frac{1}{3a}\left\{\frac{c}{2^2a^3} + \frac{1}{a}\right\} < 1$$

so that

$$|v_3(x,z)| \leq \left(\frac{c}{a}\right)^3 e^{-3ax}.$$

Suppose that

$$|v_k(x,z)| \le \left(\frac{c}{a}\right)^k e^{-kax}, \qquad 1 \le k \le n-1, \quad n \ge 4.$$

Note that the property is true for v_i, $i = 1 \ldots 3$.

We then have

$$v_n(x,z) = e^{-2izx} \int_x^{+\infty} e^{2izt} \left\{ v_{n-1}^2(t,z) + 2v_{n-1}(t,z) \sum_{k=1}^{n-2} v_k(t,z) \right\} dt,$$

so that

$$\begin{aligned}
|v_n(x,z)| &\le \int_x^{+\infty} \left\{ |v_{n-1}^2(t,z)| + 2|v_{n-1}(t,z)| \sum_{k=1}^{n-2} |v_k(t,z)| \right\} dt, \\
&\le \int_x^{+\infty} \left\{ \left(\frac{c}{a}\right)^{2(n-1)} e^{-2(n-1)at} + 2\left(\frac{c}{a}\right)^{n-1} e^{-(n-1)at} \sum_{k=1}^{n-2} \left(\frac{c}{a}\right)^k e^{-kat} \right\} dt, \\
&\le \int_x^{+\infty} \left\{ \left(\frac{c}{a}\right)^{2(n-1)} e^{-2(n-1)at} + 2\left(\frac{c}{a}\right)^{n-1} e^{-(n-1)at} \sum_{k=1}^{n-2} \left(\frac{c}{a}\right)^k e^{-at} \right\} dt, \\
&\le \int_x^{+\infty} \left\{ \left(\frac{c}{a}\right)^{2(n-1)} e^{-2(n-1)at} + 2\left(\frac{c}{a}\right)^{n-1} e^{-nat} \sum_{k=1}^{n-2} \left(\frac{c}{a}\right)^k \right\} dt, \\
&\le \int_x^{+\infty} \left\{ \left(\frac{c}{a}\right)^{2(n-1)} e^{-2(n-1)at} + 2\left(\frac{c}{a}\right)^{n} e^{-nat} \left(1 + \cdots + \frac{c^{n-3}}{a^{n-3}}\right) \right\} dt, \\
&\le \int_x^{+\infty} \left\{ \left(\frac{c}{a}\right)^{2(n-1)} e^{-2(n-1)at} + 2\left(\frac{c}{a}\right)^{n} e^{-nat} \sum_{k\ge 0} \left(\frac{c}{a}\right)^k \right\} dt, \\
&\le \int_x^{+\infty} \left\{ \left(\frac{c}{a}\right)^{2(n-1)} e^{-2(n-1)at} + 2\left(\frac{c}{a}\right)^{n} e^{-nat} \frac{1}{1-c/a} \right\} dt
\end{aligned}$$

and

$$|v_n(x,z)| \le \left(\frac{c}{a}\right)^{2(n-1)} \frac{e^{-2(n-1)ax}}{2(n-1)a} + 2\left(\frac{c}{a}\right)^n \frac{e^{-nax}}{na} \frac{1}{1-c/a}.$$

Since $a > c+1$, we also have

$$\frac{2}{na}\frac{1}{1-c/a} = \frac{2}{n(a-c)} \le \frac{1}{2} \quad \text{for } n \ge 4.$$

Moreover, for $n \geq 4$,

$$\left(\frac{c}{a}\right)^{2(n-1)} \frac{e^{-2(n-1)ax}}{2(n-1)a} \leq \frac{1}{2}\left(\frac{c}{a}\right)^{n} e^{-nax},$$

so that

$$|v_n(x,z)| \leq \left(\frac{c}{a}\right)^{n} e^{-nax}, \qquad n \geq 1$$

and $\sum_{n\geq 1} v_n(x,z)$ converges absolutely and uniformly for $x \geq 0$, $\Im z \geq 0$, $z \neq 0$,.

We then have

$$|v(x,z) - iz| \leq \sum_{n\geq 1}\left(\frac{ce^{-ax}}{a}\right)^{n} \leq \frac{ce^{-ax}}{a}\sum_{n\geq 0}\left(\frac{c}{a}\right)^{n}$$

and finally

$$|v(x,z) - iz| \leq \frac{ce^{-ax}}{a}\frac{1}{1-c/a} = \frac{ce^{-ax}}{a-c},$$

so that $|v(x,z) - iz| \to 0$ as $x \to +\infty$ and $|v(x,z) - iz| \in \mathcal{L}(\mathbb{R}^+)$.

Now, according to (4.4), we have for $n \geq 3$

$$v_n' + 2izv_n = -\left\{v_{n-1}^2 + 2v_{n-1}\sum_{k=1}^{n-2} v_k\right\}$$

so that

$$\begin{aligned} |v_n'(x,z)| &\leq 2|zv_n(x,z)| + |v_{n-1}|^2 + 2|v_{n-1}|\sum_{k=1}^{n-2}|v_k(x,z)|, \\ &\leq \left(\frac{c}{a}\right)^{n} e^{-nax}\left\{2|z| + \left(\frac{c}{a}\right)^{2} e^{-2ax} + \sum_{k\geq 0}\left(\frac{c}{a}\right)^{k}\right\} \end{aligned}$$

and finally

$$|v_n'(x,z)| \leq \left(\frac{c}{a}\right)^{n} e^{-nax}\left\{2|z| + \left(\frac{c}{a}\right)^{2} e^{-2ax} + \frac{1}{1-c/a}\right\}.$$

Hence $\sum_{n\geq 1} v_n'(x,z)$ converges absolutely and uniformly for $x \geq 0$ and $\Im z \geq 0$, so that $v(x,z)$ is a solution of the Riccati equation (4.1), which concludes our proof. □

Note that, for example, the theorem applies to potentials of the form

$$q(x) = be^{idx}e^{-ax} \quad \text{with } b \in \mathbb{C},\ a > |b| + 1, \text{ and } d \in \mathbb{R},$$

$$q(x) = \frac{e^{-ax}}{(d + x^{\gamma})} \quad \text{with } d \in \mathbb{C} \setminus \{0\},\ \gamma > 0 \text{ and } a > \tfrac{1}{|d|} + 1,$$

as well as real-valued potentials of the form

$$q(x) = c\cos(x)e^{-ax} \quad \text{with } a > c + 1.$$

Spectral concentration was investigated for the latter class of potentials, numerically in [5] and using the Riccati series (with less restrictive bounds on the real-valued potential q) described above in [17].

According to the above theorem, the solution $v(x, \sqrt{\lambda})$ of the Riccati equation (4.1) can be identified with the quotient

$$\frac{\chi'(x, \sqrt{\lambda})}{\chi(x, \sqrt{\lambda})}$$

for $\Im\lambda \geq 0$.

In particular, for $x = 0$ we have

$$m_0(\lambda) = \frac{\chi_{\pi/2}(\sqrt{\lambda})}{\chi(\sqrt{\lambda})} = v(0, \sqrt{\lambda}).$$

Since, according to theorem 4.1.1, $v(0, \sqrt{\lambda})$ remains bounded uniformly with respect to $\sqrt{\lambda}$ on every compact subset of the upper $\sqrt{\lambda}$-plane, $m_0(\lambda)$ has no poles in $\lambda \in \mathbb{C} \setminus \{\lambda > 0\}$.

We have proved the following corollary:

Corollary 4.1.2. *If q satisfies* (3.13) *with $a > c + 1$, then the operator L_0 has no eigenvalue.*

Theorem 4.1.1 also allows us to find a bound for the eigenvalues of L_α.

We recall that

$$m_\alpha(\lambda) = \frac{-\sin(\alpha) + m_0(\lambda)\cos(\alpha)}{\cos(\alpha) + m_0(\lambda)\sin(\alpha)}. \tag{4.5}$$

Corollary 4.1.3. *Suppose that* (3.13) *holds with* $a > c+1$.

Then $L_{\pi/2}$ *has no eigenvalue* $\lambda = z^2$, $\Im z > 0$, *such that*

$$|z| > \frac{c}{a-c}.$$

If $\alpha \neq \pi/2$, *then* L_α *has no eigenvalue* $\lambda = z^2$, $\Im z > 0$, *such that*

$$|z| > \frac{c}{a-c} + |\cot(\alpha)|.$$

Proof. From (4.5) we have

$$m_{\pi/2}(\lambda) = \frac{1}{m_0(\lambda)}$$

and from theorem 4.1.1 it follows that

$$\begin{aligned} |m_0(\lambda)| &\geq |z| - \sum_{n\geq 1} \left(\frac{c}{a}\right)^n, \\ &\geq |z| - \frac{c}{a}\sum_{n\geq 0} \left(\frac{c}{a}\right)^n, \\ &\geq |z| - \frac{c}{a-c}, \end{aligned}$$

so that m_0 has no zero for $|z| > c/(a-c)$, and the first part of the corollary follows.

