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Systemic Risk Assessment using a Non-stationary Fractional Dynamic Stochastic Model for the Analysis of Economic Signals

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***Abstract*— This paper considers the Fractal Market Hypothesis (FMH) for assessing the risk(s) in developing a financial portfolio based on data that is available through the Internet from an increasing number of sources. Most financial risk management systems are still based on the Efficient Market Hypothesis which often fails due to the inaccuracies of the statistical models that underpin the hypothesis, in particular, that financial data are based on stationary Gaussian processes. The FMH considered in this paper assumes that financial data are non-stationary and statistically self-affine so that a risk analysis can, in principal, be applied at any time scale provided there is sufficient data to make the output of a FMH analysis statistically significant. This paper considers a numerical method and an algorithm for accurately computing a parameter - the Fourier dimension - that serves in the assessment of a financial forecast and is applied to data taken from the Dow Jones and FTSE financial indices. A more detailed case study is then presented based on a FMH analysis of Sub-Prime Credit Default Swap Market ABX Indices.**

***Index Terms*— Risk assessment of economy, Risk assessment statistics and numerical data, Fractal Market Hypothesis, FTSE, Dow Jones and ABX index.**

I. Introduction

Attempts to develop stochastic models for financial time series are common place in financial mathematics and econometric in general. Financial time series are essentially digital signals composed of 'tick data' that provides traders with daily tick-by-tick data of trade price, trade time, and volume traded, for example, at different sampling rates [1], [2]. Stochastic financial models can be traced back to the early Twentieth Century when Louis Bachelier [3] proposed that fluctuations in the prices of stocks and shares (which appeared to be yesterday's price plus some random change) could be viewed in terms of random walks in which price changes were entirely independent of each other. Thus, one of the simplest models for price variation is based on the sum of independent random numbers. This is the basis for Brownian motion [4] in which the random numbers are considered to conform to a normal distribution. This model is the basis for the Efficient Market Hypothesis (EMH) which has a number of questionable assumptions as discussed in the following section. In this paper, we consider a method for processing financial time series data based on the Fractal Market Hypothesis. The underlying rationale for this model is discussed and example results presented to illustrate the ability for the model to provide an improved risk assessment of an economy with regard to predicting the characteristics of an economic time series based on a risk assessment statistic computed from numerical data. A case study is presented that is based on the sub-prime credit default swap market ABX index which is acknowledged as being one of the principal markets whose collapse triggered the current global recession.

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II. Brownian Motion and the Efficient Market Hypothesis

Random walk models, which underpin the so called Efficient Market Hypothesis (EMH) [5]-[12] have been the basis for financial time series analysis since the work of Bachelier in the late Nineteenth Century. Although the Black-Scholes equation [13], developed in the 1970s for valuing options, is deterministic (one of the first financial models to achieve determinism), it is still based on the EMH, i.e. stationary Gaussian statistics. The EMH is based on the principle that the current price of an asset fully reflects all available information relevant to it and that new information is immediately incorporated into the price. Thus, in an efficient market, the modelling of asset prices is concerned with modelling the arrival of new information. New information must be independent and random, otherwise it would have been anticipated and would not be new. The arrival of new information can send 'shocks' through the market (depending on the significance of the information) as people react to it and then to each other's reactions. The EMH assumes that there is a rational and unique way to use the available information and that all agents possess this knowledge. Further, the EMH assumes that this 'chain reaction' happens effectively instantaneously. These assumptions are clearly questionable at any and all levels of a complex financial system.

The EMH implies independence of price increments and is typically characterised by a normal of Gaussian Probability Density Function (PDF) which is chosen because most price movements are presumed to be an aggregation of smaller ones, the sums of independent random contributions having a Gaussian PDF. However, it has long been known that financial time series do not follow random walks. The shortcomings of the EMH model include: failure of the independence and Gaussian distribution of increments assumption, clustering, apparent non-stationarity and failure to explain momentous financial events such as 'crashes' leading to recession and, in some extreme cases, depression. These limitations have prompted a new class of methods for investigating time series obtained from a range of disciplines. For example, Re-scaled Range Analysis (RSRA), e.g. [14]-[16], which is essentially based on computing the Hurst exponent [17], is a useful tool for revealing some well disguised properties of stochastic time series such as persistence (and anti-persistence) characterized by non-periodic cycles. Non-periodic cycles correspond to trends that persist for irregular periods but with a degree of statistical regularity often associated with non-linear dynamical systems. RSRA is particularly valuable because of its robustness in the presence of noise. The principal assumption associated with RSRA is concerned with the self-affine or fractal nature of the statistical character of a time-series rather than the statistical 'signature' itself. Ralph Elliott first reported on the fractal properties of financial data in 1938 (e.g. [18] and reference therein). He was the first to observe that segments of financial time series data of different sizes could be scaled in such a way that they were statistically the same producing so called Elliot waves.