If $\alpha \neq \pi/2$ then, from (4.5), we have

$$m_\alpha(\lambda) = \frac{-1 + m_0(\lambda)\cot(\alpha)}{\cot(\alpha) + m_0(\lambda)}.$$

From corollary 4.1.2 $m_0(\lambda)$ has no pole for $\Im\sqrt{\lambda} > 0$, so that the poles of m_α are the zeros of the denominator in the above equality.

We have

$$|\cot(\alpha) + m_0(\lambda)| \geq |m_0(\lambda)| - |\cot(\alpha)| \geq |z| - \frac{c}{a-c} - |\cot(\alpha)|,$$

so that $m_\alpha(\lambda)$ has no pole for $|z| > |\cot(\alpha)| + c/(a-c)$, as required. □

We now turn our attention to the typically nonselfadjoint phenomenon of spectral singularity.

Corollary 4.1.4. *If* (3.13) *holds with* $a > c+1$ *and if* 0 *is not a spectral singularity, then the operator* L_0 *has no eigenvalues, no spectral singularities and the following expansion holds:*

For $f \in \mathcal{L}^2(\mathbb{R}^+)$

$$f(x) = \frac{1}{\pi}\int_{\mathbb{R}+} \phi(x,\lambda)g(\lambda)\frac{\sqrt{\lambda}}{\chi(\sqrt{\lambda})\chi(-\sqrt{\lambda})}d\lambda,$$

where

$$g(\lambda) = \int_{\mathbb{R}+} f(x)\phi(x,\lambda)d\lambda$$

and ϕ *is the solution of* (1.3) *satisfying* (1.2) *with* $\alpha = 0$.

Proof. According to corollary 4.1.2, L_0 has no eigenvalues. Note that, if q is real-valued, i.e. if L_0 is selfadjoint, then L_0 has no spectral singularities and the expansion in corollary 4.1.4 was proved in [23]. In this case the expansion of $f \in \mathcal{L}^2(\mathbb{R}^+)$ in generalised eigenfunctions of L_0 takes the form

$$f(x) = \frac{1}{\pi} \int_{\mathbb{R}^+} \phi(x, \lambda) g(\lambda) \frac{\sqrt{\lambda}}{\left|\chi(\sqrt{\lambda})\right|^2} d\lambda.$$

Now, the results in theorem 4.1.1 hold for $\Im z \geq 0$, $z \neq 0$, and the series (4.3) converges absolutely and uniformly for $\Im z \geq 0$, $z \neq 0$, in particular $m(x, z)$ is continuous for $\Im z = 0$, $z \neq 0$.
It follows from (2.9) with $\alpha = 0$ and theorem 4.1.1 that

$$m(0, z) = \frac{\chi_{\pi/2}(z)}{\chi(z)}, \quad \Im z = 0, \ z \neq 0.$$

Since $m(x, z)$ is continuous as a function of z for $\Im z = 0$, it follows from the equality above that $\chi(z)$ cannot vanish for $\Im z = 0$, $z \neq 0$. □

4.1.2 Examples

1. Set

$$q(x) = \frac{(1 - \beta^2)}{4 \cosh^2(x)}.$$

From (3.33), we have for the Jost solution $\chi(z)$:

$$\chi(z) = 2^{iz} \pi^{\frac{1}{2}} \frac{\Gamma(1 - iz)}{\Gamma(\frac{1}{2}(\frac{3}{2} - \frac{\beta}{2} - iz))\Gamma(\frac{1}{2}(\frac{3}{2} + \frac{\beta}{2} - iz))}.$$

Since

$$|q(x)| \leq |1 - \beta^2| e^{-2x},$$

theorem 4.1.1 holds for $|1 - \beta^2| < 1$ so that, if $|1 - \beta^2| < 1$, L_0 has no eigenvalues and no spectral singularities.

Set $\beta = 1/2$ so that $|1 - \beta^2| = 3/4 < 1$. In this case, the zeros of $\chi(z)$ are given by

$$z = -i(2M + 5/4), \quad M = 0, 1, 2, \ldots \qquad z = -i(2N + 7/4), \quad N = 0, 1, 2 \ldots$$

and L_0 has no eigenvalues, in accordance with corollary 4.1.2.

Set $\beta = 1 + i/4$ so that $|1 - \beta^2| = \sqrt{65}/16 < 1$. In this case the zeros of $\chi(z)$ are given by

$$z = -\frac{1}{8} - i(2M+1), \quad M = 0, 1, 2, \ldots \qquad z = \frac{1}{8} - i(2N+2), \quad N = 0, 1, 2, \ldots$$

and L_0 has no eigenvalues and no spectral singularities, in accordance with corollary 4.1.4.

2. Suppose now that $q(x) = ce^{-ax}$. According to (3.12), the Jost solution is

$$\chi(z) = 1 + \sum_{n \geq 0} (ca^{-2})^n \frac{1}{n!} \left(\frac{1}{(1 - 2iz/a)} \cdots \frac{1}{(n - 2iz/a)} \right),$$

where $z \in \mathbb{C} \setminus \bigcup_{n \geq 1} \{-ina/2\}$. Set $c = -1$ and $a = 2.5$, so that $a > |c| + 1$ and theorem 4.1.1 holds. The graph below was plotted using Maple 9. Note that Maple recognises in the series for the Jost function above a form of the Bessel function evaluated at 0, which is in turn interpreted as a hypergeometric function. Maple has a substantial amount of tools and an extensive library to deal with such functions and as a result we have found that Maple is both accurate and reliable when dealing with these functions. We plot $|\chi(y)|$ against y, where y is real.

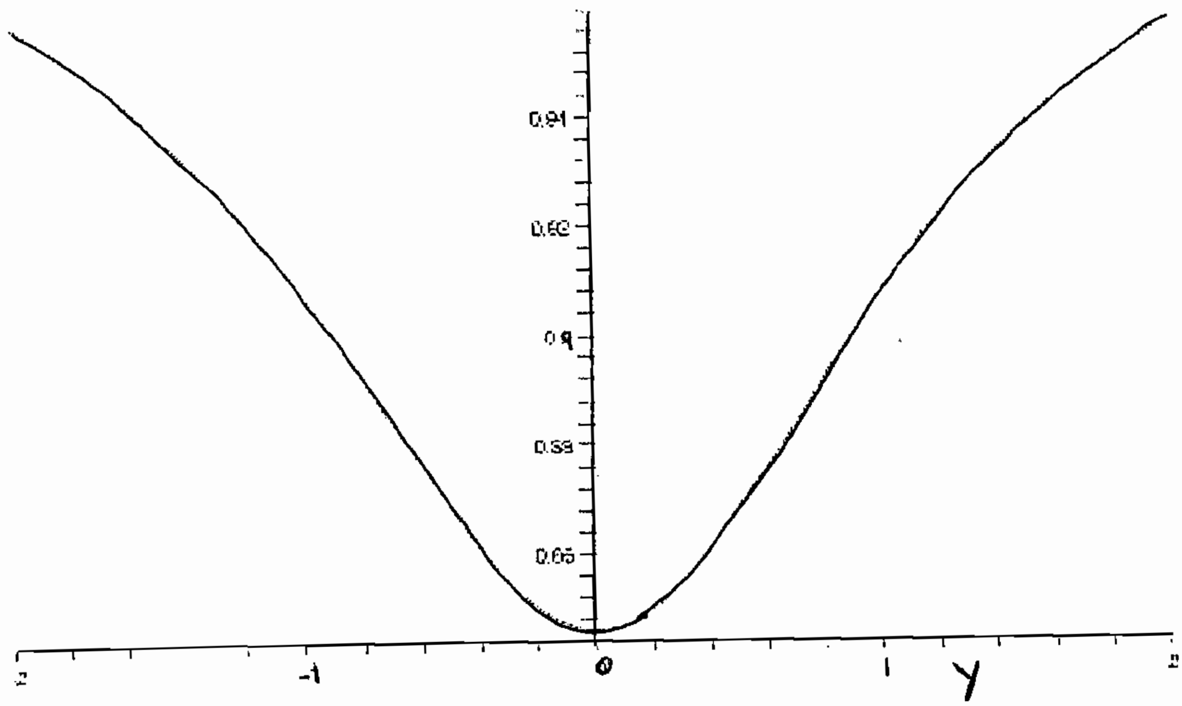


Figure 4.6 Graph of $|\chi(y)|$ plotted against y, $-2 \leq y \leq 2$, for $a = 2.5$ and $c = -1$.

The graph of $|\chi(y)|$ does not seem to intersect the real axis for real values of y, which indicates that there are no spectral singularities, in accordance with corollary 4.1.4. Note that the set of spectral singularities is in this case necessarily finite so it is likely that, if $\lambda = z^2$ is such a spectral singularity, z would not be too far away from 0.

4.1.3 Extension to the Lower-half Plane

We now give a lemma that shows that it is possible to extend analytically $v(x, z)$, as a function of z, to a part of the lower half z-plane.

Lemma 4.1.5. *If* (3.13) *holds, i.e. if* $|q(x)| \leq ce^{-ax}$, $x \geq 0$, *with* $c > 0$ *and* $a > c + 2$, *then*

$$|v_n(x,z)| \leq \left(\frac{c}{a + 2\Im z}\right)^n e^{-nax}, \quad n \geq 1,$$

and the series (4.3) *converges absolutely and uniformly for* $\Im z > c - a/2$, $x \geq 0$.