III. Risk Assessment and Repeating Economic Patterns

A good stochastic financial model should ideally consider all the observable behaviour of the financial system it is attempting to model. It should therefore be able to provide some predictions on the immediate future behaviour of the system within an appropriate confidence level. Predicting the markets has become (for obvious reasons) one of the most important problems in financial engineering. Although, at least in principle, it might be possible to model the behaviour of each individual agent operating in a financial market, one can never be sure of obtaining all the necessary information required on the agents themselves and their modus operandi. This principle plays an increasingly important role as the scale of the financial system, for which a model is required, increases. Thus, while quasi-deterministic models can be of value in the understanding of micro-economic systems (with known 'operational conditions'), in an ever increasing global economy (in which the operational conditions associated with the fiscal policies of a given nation state are increasingly open), we can take advantage of the scale of the system to describe its behaviour in terms of functions of random variables.

A. Elliot Waves

The stochastic nature of financial time series is well known from the values of the stock market major indices such as the FTSE (Financial Times Stock Exchange) in the UK, the Dow Jones in the US which are frequently quoted. A principal aim of investors is to attempt to obtain information that can provide some confidence in the immediate future of the stock markets often based on patterns of the past. One of the principal components of this aim is based on the observation that there are 'waves within waves' and 'events within events' that appear to permeate financial signals when studied with sufficient detail and imagination. It is these repeating patterns that occupy both the financial investor and the systems modeller alike and it is clear that although economies have undergone many changes in the last one hundred years, the dynamics of market data do not appear to change significantly (ignoring scale). For example, with data obtained from [19], Figure 1 shows the re-scaled signals and associated 'macrotrends' (i.e. normalised time series and associated time series after application of a Gaussian lowpass filter) associated with FTSE Close-of-Day (COD) illustrating the 'development' of three different 'crashes'; those of 1987, 1997 and the most recent crash of 2007. The macrotrends are computed by filtering each signal in Fourier space using a Gaussian lowpass filter $\exp(-\beta\omega^2)$ with $\beta = 0.1$ where ω is the angular frequency.

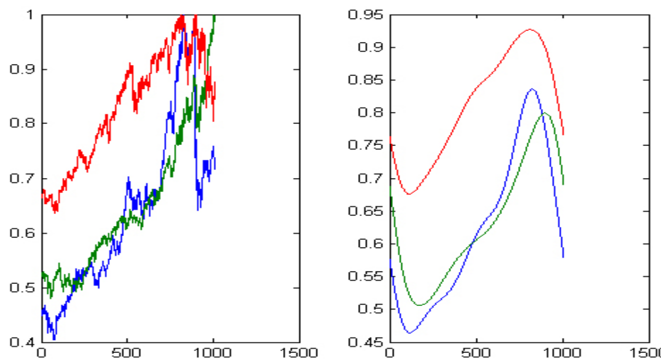

Fig. 1. Evolution of the 1987, 1997 and 2007 financial crashes. Normalised data (left) and macrotrends (right) where the data has been smoothed and rescaled to values between 0 and 1 inclusively) of the daily FTSE value (close-of-day) for 02-04-1984 to 24-12-1987 (blue), 05-04-1994 to 24-12-1997 (green) and 02-04-2004 to 24-09-2007 (red).

The similarity in behaviour of these signals is remarkable and clearly indicates a wavelength of approximately 1000 days. This is indicative of the quest to understand economic signals in terms of some universal phenomenon from which appropriate (macro) economic models can be generated. In an efficient market, only the revelation of some dramatic information can cause a crash, yet post-mortem analysis of crashes typically fail to (convincingly) tell us what this information must have been.

One cause of correlations in market price changes (and volatility) is mimetic behaviour, known as herding. In general, market crashes happen when large numbers of agents place sell orders simultaneously creating an imbalance to the extent that market makers are unable to absorb the other side without lowering prices substantially. Most of these agents do not communicate with each other, nor do they take orders from a leader. In fact, most of the time they are in disagreement, and submit roughly the same amount of buy and sell orders. This is a healthy non-crash situation; it is a diffusive (random-walk) process which underlies the EMH and financial portfolio rationalization.

B. Non-equilibrium Systems

Financial markets can be considered to be non-equilibrium systems because they are constantly driven by transactions that occur as the result of new fundamental information about firms and businesses. They are complex systems because the market also responds to itself, often in a highly non-linear fashion, and would carry on doing so (at least for some time) in the absence of new information. The 'price change field' is highly non-linear and very sensitive to exogenous shocks and it is probable that all shocks have a long term effect. Market transactions generally occur globally at the rate of hundreds of thousands per second. It is the frequency and nature of these transactions that dictate stock market indices, just as it is the frequency and nature of the sand particles that dictates the statistics of the avalanches in a sand pile. These are all examples of random scaling fractals [21]-[26].

IV. The Fractal Market Hypothesis

Developing mathematical models to simulate stochastic processes has an important role in financial analysis and information systems in general where it should be noted that information systems are now one of the most important aspects in terms of regulating financial systems, e.g. [27]-[30]. A good stochastic model is one that accurately predicts the statistics we observe in reality, and one that is based upon some well defined rationale. Thus, the model should not only describe the data, but also help to explain and understand the system.