Proof. We suppose that (3.13) holds and that

$$\Im z > \frac{1}{2}(2c - a).$$

We prove by induction that

$$|v_n(x,z)| \leq \left(\frac{c}{a + 2\Im z}\right)^n e^{-nax}.$$

Note that

$$\Im z > c - \frac{a}{2} \quad \Rightarrow \quad \frac{c}{a + 2\Im z} < 1.$$

From (4.3),

$$|v_1(x,z)| \leq ce^{2x\Im z} \int_x^{+\infty} e^{-t(a+2\Im z)} dt,$$

so that

$$|v_1(x,z)| \leq \frac{c}{a + 2\Im z} e^{-ax}. \tag{4.7}$$

For v_2, we have therefore

$$|v_2(x,z)| \leq \left(\frac{c}{a + 2\Im z}\right)^2 e^{2x\Im z} \int_x^{+\infty} e^{-t(2a+2\Im z)} dt,$$

so that

$$|v_2(x,z)| \leq \left(\frac{c}{a + 2\Im z}\right)^2 \frac{e^{-2ax}}{2a + 2\Im z}$$

and finally

$$|v_2(x,z)| \leq \left(\frac{c}{a+2\Im z}\right)^2 e^{-2ax} \tag{4.8}$$

since $2a + 2\Im z > 2c + a > 1$.
We also prove that the result is true for v_3. We will later explain why we chose to do so. It follows from (4.3), (4.7) and (4.8) that

$$|v_3(x,z)| \leq e^{2x\Im z}\left(\frac{c}{a+2\Im z}\right)^3 \int_x^{+\infty} e^{-2t\Im z}\left\{\left(\frac{c}{a+2\Im z}\right)e^{-4at} + 2e^{-3at}\right\} dt,$$

$$\leq e^{2x\Im z}\left(\frac{c}{a+2\Im z}\right)^3 \int_x^{+\infty} e^{-t(3a+2\Im z)}\left\{\left(\frac{c}{a+2\Im z}\right) + 2\right\} dt,$$

$$|v_3(x,z)| \leq \left(\frac{c}{a+2\Im z}\right)^3 \frac{e^{-3ax}}{3a+2\Im z}\left\{\left(\frac{c}{a+2\Im z}\right) + 2\right\}.$$

On the other hand, since $2\Im z > 2c - a$ and $a > c + 2$, we have

$$\frac{1}{3a+2\Im z}\left\{\left(\frac{c}{a+2\Im z}\right) + 2\right\} < \frac{1}{2(c+a)}\left(\frac{1}{2} + 2\right)$$

so that

$$\frac{1}{3a+2\Im z}\left\{\left(\frac{c}{a+2\Im z}\right) + 2\right\} < \frac{5}{8} < 1.$$

This yields for v_3:

$$|v_3(x,z)| \leq \left(\frac{c}{a+2\Im z}\right)^3 e^{-3ax},$$

as required.

We suppose that the property is true for $1 \leq k \leq n-1$, $n \geq 4$. It follows from (4.3) and the induction hypothesis that

$$|v_n(x,z)| \leq e^{2x\Im z}\int_x^{+\infty} e^{-2t\Im z}\left\{\left(\frac{c}{a+2\Im z}\right)^{2(n-1)} e^{-2(n-1)at}\right.$$

$$\left. +2e^{-(n-1)at}\left(\frac{c}{a+2\Im z}\right)^{n-1}\sum_{k=1}^{n-2}\left(\frac{c}{a+2\Im z}\right)^k e^{-kt}\right\} dt$$

and

$$|v_n(x,z)| \le e^{2x\Im z}\left(\frac{c}{a+2\Im z}\right)^n \int_x^{+\infty} e^{-2t\Im z}\left\{e^{-nat}\right.$$

$$\left. +2e^{-nat}\left(\frac{c}{a+2\Im z}\right)^n \sum_{k\ge 0}\left(\frac{c}{a+2\Im z}\right)^k e^{-kt}\right\} dt,$$

since $2(n-1) \ge n$ for $n \ge 2$ and since

$$\sum_{k=1}^{n-2}\left(\frac{c}{a+2\Im z}\right)^k e^{-kt} \le \left(\frac{c}{a+2\Im z}\right) e^{-t} \sum_{k\ge 0}\left(\frac{c}{a+2\Im z}\right)^k e^{-kt}.$$

We also have

$$\sum_{k\ge 0}\left\{\left(\frac{c}{a+2\Im z}\right)^k e^{-kt}\right\} \le \sum_{k\ge 0}\left(\frac{c}{a+2\Im z}\right)^k = \frac{1}{1-c/(a+2\Im z)},$$

so that

$$|v_n(x,z)| \le e^{2x\Im z}\left(\frac{c}{a+2\Im z}\right)^n \int_x^{+\infty} e^{-t(na+2\Im z)}\left(1+\frac{1}{1-c/(a+2\Im z)}\right) dt$$

and

$$|v_n(x,z)| \le \left(\frac{c}{a+2\Im z}\right)^n e^{-nax}\frac{1}{na+2\Im z}\left(1+\frac{2}{1-c/(a+2\Im z)}\right).$$

Since $2\Im z > 2c - a$ we have $c/(a+2\Im z) < 1/2$ so that

$$\frac{2}{1-c/(a+2\Im z)} < 4.$$

Also

$$1/(na+2\Im z) \le 1/((n-1)a+2c) \le \frac{1}{2(n-1)},$$

since $na + 2\Im z > (n-1)a + 2c$ and $a > 2$. We have therefore

$$\frac{1}{na+2\Im z}\left(1+\frac{2}{1-c/(a+2\Im z)}\right) < \frac{5}{2(n-1)} < 1 \quad \text{for } n \ge 4$$

which is why we also proved that the result holds for $n = 3$.
It follows that

$$|v_n(x, z)| \le \left(\frac{c}{a + 2\Im z}\right)^n e^{-nax}, \quad n \ge 4,$$

which concludes our proof. □

The above lemma enables us to show that, under strong restrictions on the bound for q, the operator L_0 has no resonances close to the real axis in the lower half of the z-plane.

Theorem 4.1.6. *If* (3.13) *holds, i.e. if* $|q(x)| \le ce^{-ax}$, $x \ge 0$, *with* $c > 0$ *and* $a > \max\{2c,\ c+2\}$, *then* $c - a/2 < 0$ *and the operator* L_0 *has no eigenvalues and no resonances in the strip* $c - a/2 < \Im z < 0$.

Proof: If $a > \max\{2c, c+2\}$, then

$$c - \frac{a}{2} < 0 \quad \text{and} \quad a > c + 2,$$

so that lemma 4.1.5 applies and

$$v(x, z) = iz + \sum_{n \ge 1} v_n(x, z)$$

satisfies the Riccati equation (4.1) and can be extended analytically as a function of z to the half plane $\Im z > c - a/2$.

In particular, $v(0, z)$ is analytic and bounded in every compact subset of the strip $c - a/2 < \Im z < 0$, so that $v(0, z)$ has no poles in this strip.

Since $m_0(z^2) = v(0, z)$ for $\Im z \ge 0$ and since the analytic continuation is unique, it follows that L_0 has no resonance in the strip. □

We note that in [4] (see theorem 3.1), it was shown that if $q(x) = ce^{-ax}$ and if $c > 0$, $a > 0$, then L_0 has no resonances for $-2M < 2(\Im z)/a < -(2M+1)$.

Our results here are complementary to the results of [4], since we do not assume that q is of one sign or indeed that q is real-valued and our results are valid for

$$|q(x)| \leq ce^{-ax}, \quad x \geq 0,$$

admittedly with stringent conditions on a and c.

4.2 SPECTRAL CONCENTRATION

We suppose throughout this section that q is real-valued.

4.2.1 Definition and Example

We begin by defining the notion of spectral concentration and then provide a simple example for $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$.

We denote by $\rho_\alpha(\lambda)$ the spectral function associated with the operator L_α. Note that, according to theorem 2.2.4 with $\Im q \equiv 0$ (or according to [23]) we have, if (2.2) holds (i.e. if $q(x) = O(e^{-ax})$ as $x \to +\infty$ for some $a > 0$),

$$\rho'_\alpha(\lambda) = \frac{1}{\pi} \frac{\sqrt{\lambda}}{|\chi_\alpha(\sqrt{\lambda})|^2},$$

so that $\rho'(\lambda)$ is continuously differentiable for $\lambda > 0$.

Definition 4.2.1. If q satisfies (2.2), a point of spectral concentration is defined to be a local maximum for $\rho'_\alpha(\lambda)$, $\lambda > 0$.

Note that a point of spectral concentration is, in particular, a zero of $\rho''_\alpha(\lambda)$.

The intuitive idea behind the mathematical notion of points of spectral concentration is that, if the spectral function increases suddenly around some point λ_0, then it might appear during practical experiments as an eigenvalue.

Since, if q satisfies (2.2),

$$\rho'_\alpha(\lambda) = \frac{1}{\pi}\frac{\sqrt{\lambda}}{|\chi_\alpha(\sqrt{\lambda})|^2},$$

it seems therefore reasonable to expect that resonances situated close to the real axis and eigenvalues close to 0 are most likely to result in a local maximum of $\rho'_\alpha(\lambda)$.