There are two principal criteria used to define the characteristics of a stochastic field: (i) The PDF or the Characteristic Function (i.e. the Fourier transform of the PDF); the Power Spectral Density Function (PSDF). The PSDF is the function that describes the envelope or shape of the power spectrum of a signal. In this sense, the PSDF is a measure of the field correlations. The PDF and the PSDF are two of the most fundamental properties of any stochastic field and various terms are used to convey these properties. For example, the term 'zero-mean white Gaussian noise' refers to a stochastic field characterized by a PSDF that is effectively constant over all frequencies (hence the term 'white' as in 'white light') and has a PDF with a Gaussian profile whose mean is zero.

Stochastic fields can of course be characterized using transforms other than the Fourier transform (from which the PSDF is obtained) but the conventional PDF-PSDF approach serves many purposes in stochastic systems theory. However, in general, there is no general connectivity between the PSDF and the PDF either in terms of theoretical prediction and/or experimental determination. It is not generally possible to compute the PSDF of a stochastic field from knowledge of the PDF or the PDF from the PSDF. Hence, in general, the PDF and PSDF are fundamental but non-related properties of a stochastic field. However, for some specific statistical processes, relationships between the PDF and PSDF can be found, for example, between Gaussian and non-Gaussian fractal processes [31] and for differentiable Gaussian processes [32].

There are two conventional approaches to simulating a stochastic field. The first of these is based on predicting the PDF (or the Characteristic Function) theoretically (if possible). A pseudo random number generator is then designed whose output provides a discrete stochastic field that is characteristic of the predicted PDF. The second approach is based on considering the PSDF of a field which, like the PDF, is ideally derived theoretically. The stochastic field is then typically simulated by filtering white noise. A 'good' stochastic model is one that accurately predicts both the PDF and the PSDF of the data. It should take into account the fact that, in general, stochastic processes are non-stationary. In addition, it should, if appropriate, model rare but extreme events in which significant deviations from the norm occur.

One explanation for crashes involves a replacement for the EMH by the Fractal Market Hypothesis (FMH) which is the basis of the model considered in this paper. The FMH proposes the following: (i) The market is stable when it consists of investors covering a large number of investment horizons which ensures that there is ample liquidity for traders; (ii) information is more related to market sentiment and technical factors in the short term than in the long term - as investment horizons increase and longer term fundamental information dominates; (iii) if an event occurs that puts the validity of fundamental information in question, long-term investors either withdraw completely or invest on shorter terms (i.e. when the overall investment horizon of the market shrinks to a uniform level, the market becomes unstable); (iv) prices reflect a combination of short-term technical and long-term fundamental valuation and thus, short-term price movements are likely to be more volatile than long-term trades - they are more likely to be the result of crowd behaviour; (v) if a security has no tie to the economic cycle, then there will be no long-term trend and short-term technical information will dominate. Unlike the EMH, the FMH states that information is valued according to the investment horizon of the investor. Because the different investment horizons value information differently, the diffusion of information will also be uneven. Unlike most complex physical systems, the agents of the economy, and perhaps to some extent the economy itself, have an extra ingredient, an extra degree of complexity. This ingredient is consciousness.

V. Mathematical Model for the FMH

We consider an economic times series to be a solution to the fractional diffusion equation [33]-[38]

$$\left(\frac{\partial^2}{\partial x^2}-\sigma^q\frac{\partial^q}{\partial t^q}\right)u(x,t)=\delta(x)n(t) \tag{1}$$

where σ is the fractional diffusion coefficient, $q>0$ is the 'Fourier dimension' and $n(t)$ is 'white noise'. Let

$$u(x,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}U(x,\omega)\exp(i\omega t)d\omega$$

and

$$n(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}N(\omega)\exp(i\omega t)d\omega.$$

Using the result

$$\frac{\partial^q}{\partial t^q}u(x,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}U(x,\omega)(i\omega)^q\exp(i\omega t)d\omega$$

we can then transform the fractional diffusion equation to the form

$$\left(\frac{\partial^2}{\partial x^2}+\Omega_q^2\right)U(x,\omega)=\delta(x)N(\omega)$$

where we take

$$\Omega_q=i(i\omega\sigma)^{\frac{q}{2}}$$

Defining the Green's function g [39] to be the solution of

$$\left(\frac{\partial^2}{\partial x^2}+\Omega_q^2\right)g(\mid x-y\mid,\omega)=\delta(x-y)$$

where δ is the delta function, we obtain the solution

$$U(x,\omega)=N(\omega)\int_{-\infty}^{\infty}g(\mid x-y\mid,\omega)\delta(y)dy=N(\omega)g(\mid x\mid,\omega)$$

where [40]

$$g(\mid x\mid,\omega)=\frac{i}{2\Omega_q}\exp(i\Omega_q\mid x\mid)$$

under the assumption that u and $\partial u/\partial x\to 0$ as $x\to\pm\infty$. The Green's function characterises the response a system modelled by equation (1) due to an impulse at $x=y$ and it is clear that

$$\lim_{x\to 0}U(x,\omega)=\frac{iN(\omega)}{2\Omega_q}$$

or

$$U(\omega)=\frac{1}{2\sigma^{\frac{q}{2}}}\frac{N(\omega)}{(i\omega)^{\frac{q}{2}}}$$

The time series associated with this asymptotic solution is then obtained by Fourier inversion giving (ignoring scaling by $[2\sigma^{q/2}\Gamma(q/2)]^{-1}$)

$$u(t)=\frac{1}{t^{1-q/2}}\otimes n(t) \qquad (2)$$

where $\otimes$ defines the convolution integral. This equation is the Riemann - Liouville transform (ignoring scaling by $[\Gamma^{-1}(q/2)]^{-1}$) [41] which is a fractional integral and defines a function $u(t)$ which is statistically self-affine, i.e. for a scaling parameter $\lambda>0$,

$$\lambda^{q/2}\Pr[u(\lambda t)]=\Pr[u(t)]$$

where $\Pr[u(t)]$ denotes the Probability Density Function of $u(t)$. Thus, equation (2) can be considered to be the temporal solution of equation (1) as $x\to 0$ and $u(t)$ is taken to be a random scaling fractal signal. Note that for $\mid x\mid>0$ the phase $\Omega_q\mid x\mid$ does not affect the ω^{-q} scaling law of the power spectrum, i.e. $\forall x$,

$$\mid U(x,\omega)\mid^2=\frac{\mid N(\omega)\mid^2}{4\sigma^q\omega^q},\quad \omega>0$$

Thus for a uniformly distributed spectrum $N(\omega)$ the Power Spectrum Density Function of U is determined by ω^{-q} and the algorithm developed to compute q given in Section 6 applies $\forall x$ and not just for the case when $x\to 0$. However, since we can write

$$U(x,\omega)=N(\omega)\frac{i}{2\Omega_q}\exp(i\Omega_q\mid x\mid)$$

$$=N(\omega)\frac{1}{2(i\omega\sigma)^{q/2}}\left(1+i(i\omega\sigma)^{q/2}\mid x\mid-\frac{1}{2!}(i\omega\sigma)^q\mid x\mid^2+...\right)$$

unconditionally, by inverse Fourier transforming, we obtain the following expression for $u(x,t)$ (ignoring scaling factors):

$$u(x,t)=n(t)\otimes\frac{1}{t^{1-q/2}}+i\mid x\mid n(t)$$

$$+\sum_{k=1}^{\infty}\frac{i^{k+1}}{(k+1)!}\mid x\mid^{2k}\frac{d^{kq/2}}{dt^{kq/2}}n(t)$$

Here, the solution is composed of three terms composed of (i) a fractional integral, (ii) the source term $n(t)$; (iii) an infinite series of fractional differentials of order $kq/2$.

A. *Rationale for the Model - Hurst Processes*

A Hurst process describes fractional Brownian motion and is based on the generalization of Brownian motion quantified by the equation $A(t)=\sqrt{t}$ to

$$A(t)=t^H,\quad H\in(0,1]$$

for a unit random step length in the plane where A is the most likely position in the plane after time t with respect to an initial position in the plane at $t=0$. This scaling law makes no prior assumptions about any underlying distributions. It simply tells us how the system is scaling with respect to time. Processes of this type appear to exhibit cycles, but with no predictable period. The interpretation of such processes in terms of the Hurst exponent H is as follows: We know that $H=0.5$ is consistent with an independently distributed system. The range $0.5<H\leq 1$, implies a persistent time series, and a persistent time series is characterized by positive correlations. Theoretically, what happens today will ultimately have a lasting effect on the future. The range $0<H\leq 0.5$ indicates anti-persistence which means that the time series covers less ground than a random process. In other words, there are negative correlations. For a system to cover less distance, it must reverse itself more often than a random process.

Given that random walks with $H=0.5$ describe processes whose macroscopic behaviour is characterised by the diffusion equation, then, by induction, Hurst processes should be characterised by generalizing the diffusion operator

$$\frac{\partial^2}{\partial x^2}-\sigma\frac{\partial}{\partial t}$$

to the fractional form

$$\frac{\partial^2}{\partial x^2}-\sigma^q\frac{\partial^q}{\partial t^q}$$

where $q\in(0,2]$ Fractional diffusive processes can therefore be interpreted as intermediate between classical diffusive (random phase walks with $H=0.5$; diffusive processes with $q=1$) and 'propagative process' (coherent phase walks for $H=1$; propagative processes with $q=2$), e.g. [42] and [43]. The relationship between the Hurst exponent H, the Fourier dimension q and the Fractal dimension D_F is given by [44]

$$D_F=D_T+1-H=1-q+\frac{3}{2}D_T$$

where D_T is the topological dimension. Thus, a Brownian process, where $H = 1/2$, has a fractal dimension of 1.5.