However, the following example seems to indicate that, at least for $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$, it is possible for large negative eigenvalues and resonances far below the real axis to affect the local maximum of $\rho'_\alpha(\lambda)$.

Example
The Jost solution associated with (1.3) with $q \equiv 0$ is $\chi(x, z) = e^{izx}$, where $z = \sqrt{(}\lambda)$, $\Im z > 0$, so that

$$\chi_\alpha(z) = \chi(z)\cos(\alpha) + \chi_{\pi/2}(z)\sin(\alpha).$$

In this case

$$\chi_\alpha(z) = \cos(\alpha) + iz\sin(\alpha). \tag{4.9}$$

It is known that the derivative of the spectral function $\rho_\alpha(\lambda)$ in this case, for $\lambda > 0$, is

$$\rho'_\alpha(\lambda) = \frac{d}{d\lambda}\rho_\alpha(\lambda) = \frac{\sqrt{\lambda}}{\pi|\chi_\alpha(\sqrt{\lambda})|^2},$$

so that, according to (4.9),

$$\rho'_\alpha(\lambda) = \frac{\sqrt{\lambda}}{\pi|\cos(\alpha) + i\sqrt{\lambda}\sin(\alpha)|^2}$$

and finally for $\alpha \in [0, \pi)$

$$\rho'_\alpha(\lambda) = \frac{1}{\pi}\frac{\sqrt{\lambda}}{\cos^2(\alpha) + \lambda\sin^2(\alpha)}. \tag{4.10}$$

From (4.10) we can actually get the spectral function itself for $\lambda > 0$:

$$\rho_\alpha(\lambda) = \frac{2}{\pi\sin^2(\alpha)}\left\{\sqrt{\lambda} - \frac{\cos(\alpha)}{\sin(\alpha)}\arctan\left(\frac{\sqrt{\lambda}\sin(\alpha)}{\cos(\alpha)}\right)\right\}, \quad \alpha \neq 0.$$

It is also known that L_α has a unique eigenvalue for $\alpha \in (0, \pi/2)$ and that this eigenvalue λ_α is such that $\lambda_\alpha = z_\alpha^2$, where z_α is the zero of $\chi_\alpha(z)$ with $\Im z_\alpha > 0$.

According to (4.9),

$$z_\alpha = i\frac{\cos(\alpha)}{\sin(\alpha)} \quad \alpha \in (0, \pi/2).$$

Now, if $\Im z_\alpha < 0$, i.e. if $\alpha \in (\pi/2, \pi)$, $\chi_\alpha(z)$ has a zero with negative imaginary part, i.e. a resonance for L_α. Also

$$\rho_\alpha''(\lambda) = \frac{1}{\pi}\frac{\lambda^{-1/2}(\cos^2(\alpha) + \lambda\sin^2(\alpha)) - 2\lambda^{1/2}\sin^2(\alpha)}{2(\cos^2(\alpha) + \lambda\sin^2(\alpha))^2},$$

so that

$$\rho_\alpha''(\lambda) = \frac{1}{\pi}\frac{\lambda^{-1/2}\cos^2(\alpha) - \lambda^{1/2}\sin^2(\alpha)}{2(\cos^2(\alpha) + \lambda\sin^2(\alpha))^2}.$$

Hence $\rho_\alpha'(\lambda)$ has a unique local maximum at

$$\lambda_{e,\alpha} = \frac{\cos^2(\alpha)}{\sin^2(\alpha)}.$$

Note that $\lambda_{e,\alpha}$ is a point of spectral concentration, whether

$$z_\alpha = i\frac{\cos(\alpha)}{\sin(\alpha)}$$

is a resonance or gives rise to an eigenvalue. Note that, depending on the value of α, the eigenvalue can be far below zero or the resonance can be far below the real axis.

We now examine the spectral concentration phenomenon for a class of self-adjoint operators, using the methods of [17]. If q is real valued and if $\rho_0(\lambda)$, $\lambda > 0$, is differentiable, then

$$\rho_0'(\lambda) = \frac{1}{\pi}\Im m_0(\lambda + i0^+).$$

On the other hand, we have shown that, under certain conditions on q, $m_0(\lambda) = v(0, \sqrt{\lambda})$ so that, provided $v(0, \sqrt{\lambda})$ is differentiable,

$$\rho_0''(\lambda) = \frac{1}{\pi}\Im\left\{\frac{\partial v(x, \sqrt{\lambda})}{\partial \lambda}|_{x=0}\right\},$$

which should allow us to examine the properties of $\rho_0''(\lambda)$.

We recall that if q satisfies (3.13) and if $a > \max\{2c, c+2\}$ then, according to corollary 4.1.6, L_0 has no eigenvalues and no resonances in the strip $c - a/2 < \Im z < 0$, so that it is unlikely that L_0 has points of spectral concentration if, as is widely believed, resonances close to the real axis induce points of spectral concentration. See for example [12], [16] and [5].

4.2.2 Spectral Concentration for $\alpha = 0$

To begin with, we suppose that

$$q(x) = ce^{-ax}, \quad c \in \mathbb{R}, \quad a > 0. \tag{4.11}$$

Lemma 4.2.2. *If* (4.11) *holds with* $a > |c| + 1$*, we have the following bounds for the terms of the series in theorem 4.1.1:*

$$|v_n(x,z)| \leq \left(\frac{|c|}{|a-2iz|}\right)^n e^{-nax}, \quad n \geq 1, \quad \Im z = 0, \quad z \neq 0.$$

Proof. We note that, since $\Im z = 0$, $|e^{\pm 2izx}| = 1$. From (4.3), we have

$$|v_1(x,z)| = \left|\int_x^{+\infty} ce^{-t(a-2iz)}dt\right| = \frac{|c|}{|a-2iz|}e^{-ax}$$

and

$$|v_2(x,z)| = \left(\frac{|c|}{|a-2iz|}\right)^2 \left|\int_x^{+\infty} e^{-t(2a-2iz)}dt\right| \leq \left(\frac{|c|}{|a-2iz|}\right)^2 e^{-2ax}.$$

Suppose that for $1 \leq k \leq n-1$, $n \geq 3$,

$$|v_k(x,z)| \leq \left(\frac{|c|}{|a-2iz|}\right)^k e^{-kax}.$$

From (4.3)

$$|v_n(x,z)| \le \int_x^{+\infty} \left\{ \left(\frac{|c|}{|a-2iz|}\right)^{2(n-1)} e^{-2(n-1)at} \right.$$

$$\left. +2\left(\frac{|c|}{|a-2iz|}\right)^{n-1} e^{-(n-1)at} \sum_{k=1}^{n-2} \left(\frac{|c|}{|a-2iz|}\right)^k e^{-kat} \right\} dt,$$

$$|v_n(x,z)| \le \left(\frac{|c|}{|a-2iz|}\right)^n \int_x^{+\infty} e^{-nat} \left\{ 1 + 2\sum_{k\ge 0} \left(\frac{|c|}{|a-2iz|}\right)^k e^{-kat} \right\} dt,$$

so that

$$|v_n(x,z)| \le \left(\frac{|c|}{|a-2iz|}\right)^n \frac{e^{-nax}}{na} \left\{ 1 + \frac{2}{1-|c|/|a-2iz|} \right\}.$$

Now,

$$\frac{1}{na}\left(1+\frac{2}{1-|c|/|a-2iz|}\right) = \frac{1}{na} + \frac{2}{na(1-|c|/|a-2iz|)}$$

and since

$$a\left(1-\frac{|c|}{|a-2iz|}\right) > 1 \quad \Rightarrow \quad c < \left(1-\frac{1}{a}\right)|a-2iz| \le a-1,$$

we have for $a > c+1$

$$\frac{1}{na}\left(1+\frac{2}{1-|c|/|a-2iz|}\right) \le \frac{1}{n} + \frac{2}{n} \le 1, \quad n \ge 3,$$

so that

$$|v_n(x,z)| \le \left(\frac{|c|}{|a-2iz|}\right)^n e^{-nax},$$

as required. □

We now set

$$w_n(x,z) = \frac{\partial}{\partial\lambda} v_n(x,z), \quad n \ge 1.$$

We note that the conditions of lemma 4.2.2 ensure that assumptions (I)-(III) and (1.3) of [17] can be satisfied, so that we have formally (see [17], proof of lemma 3, page 334)

$$w_1(x,z) = \frac{i}{z} e^{-2izx} \int_x^{+\infty} e^{2izt} v_1(t,z)dt, \tag{4.12}$$

$$\begin{aligned} w_2(x,z) &= e^{-2izx} \int_x^{+\infty} e^{2izt} \left(\frac{i}{z} v_2 + 2v_1 w_1 \right) dt, \\ w_n(x,z) &= e^{-2izx} \int_x^{+\infty} e^{2izt} \left(\frac{i}{z} v_n + 2v_{n-1} w_{n-1} + 2w_{n-1} \sum_{k=1}^{n-2} v_k + 2v_{n-1} \sum_{k=1}^{n-2} w_k \right) dt \end{aligned}$$

Lemma 4.2.3. *If* (4.11) *holds with* $a \geq |c| + 2$, *then we have for the terms of the series* (4.12)

$$|w_n(x,z)| \leq \frac{1}{z} \left(\frac{|c|}{|a-2iz|} \right)^n e^{-nax}, \quad n \geq 1, \ \Im z = 0, \ z \neq 0, \ x \geq 0.$$

Proof. From (4.12) and lemma 4.2.2,

$$|w_1(x,z)| \leq \frac{1}{z} \frac{|c|}{|a-2iz|} \frac{e^{-ax}}{a}$$

and since $a > 1$, the result holds for $n = 1$. We also have

$$\begin{aligned} |w_2(x,z)| &\leq \frac{1}{z} \int_x^{+\infty} \left\{ \left(\frac{|c|}{|a-2iz|} \right)^2 e^{-2at} + 2 \left(\frac{|c|}{|a-2iz|} \right)^2 e^{-2at} \right\} dt, \\ &\leq \frac{1}{z} \left(\frac{|c|}{|a-2iz|} \right)^2 \int_x^{+\infty} 3e^{-2at} dt, \\ &\leq \frac{1}{z} \left(\frac{|c|}{|a-2iz|} \right)^2 \frac{3}{2a} e^{-2ax} \end{aligned}$$

and, since $3/(2a) < 1$, the result holds for $n = 2$.