Fractional diffusion processes are based on random walks which exhibit a bias with regard to the distribution of angles used to change the direction. By induction, it can be expected that as the distribution of angles reduces, the corresponding walk becomes more and more coherent, exhibiting longer and longer time correlations until the process conforms to a fully coherent walk. A simulation of such an effect is given in Figure 2 which shows a random walk in the (real) plane as the (uniform) distribution of angles decreases. The walk becomes less and less random as the width of the distribution is reduced. Each position of the walk $(x_j, y_j), \quad j = 1, 2, 3, ..., N$ is computed using

$$x_j = \sum_{i=1}^{j} \cos(\theta_i), \quad y_j = \sum_{i=1}^{j} \sin(\theta_i)$$

where

$$\theta_i = \alpha\pi \frac{n_i}{\|\mathbf{n}\|_\infty}$$

and n_i are random numbers computed using the linear congruential pseudo random number generator

$$n_{i+1} = a n_i \text{mod} P, \quad i = 1, 2, ..., N, \quad a = 7^7, \quad P = 2^{31} - 1$$

The parameter $0 \leq \alpha \leq 2\pi$ defines the width of the distribution of angles such that as $\alpha \to 0$, the walk becomes increasingly coherent or 'propagative'

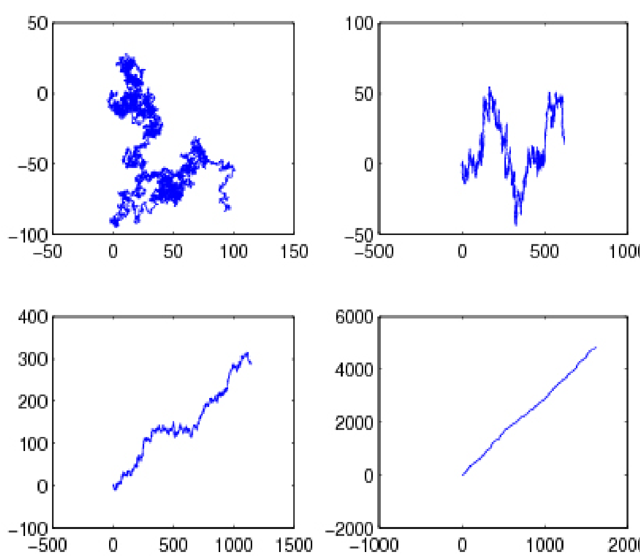

Fig. 2. Random phase walks in the plane for a uniform distribution of angles $\theta_i \in [0, 2\pi]$ (top left), $\theta_i \in [0, 1.9\pi]$ (top right), $\theta_i \in [0, 1.8\pi]$ (bottom left) and $\theta_i \in [0, 1.2\pi]$ (bottom right).

In considering a t^H scaling law with Hurst exponent $H \in (0, 1]$, Hurst paved the way for an appreciation that most natural stochastic phenomena which, at first site, appear random, have certain trends that can be identified over a given period of time. In other words, many natural random patterns have a bias to them that leads to time correlations in their stochastic behaviour, a behaviour that is not an inherent characteristic of a random walk model and fully diffusive processes in general. This aspect of stochastic field theory is the basis for Lévy processes [45].

B. Lévy Processes

Lévy processes are random walks whose distribution has infinite moments. The statistics of (conventional) physical systems are usually concerned with stochastic fields that have PDFs where (at least) the first two moments (the mean and variance) are well defined and finite. Lévy statistics is concerned with statistical systems where all the moments (starting with the mean) are infinite. Many distributions exist where the mean and variance are finite but are not representative of the process, e.g. the tail of the distribution is significant, where rare but extreme events occur. These distributions include Lévy distributions. Lévy's original approach to deriving such distributions is based on the following question: Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine. The characteristic function (i.e. the Fourier transform) $P(k)$ of such a distribution $p(x)$ was first shown by Lévy to be given by (for symmetric distributions only)

$$P(k) = \exp(-a \mid k \mid^\gamma), \quad 0 < \gamma \leq 2 \tag{3}$$

where a is a constant and γ is the Lévy index. For $\gamma \geq 2$, the second moment of the Lévy distribution exists and the sums of large numbers of independent trials are Gaussian distributed. For example, if the result were a random walk with a step length distribution governed by $p(x), \quad \gamma > 2$, then the result would be normal (Gaussian) diffusion, i.e. a Brownian process. For $\gamma < 2$ the second moment of this PDF (the mean square), diverges and the characteristic scale of the walk is lost. For values of γ between 0 and 2, Lévy's characteristic function corresponds to a PDF of the form

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty.$$

This type of random walk is called a Leúy flight and is an example of a non-stationary fractal walk.

Lévy process are consistent with a fractional diffusion equation [46]. The basic evolution equation for a random Brownian particle process is given by

$$u(x, t + \tau) = \int_{-\infty}^{\infty} u(x + \lambda, t) p(\lambda) d\lambda$$

where $u(x, t)$ is the concentration of particles and τ is the interval of time in which a particle moves some distance between λ and $\lambda + d\lambda$ with a probability $p(\lambda)$ satisfying the condition $p(\lambda) = p(-\lambda)$. We note that

$$u(x, t + \tau) = u(x, t) \otimes p(x)$$

and that in Fourier space, this equation is

$$U(k, t + \tau) = U(k, t) P(k)$$

where U and P are the Fourier transforms of u and p respectively. From equation (3),

$$P(k) \simeq 1 - a \mid k \mid^\gamma$$

so that we can write

$$\frac{U(k,t+\tau) - U(k,t)}{\tau} \simeq -\frac{a}{\tau} \mid k \mid^\gamma U(k,t)$$

which for $\tau \to 0$ gives the fractional diffusion equation

$$\sigma \frac{\partial}{\partial t} u(x,t) = \frac{\partial^\gamma}{\partial x^\gamma} u(x,t), \quad \gamma \in (0,2] \tag{4}$$

where $\sigma = \tau / a$ and we have used the result

$$\frac{\partial^\gamma}{\partial x^\gamma} u(x,t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mid k \mid^\gamma U(k,t) \exp(ikx) dk$$

The solution to this equation with the singular initial condition $u(x,0) = \delta(x)$ is given by

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx - t \mid k \mid^\gamma / \sigma) dk$$

which is itself Lévy distributed. This derivation of the fractional diffusion equation reveals its physical origin in terms of Lévy statistics, i.e. Lévy's characteristic function. Note that the diffusion equation is fractional in the spatial derivative rather than the temporal derivative as given in equation (1). However, since the Green's function for equation (4) is given by

$$g(\mid x \mid, \omega) = \frac{i}{2\Omega_\gamma} \exp(i\Omega_\gamma \mid x \mid)$$

where

$$\Omega_\gamma = i^{\frac{2}{\gamma}} (i\omega\sigma)^{\frac{1}{\gamma}},$$

by induction, we obtain a relationship between the Lévy index γ and the Fourier dimension q given by

$$\frac{1}{\gamma} = \frac{q}{2}$$

Gaussian processes associated with the classical diffusion equation are thus recovered when $\gamma = 2$ and $q = 1$.

C. Fractional Differentials

Fractional differentials of any order need to be considered in terms of the definition for a fractional differential given by

$$\hat{D}^q f(t) = \frac{d^m}{dt^m} [\hat{I}^{m-q} f(t)], \quad m - q > 0$$

where m is an integer and $\hat{I}$ is the fractional integral operator (the Riemann-Liouville transform) given by

$$\hat{I}^p f(t) = \frac{1}{\Gamma(p)} f(t) \otimes \frac{1}{t^{1-p}}, \quad p > 0$$

The reason for this is that direct fractional differentiation can yield divergences. However, there is a deeper interpretation of this result that has a synergy with the issue over a macroeconomic system having 'memory' and is based on observing that the evaluation of a fractional differential operator depends on the history of the function in question. Thus, unlike an integer differential operator of order m, a fractional differential operator of order q has 'memory' because the value of $\hat{I}^{m-q} f(t)$ at a time t depends on the behaviour of $f(t)$ from $-\infty$ to t via the convolution of $f(t)$ with $t^{(m-q)-1} / \Gamma(m-q)$. The convolution process is dependent on the history of a function $f(t)$ for a given kernel and thus, in this context, we can consider a fractional derivative defined by $\hat{D}^q$ to have memory. In this sense, the operator

$$\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}$$

decribes a process, compounded in a field $u(x,t)$, that has memory association with regard to the temporal characteristics of the system it is attempting to model. This is not an intrinsic charcteristic of systems that are purely diffusive $q = 1$ or propagative $q = 2$.

D. Non-stationary Model

The fractional diffusion operator used in equation (1) is appropriate for modelling fractional diffusive processes that are stationary. For non-stationary fractional diffusion, we could consider the case where the diffusivity is time variant as defined by the function $\sigma(t)$. However, a more interesting case arises when the characteristics of the diffusion processes change over time becoming less or more diffusive. This is illustrated in terms of the random walk in the plane given in Figure 3. Here, the walk starts off being fully diffusive (i.e. $H = 0.5$ and $q = 1$), changes to being fractionally diffusive ($0.5 < H < 1$ and $1 < q < 2$) and then changes back to being fully diffusive. In terms of fractional diffusion, this is equivalent to having an operator

$$\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q}$$

where $q = 1, t \in (0, T_1]$; $q > 1, t \in (T_1, T_2]$; $q = 1, t \in (T_2, T_3]$ where $T_3 > T_2 > T_1$. If we want to generalise such processes over arbitrary periods of time, then we should consider q to be a function of time. We can then introduce a non-stationary fractional diffusion operator given by

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}}.$$

This operator is the theoretical basis for the Fractal Market Hypothesis considered in this paper. In terms of using this model to develop a FMH risk management metric based on the analysis of economic time series, the principal Hypothesis is that a change in $q(t)$ precedes a change in a macroeconomic index. This requires accurately numerical methods for computing $q(t)$ for a given index which are discussed later. Real economic signals exhibit non-stationary fractal walks. An example of this is illustrated in Figure 4 which shows a non-stationary walk in the complex plane obtained by taking the Hilbert transform of an economic signal, i.e. computing the analytic signal

$$s(t) = u(t) + \frac{i}{\pi t} \otimes u(t)$$

and plotting the real and imaginary component of this signal in the complex plane.