We also need to prove the result for w_3. From (4.12), we get

$$|w_3(x,z)| \le \int_x^{+\infty} \left\{ \frac{1}{z}|v_3| + 2|v_2 w_2| + 2|w_2||v_1| + 2|v_2||w_1| \right\} dt,$$

$$\begin{aligned} |w_3(x,z)| &\le \frac{1}{z}\int_x^{+\infty} \left\{ \left(\frac{|c|}{|a-2iz|}\right)^3 e^{-3at} + 2\left(\frac{|c|}{|a-2iz|}\right)^4 \frac{3}{2a} e^{-4at} \right. \\ &+ \left. 2\left(\frac{|c|}{|a-2iz|}\right)^3 \frac{3e^{-3ax}}{2a} + 2\left(\frac{|c|}{|a-2iz|}\right)^3 \frac{e^{-3at}}{a} \right\} dt, \end{aligned}$$

which yields

$$|w_3(x,z)| \le \frac{1}{z}\left(\frac{|c|}{|a-2iz|}\right)^3 \int_x^{+\infty} e^{-3at}\left\{1+\frac{8}{a}\right\} dt$$

and finally

$$|w_3(x,z)| \le \frac{1}{z}\left(\frac{|c|}{|a-2iz|}\right)^3 e^{-3ax}$$

since

$$\frac{1}{3a}\left\{1+\frac{8}{a}\right\} \le \frac{5}{6}, \quad \text{for } a \ge 2.$$

Suppose that the result holds for $1 \ge k \ge n-1$, $n \ge 4$. Then

$$|w_n(x,z)| \le \int_x^{+\infty} \left\{ \frac{1}{z}|v_n| + 2|v_{n-1}w_{n-1}| + 2|w_{n-1}|\sum_{k=1}^{n-2}|v_k| + 2|v_{n-1}|\sum_{k=1}^{n-2}|w_k| \right\} dt,$$

$$\begin{aligned} |w_n(x,z)| \le \frac{1}{z}\int_x^{+\infty} &\left\{ \left(\frac{|c|}{|a-2iz|}\right)^n e^{-nat} + 2\left(\frac{|c|}{|a-2iz|}\right)^{2(n-1)} e^{-2(n-1)at} \right. \\ &\left. +4\left(\frac{|c|}{|a-2iz|}\right)^n e^{-nat}\sum_{k\ge 0}\left(\frac{|c|}{|a-2iz|}\right)^k \right\} dt \end{aligned}$$

and

$$|w_n(x,z)| \le \frac{1}{z}\left(\frac{|c|}{|a-2iz|}\right)^n \frac{e^{-nax}}{na}\left\{3+\frac{4}{1-|c|/|a-2iz|}\right\}.$$

We also have

$$\frac{1}{na}\left\{3+\frac{4}{1-|c|/|a-2iz|}\right\}=\frac{3}{na}+\frac{4}{n}\left(\frac{1}{a(1-|c|/|a-2iz|)}\right)$$

and since

$$a\geq|c|+2 \quad \Rightarrow \left(\frac{1}{a(1-|c|/|a-2iz|)}\right)\leq\frac{1}{2}$$

we get

$$\frac{1}{na}\left\{3+\frac{4}{1-|c|/|a-2iz|}\right\}\leq\frac{3}{8}+\frac{1}{2}<1, \quad n\geq 4.$$

It follows that

$$|w_n(x,z)|\leq\frac{1}{z}\left(\frac{|c|}{|a-2iz|}\right)^n e^{-nax}, \quad n\geq 1,$$

as required. □

We collect the consequences of the above lemmata in the following theorem:

Theorem 4.2.4. *If* (4.11) *holds with* $a\geq|c|+2$*, then* $\rho_0''(\lambda)$ *exists and is continuous for* $\lambda>0$*. Moreover,*

$$|\rho_0''(\lambda)-\frac{1}{2\pi\sqrt{\lambda}}|\leq\frac{1}{\pi\sqrt{\lambda}}\left(\frac{|c|}{|a-2i\sqrt{\lambda}|-|c|}\right), \quad \lambda>0.$$

In particular, if $3|c|\leq a$*, then there is no point of spectral concentration for* $\lambda>0$ *and if* $3|c|>a$ *then for*

$$\lambda>\frac{1}{4}(9|c|^2-a^2)$$

there is no point of spectral concentration.

Proof. We have

$$\rho_0''(\lambda) = \frac{1}{\pi}\Im\left\{\frac{\partial v(x,\sqrt{\lambda})}{\partial\lambda}|_{x=0}\right\} = \frac{1}{\pi}\left\{\Im w\left(0,\sqrt{\lambda}\right)\right\}$$

and

$$\Im w(0,z) = \frac{1}{2z} + \Im\left(\sum_{n\geq 1} w_n(0,z)\right) \quad z > 0$$

which yields

$$|\rho_0''(\lambda) - \frac{1}{2\pi\sqrt{\lambda}}| \leq \frac{1}{\pi}\sum_{n\geq 1}|w_n(0,\sqrt{\lambda})|, \quad \lambda > 0. \tag{4.13}$$

From lemma 4.2.3, we get

$$\begin{aligned}\sum_{n\geq 1}|w_n(0,\sqrt{\lambda})| &\leq \frac{|c|}{\sqrt{\lambda}|a-2i\sqrt{\lambda}|}\sum_{n\geq 0}\left(\frac{|c|}{|a-2i\sqrt{\lambda}|}\right)^n, \\ &= \frac{|c|}{\sqrt{\lambda}|a-2i\sqrt{\lambda}|}\frac{1}{1-|c|/|a-2i\sqrt{\lambda}|}\end{aligned}$$

and finally

$$\sum_{n\geq 1}|w_n(0,\sqrt{\lambda})| \leq \frac{|c|}{\sqrt{\lambda}(|a-2i\sqrt{\lambda}|-|c|)}.$$

Note that if $\rho_0''(\lambda) > 0$ for λ sufficiently large, then there is no point of spectral concentration for such λ since a point of spectral concentration is in particular a zero of $\rho''(\lambda)$. From (4.13), it follows that $\rho_0''(\lambda) > 0$ for

$$0 < \frac{1}{2\pi\sqrt{\lambda}} - \frac{|c|}{\pi\sqrt{\lambda}(|a-2i\sqrt{\lambda}|-|c|}, $$

i.e. for

$$|a-2i\sqrt{\lambda}| > 3|c|,$$

so that $\rho_0''(\lambda) > 0$ for

$$\lambda > \frac{1}{4}(9|c|^2 - a^2)$$

and the theorem follows. □

We note that our hypotheses are of course a lot more restrictive than those of [17]. One of the interests of the theorem is that the bound on the second derivative of the spectral function holds for $\lambda > 0$ if $3|c| \leq a$, allowing us to decide whether there is any point of spectral concentation at all.

Since the hypotheses have been strengthened here, we might expect a tighter estimate than the one given in [17].
Note that, in the notation of [17], we can by lemma 4.2.2 choose

$$\eta(\lambda) = \frac{|c|}{|a - 2i\sqrt{\lambda}|} \quad \text{and} \quad a(t) = e^{-at}$$

to satisfy the condition (1.3) in [17].
Theorem 1 in [17] says that there is no point of spectral concentration for

$$\eta(\lambda) \int_{\mathbb{R}^+} a(t)dt \leq \frac{1}{32},$$

giving for $q(x) = ce^{-ax}$

$$\lambda \geq \left(\frac{32c}{2a}\right)^2 - \frac{a^2}{4}.$$

Taking for example $c = 1$ and $a = 4$, [17] theorem 1, says that there is no point of spectral concentration for $\lambda \geq 12$, while the above theorem says that there is no point of spectral concentration for $\lambda > 0$.

If we take another example with $a = 5$ and $c = 2$, so that $3c > a$, [17] theorem 1 says that there is no point of spectral concentration for $\lambda \geq 34.71$ and the above theorem says that there is no point of spectral concentration for $\lambda > 2.75$.

On the other hand, taking $a = 10$ and $c = 5$, [17] theorem 1 gives $\lambda \geq 39$ and the above theorem gives $\lambda > 31.25$, which is in this case a rather disappointing improvement considering our hypotheses.