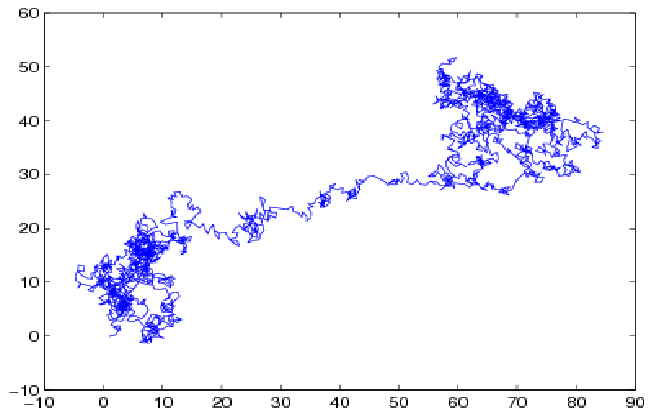

Fig. 3. Non-stationary random phase walk in the plane.

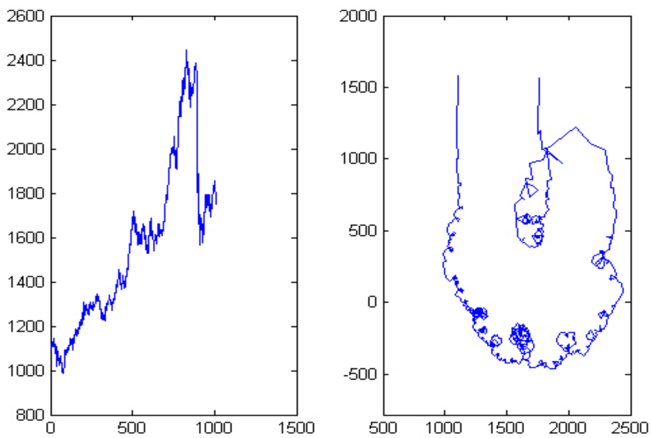

Fig. 4. Non-stationary fractal walk in the complex plane (right) obtained by computig the Hilbert transform of the economic signal (left) - FTSE Close-of-Day from 02-04-1984 to 24-12-1987.

The non-stationary model considered here exhibits behaviour that is similar to Lévy processes. However, the aim is not to derive a statistical model for a stochastic process using a stationary fractional diffusion of the type given by equation (4) but to be able to compute a function - namely $q(t)$ - which is a measure of the non-stationary behaviour especially with regard to a 'future flight'. This is because, in principle, the value of $q(t)$ should reflect the early stages of a change in the behaviour of $u(t)$, a principle that is the basis for the financial data processing and analysis discussed in the following section.

VI. Financial Data Analysis

If we consider the case where the Fourier dimension is a relatively slowly varying function of time, then we can legitimately consider $q(t)$ to be composed of a sequence of different states $q_i = q(t_i)$. This approach allows us to develop a stationary solution for a fixed q over a fixed period of time. Non-stationary behaviour can then be introduced by using the same solution for different values of q over fixed (or varying) periods of time and concatenating the solutions for all q to produce an output digital signal.

The FMH model for a quasi-stationary segment of a financial signal is given by

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t), \quad q > 0$$

which has characteristic spectrum

$$U(\omega) = \frac{N(\omega)}{(i\omega)^{q/2}}$$

The PSDF is thus characterised by $\omega^{-q}, \omega \geq 0$ and our problem is thus, to compute q from the data $P(\omega) = \mid U(\omega) \mid^2$ $, \omega \geq 0$. For this data, we consider the PSDF

$$\hat{P}(\omega) = \frac{c}{\omega^q}$$

or

$$\ln \hat{P}(\omega) = C + q \ln \omega$$

where $C = \ln c$. The problem is therefore reduced to implementing an appropriate method to compute q (and C) by finding a best fit of the line $\ln \hat{P}(\omega)$ to the data $\ln P(\omega)$. Application of the least squares method for computing q, which is based on minimizing the error

$$e(q, C) = \| \ln P(\omega) - \ln \hat{P}(\omega, q, C) \|_2^2$$

with regard to q and C, leads to errors in the estimates for q which are not compatible with market data analysis. The reason for this is that relative errors at the start and end of the data $\ln P$ may vary significantly especially because any errors inherent in the data P will be 'amplified' through application of the logarithmic transform required to linearise the problem. In general, application of a least squares approach is very sensitive to statistical heterogeneity [47] and in this application, may provide values of q that are not compatible with the rationale associated with the FMH (i.e. values of $1 < q < 2$ that are intermediate between diffusive and propagative processes). For this reason, an alternative approach must be considered which, in this paper, is based on Orthogonal Linear Regression (OLR) [48] [49].

Applying a standard moving window, $q(t)$ is computed by repeated application of OLR based on the m-code available from [51]. This provides a numerical estimate of the function $q(t)$ whose values reflect the state of a financial signals (assumed to be a non-stationary random fractal) in terms of a stable or unstable economy, from which a risk analysis can be performed. Since q is, in effect, a statistic, its computation is only as good as the quantity (and quality) of data that is available for its computation. For this reason, a relatively large window is required whose length is compatible with the number of samples available.