The latter example suggests that theorem 4.2.4 is essentially useful for relatively small values of $|c|$.

It is possible to broaden a bit our horizon and to use the theorem to get results for a larger class of potentials, as illustrated in the following corollary:

Corollary 4.2.5. *Suppose that*

$$q(x) = cp(x)e^{-ax},$$

with $a \geq |c| + 2$ *and where* $p \in C^1(\mathbb{R}^+)$ *is such that*

$$\|p\|_\infty + \frac{1}{2}\|p'\|_\infty \leq 1.$$

Then theorem 4.2.4 continues to hold.

Proof. We have

$$\int_x^{+\infty} e^{2izt}q(t)dt = c\int_x^{+\infty} p(t)e^{-t(a-2iz)}dt.$$

Integration by parts gives

$$\int_x^{+\infty} e^{2izt}q(t)dt = \frac{-c}{a-2iz}\left[p(t)e^{-t(a-2iz)}\right]_x^{+\infty} + \frac{c}{a-2iz}\int_x^{+\infty} p'(t)e^{-t(a-2iz)}dt,$$

$$|\int_x^{+\infty} e^{2izt}q(t)dt| \leq \|p\|_\infty \frac{|c|e^{-ax}}{|a-2iz|} + \|p'\|_\infty \left(\frac{|c|}{|a-2iz|}\right)\int_x^{+\infty} e^{-at}dt$$

and

$$|\int_x^{+\infty} e^{2izt}q(t)dt| \leq \|p\|_\infty \frac{|c|e^{-ax}}{|a-2iz|} + \|p'\|_\infty \left(\frac{|c|}{|a-2iz|}\right)\frac{e^{-ax}}{a}.$$

Since $|a| \geq 2$ it follows that

$$|\int_x^{+\infty} e^{2izt}q(t)dt| \leq \frac{|c|e^{-ax}}{|a-2iz|}\left(\|p\|_\infty + \frac{1}{2}\|p'\|_\infty\right),$$

which shows that

$$|\int_x^{+\infty} e^{2izt} q(t)dt| \le \frac{|c|e^{-ax}}{|a-2iz|}.$$

Lemma 4.2.2 and lemma 4.2.3 therefore hold as they stand for $q(x) = cp(x)e^{-ax}$, and the results of theorem 4.2.4 are still valid under the assumptions made on q in this corollary. $\square$

Taking for example $q(x) = \cos(2x)e^{-(3\pi/2)x}$, we can set

$$c = 2, \quad p(x) = \frac{1}{2}\cos(2x), \qquad \text{and } a = \frac{3\pi}{2},$$

so that corollary 4.2.5 holds with $3c > a$ and there is no point of spectral concentration for $\lambda > 3.45$.
In the notation of [17], we can set

$$a(t) = e^{-(3\pi/2)t}, \quad \eta(\lambda) = \frac{1}{\sqrt{(3\pi/2)^2 + 4(\sqrt{\lambda}+1)^2}}$$

which yields $\lambda \ge 1.7$, suggesting that theorem 4.2.5 is not necessarily an improvement on [17] if $3c > a$.

On the other hand, if we take $q(x) = \frac{1}{2}\cos(2x)e^{-\pi x}$, we can set

$$c = 1, \quad p(x) = \frac{1}{2}\cos(2x), \qquad \text{and } a = \pi$$

and corollary 4.2.5 implies that there is no point of spectral concentration for $\lambda > 0$, while [17] yields $\lambda \ge 10.3$

4.2.3 Computer Generated Examples

We suppose throughout this section that $q(x) = ce^{-ax}$ for some $c < 0$ and $a > 0$.

In section 3.2.3, we have seen that, when $q(x) = ce^{-ax}$ for some $c < 0$ and $a > 0$, it is possible to approximate some of the anti-bound states and eigenvalues of L_0. In this section, we propose to explore the relationship between anti-bound states, eigenvalues and points of spectral concentration using the mathematical software Maple 9. Maple is usually an efficient tool for dealing with special functions like the Gamma function, the Bessel function and the Hypergeometic function.

Intuitively, if an anti-bound state is close enough to the real axis, it should induce a point of spectral concentration. We are going to use Maple 9 to get an indication as to how close the anti-bound state should to be to the real axis in order to induce a point of spectral concentration.

If $\lambda = (it)^2$ with $t > 0$ is an eigenvalue of L_0, then it is a zero of the Jost function $\chi(z)$ and it might induce a point of spectral concentration if it is close enough to the real axis. We will also illustrate this phenomenon.

Let $q(x) = ce^{-ax}$. According to (3.12), we have

$$\chi(x,z) = e^{izx}\left\{1 + \sum_{n\geq 0}(ca^{-2}e^{-ax})^n \frac{1}{n!}\left(\frac{1}{(1-2iz/a)}\cdots\frac{1}{(n-2iz/a)}\right)\right\},$$

so that

$$\chi(z) = 1 + \sum_{n\geq 0}(ca^{-2})^n \frac{1}{n!}\left(\frac{1}{(1-2iz/a)}\cdots\frac{1}{(n-2iz/a)}\right).$$

Note that Maple identifies the series above as a form of the Bessel function. We use the above form of the Jost function with $z = it$, $t \in \mathbb{R}$, to produce the following graphs.

Example 1. Set $q(x) = -e^{-x}$ (i.e. set $a = 1$ and $c = -1$). The corresponding Jost solution $\chi(it)$ is plotted below against t for $-0.2 \leq t \leq 0.1$. According to the results obtained in section 4.2.3, $\chi(it)$ has a zero it_0 with $t_0 \in [-0.13, \ -0.12]$.

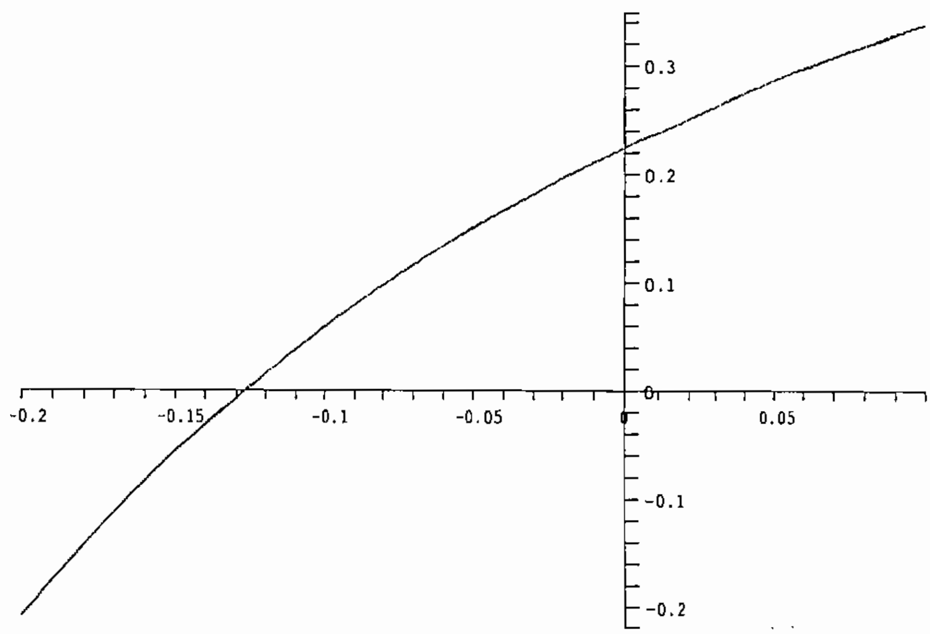


Figure 4.14 Graph of $\chi(it)$ plotted against t, $-0.2 \leq t \leq 0.1$, for Example 1

The graph above seems to indicate that the results given by Maple are in this case accurate since, from the graph, we get $t_0 \approx -0.126$.

The graph below is the graph of the spectral density

$$\rho'(\lambda) = \frac{\sqrt{\lambda}}{\pi \left|\chi\left(\sqrt{\lambda}\right)\right|^2}$$

(corresponding to the potential $q(x) = -e^{-x}$) plotted against λ for $0 \leq \lambda \leq 0.6$.

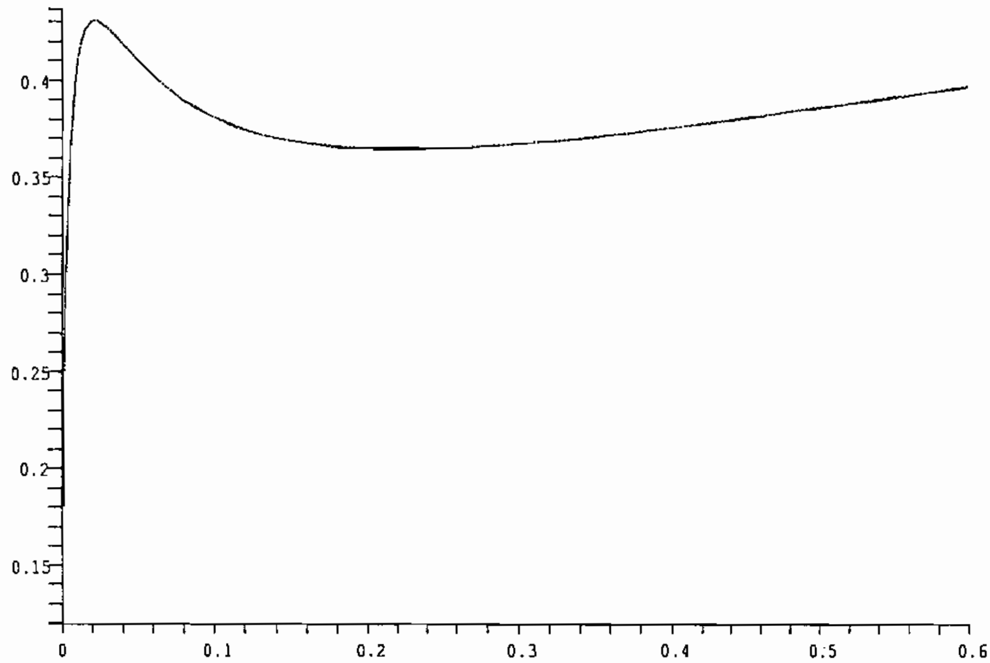


Figure 4.15 Graph of $\rho'(\lambda)$ plotted against λ, $0 \leq \lambda \leq 0.6$, for Example 1

Note that it_0 is close to the real axis and that Maple does indeed indicate that the anti-bound state induces a point of spectral concentration. Also, Maple does not show any evidence of another resonance it with $t \in (-1/2, 0)$ or of an eigenvalue.

Example 2. Setting $a = 1$ and $c = -1.5$ (i.e. setting $q(x) = -1.5e^{-x}$) and proceeding as above, we get

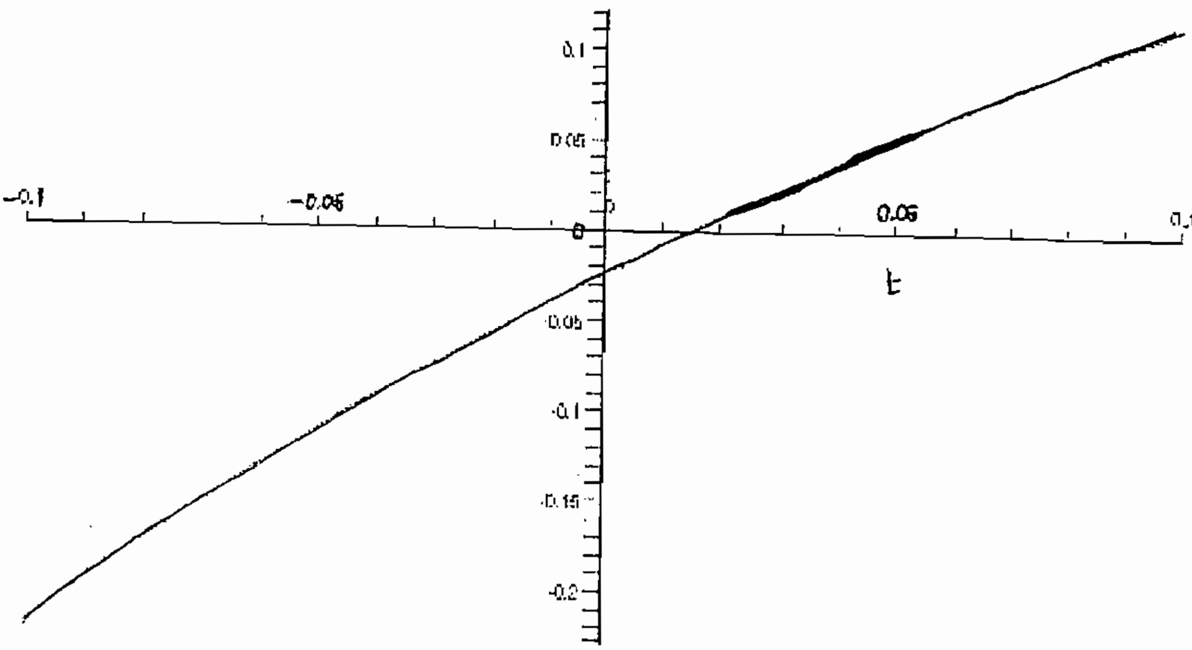


Figure 4.16 Graph of $\chi(it)$ plotted against t, $-0.1 \leq t \leq 0.1$, for Example 2

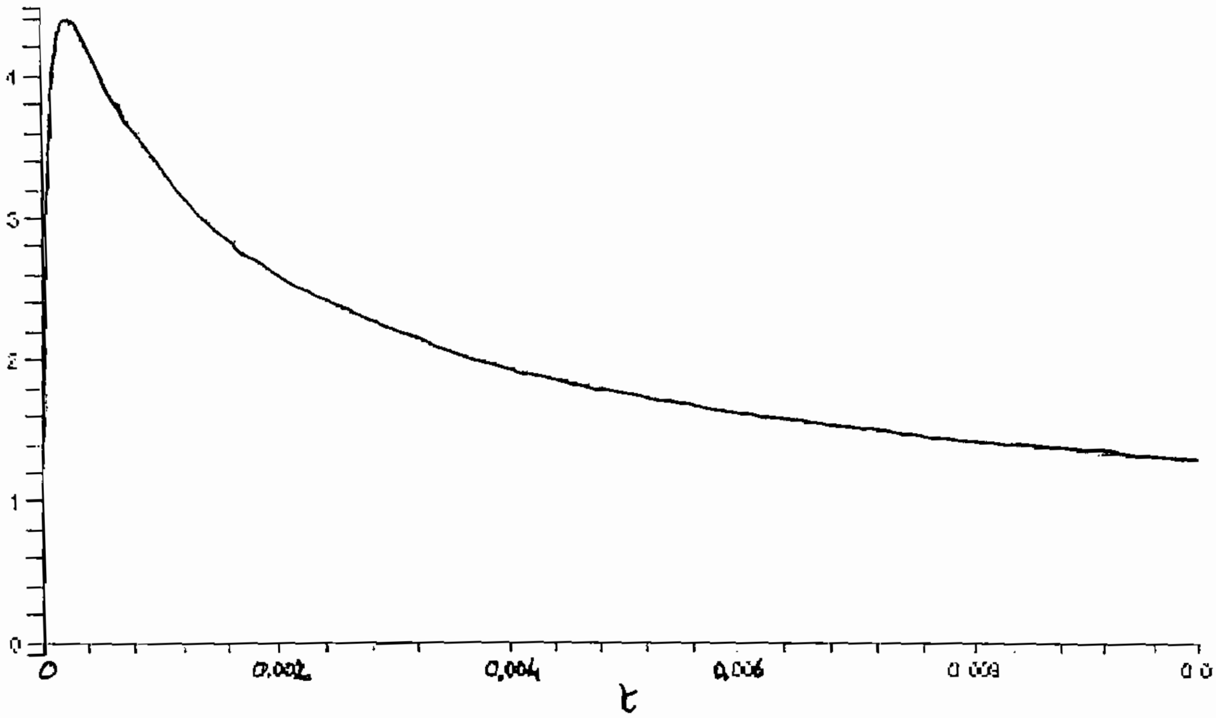


Figure 4.17 Graph of $\rho'(\lambda)$ plotted against λ, $0 \leq \lambda \leq 0.01$, for Example 2

The graphs (4.16) and (4.17) indicate that an eigenvalue $\lambda = (iz)^2$ with iz close enough to the real axis might induce a point of spectral concentration. Note that Maple does not show any evidence of a resonance it with $t \in (-1/2, 0)$ or of another eigenvalue.

Note that none of the potentials in the examples above satisfy the conditions of theorem 4.1.6 or the conditions of corollary 4.1.2. We now give an example of a potential that satisfies the criterion of theorem 4.1.6, i.e. $|q(x)| \leq ce^{-ax}$, $x \geq 0$ with $a > \max\{2c, c+2\}$.

Example 3. Let $q(x) = -e^{-4x}$. The hypotheses of theorem 4.1.6 are satisfied and theorem 4.1.6 predicts in particular that there are no antibound states on the segment line it, $-1 < t < 0$. The graph below seems to indicate that, in accordance with theorem 4.1.6, there are no antibound states on the segment

$-1 < t < 0$:

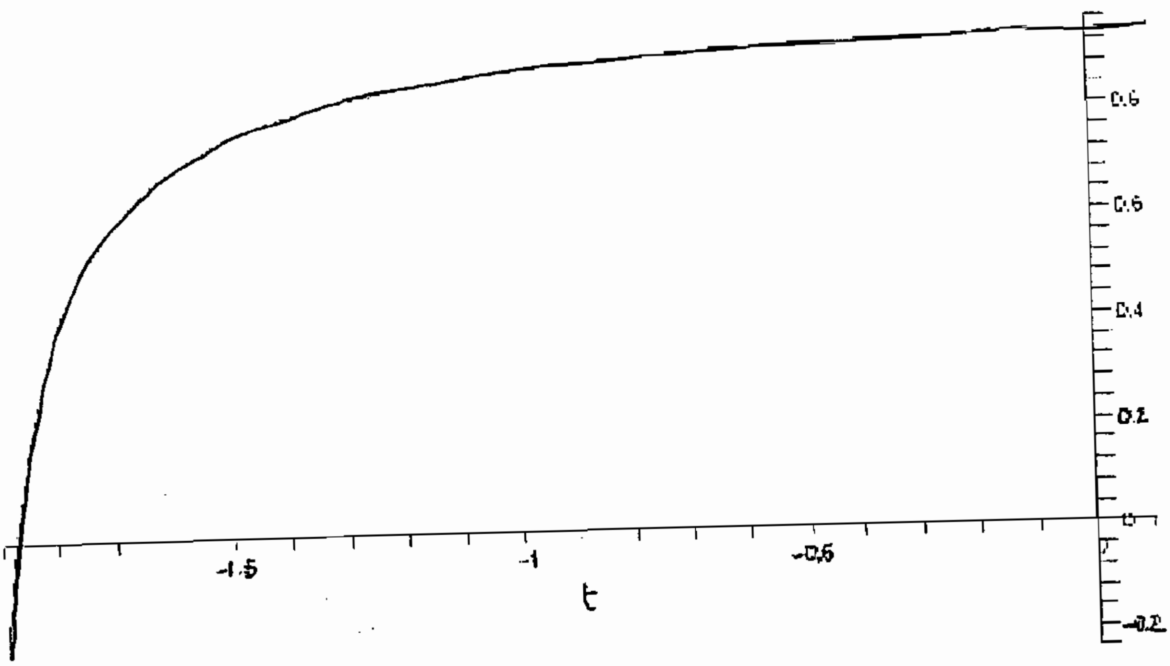


Figure 4.18 Graph of $\chi(it)$ plotted against t, $-1.9 \leq t \leq 0$, for Example 3

Note that the potential $q(x) = -e^{-4x}$ also satisfies the hypotheses of theorem 4.2.4 (with $|c| = 1$ and $a = 4$ so that $a > |c| + 2$ and $3|c| \leq a$) and that theorem 4.2.4 predicts that there are no points of spectral concentration for $\lambda > 0$. The graph below seems to confirm this assertion:

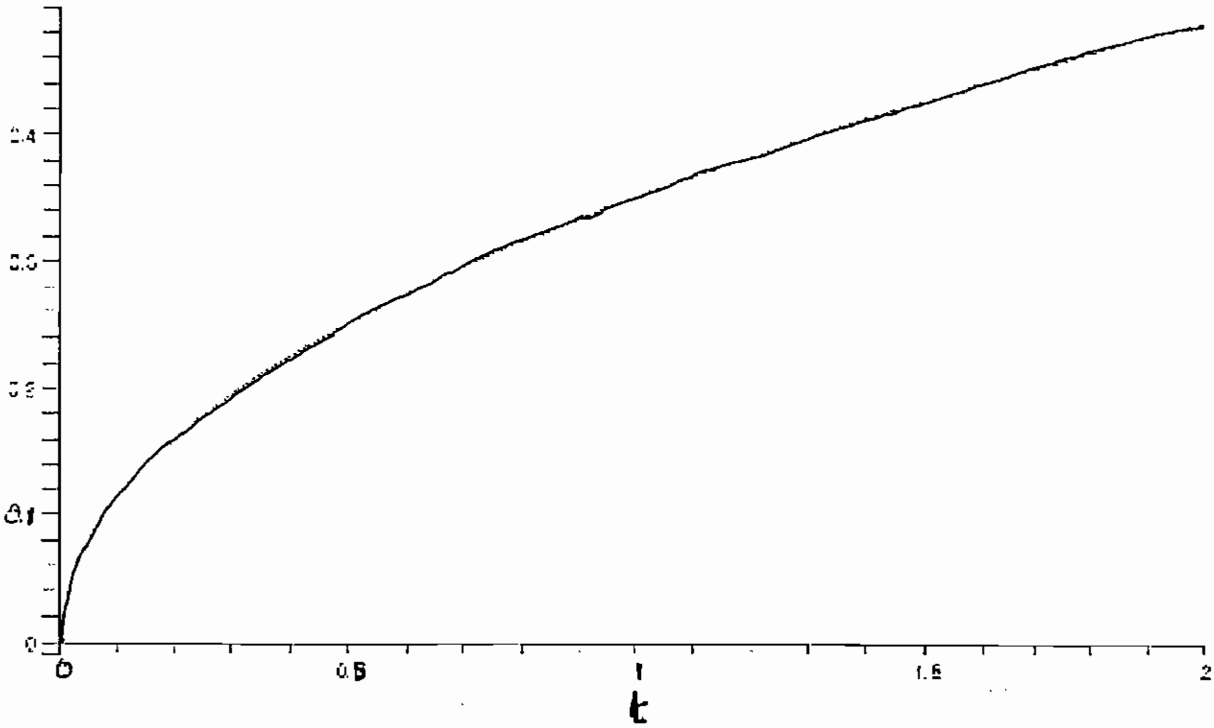


Figure 4.19 Graph of $\rho'(\lambda)$ plotted against λ, $0 \leq \lambda \leq 2$, for Example 3

We have illustrated in Example 1 how an antibound state which is close to the real axis can induce a point of spectral concentration. Example 2, however, seems to indicate that an eigenvalue can also give rise to a point of spectral concentration. Although it is likely that the condition in corollary 4.1.2 is not exact, in the sense that it may be possible to find a potential satisfying $|q(x)| \leq ce^{-ax}$, $x \geq 0$, with $0 < a \leq c+1$ for which there are no eigenvalues, Example 2 seems to indicate that, if the condition $a > c+1$ can be relaxed, it probably cannot be significantly relaxed.

Conclusion

We have shown that, for suitable conditions on a and c, the operator associated with (1.3) where $|q(x)| \leq ce^{-ax}$ shares most of the properties of the operator associated with the corresponding differential equation with $q \equiv 0$. The literature available on eigenvalues, resonances and spectral singularities is abundant, but we have shown that it is possible to study those notions as a single mathematical phenomenon if $|q(x)| \leq ce^{-ax}$, $x \geq 0$, namely as arising from the zeros of the Jost function. We have also tried, as far as possible, to show that under the hypotheses considered in this thesis, self- and nonself-adjoint Sturm-Liouville operators could be successfully studied using the same tools.

The results we have obtained in chapter 3 concerning the bounds on the set of zeros of the Jost solution apply to a class of real or complex valued potentials satsifying $|q(x)| \leq ce^{-ax}$, $x \geq 0$, while in [12] real valued potentials satisfying $q(x) = O(e^{-ax})$ as $x \to +\infty$ were considered. We also note that, in most of the papers we have read concerning resonances associated with Sturm-Liouville operators, only compactly supported potentials and super- exponentially decaying potentials were considered (see for example [21], [37] and [16]), while in this thesis we have considered potentials satisfying $|q(x)| \leq ce^{-ax}$, $x \geq 0$. Our method (see §3.2.3) for locating the eigenvalues and anti-bound states associated with Sturm-Liouville operators with potentials satisfying $q(x) = ce^{-ax}$, $x \geq 0$, for some $c < 0$ appears to be new and provides a relatively straightforward method for generating examples, as illustrated in §4.2.3. However, we have not yet been able to show that for $\alpha \neq 0$, $\alpha \neq \pi/2$, L_α has an eigenvalue $\lambda = z^2$, where z is close to $i\cot(\alpha)$, for suitable conditions on a and c. We believe that, using Rouche's theorem, it should be possible to prove such a result.

In chapter 4, our results on eigenvalues, resonances and spectral singularities are more precise than those obtained in chapter 3 as we imposed different and in general tighter conditions on the potential q. We have proved that under suitable conditions on the potential q, it is possible to show that there are no eigenvalues, no spectral singularities and no resonances close to the real axis. In particular, we have shown that a relatively simple form of Naimark's expansion theorem holds for a class of nonselfadjoint Sturm-

Liouville operators. The last part of the fourth chapter was dedicated to the study of the points of spectral concentration associated with a class of selfadjoint Sturm-Liouville operators, using a series derived by Harris and Gilbert (see [17]). We have obtained, in general, tighter bounds on the set of points of concentration (admitedly with more stringent restrictions on q) and showed that, for a class of potentials q, there is no point of spectral concentration at all. We have not systematically investigated the phenomenon of spectral concentration in the case $\alpha \neq 0$. We believe, on the basis of the example given in section 4.2.1 for $q \equiv 0$, that the cases $\alpha = 0$ and $\alpha \neq 0$ are rather dissimilar, and that for $\alpha \neq 0$ the zero of $\chi_\alpha(z)$ induced by the boundary condition could dominate the other zeros and create a single point of spectral concentration (provided q is small in some sense). This notion is supported by computer simulations hinting that, for some real-valued hypergeometric potentials of the kind considered in section 3.3, there is a single point of spectral concentration situated very close to the point of spectral concentration occuring for $q = 0$, $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$.

We also note that if the first derivative of the spectral function has a local minimum, then it might appear in experimental conditions as part of the resolvent set since, around this point, the spectral function could increase very slowly. We are not aware of any investigation on the subject and we do not know whether it is mathematically possible or whether it would be of interest to physicists. Finally, we believe that it should be possible to prove directly, at least for a small class of potentials, that eigenvalues and resonances can induce points of spectral concentration. It seems probable that this is the case but, to the best of our knowledge, no proof has been given so far apart from the one given in the example in section 4.2.1.

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