A. Numerical Algorithm

The principal algorithm associated with the application of the FMH analysis is as follows:

Step 1: Read data (financial time series) from file into operating array $a[i], i = 1, 2, ..., N$.

Step 2: Set length $L < N$ of moving window w to be used.

Step 3: For $j = 1$ assign $L + j - 1$ elements of $a[i]$ to array $w[i], i = 1, 2, ..., L$.

Step 4: Compute the power spectrum $P[i]$ of $w[i]$ using a Discrete Fourier Transform (DFT).

Step 5: Compute the logarithm of the spectrum excluding the DC, i.e. compute $\log(P[i])\forall i \in [2, L/2]$.

Step 6: Compute $q[j]$ using the OLR algorithm whose m-code is given in Appendix I.

Step 7: For $j = j + 1$ repeat Step 3 - Step 5 stopping when $j = N - L$.

Step 8: Write the signal $q[j]$ to file for further analysis and post processing.

The following points should be noted:

(i) The DFT is taken to generate an output in standard form where the zero frequency component of the power spectrum is taken to be $P[1]$.

(ii) With $L = 2^m$ for integer m, a Fast Fourier Transform can be used

(iii) The minimum window size that should be used in order provide statistically significant values of $q[j]$ is $L = 64$ when q can be computed accurate to 2 decimal places.

An example of the output generated by this algorithm for a 1024 element window is given in Figure 5 using Dow Jones Close-of-Day data obtained from [20]. Inspection of the signals illustrates a qualitative relationship between trends in the financial data and $q(t)$ in accordance with the theoretical model considered. In particular, over periods of time in which q increases in value, the amplitude of the financial signal $u(t)$ decreases. Moreover, and more importantly, an upward trend in q appears to be a precursor to a downward trend in $u(t)$, a correlation that is compatible with the idea that a rise in the value of q relates to the 'system' becoming more propagative, which in stock market terms, indicates the likelihood for the markets becoming 'bear' dominant in the future.

The results of using the method discussed above not only provides for a general appraisal of different macroeconomic financial time series, but, with regard to the size of selected window used, an analysis of data at any point in time. The output can be interpreted in terms of 'persistence' and 'anti-persistence' and in terms of the existence or absence of after-effects (macroeconomic memory effects). For those periods in time when $q(t)$ is relatively constant, the existing market tendencies usually remain. Changes in the existing trends tend to occur just after relatively sharp changes in $q(t)$ have developed. This behaviour indicates the possibility of using the time series $q(t)$ for identifying the behaviour of a macroeconomic financial system in terms of both inter-market and between-market analysis. These results support the possibility of using $q(t)$ as an independent volatility predictor to give a risk assessment associated with the likely future behaviour of different economic time series. Further, because this analysis is based on the equation (2) which defines a (stationary) random scaling fractal signal, the results are, in principle, scale invariant.

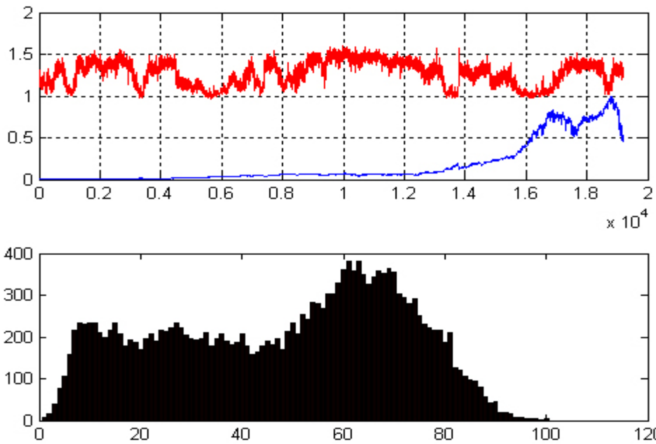

Fig. 5. Application of the FMH using a 1024 element window for analysing financial time series composed of Dow Jones Close-of-Day data from from 02-11-1932 to 25-03-2009. Above: Dow Jones Close-of-Day data (blue) and q(t) (red) computed using a window of 1024; Below: Histogram of q(t) for 100 bins.

B. Equivalence with a Wavelet Transform

The wavelet transform is defined in terms of projections of $f(t)$ onto a family of functions that are all normalized dilations and translations of a prototype 'wavelet' function w [50], i.e.

$$\mathcal{W}[f(t)] = F_L(t) = \int_{-\infty}^{\infty} f(\tau) w_L(\tau, t) d\tau$$

where

$$w_L(\tau, t) = \frac{1}{\sqrt{L}} w\left(\frac{\tau - t}{L}\right), \quad L > 0.$$

The independent variables L and t are continuous dilation and translation parameters respectively. The wavelet transformation is essentially a convolution transform where $w_L(t)$ is the convolution kernel with dilation variable L. The introduction of this factor provides dilation and translation properties into the convolution integral that gives it the ability to analyse signals in a multi-resolution role (the convolution integral is now a function of L), i.e.

$$F_L(t) = w_L(t) \otimes f(t), \quad L > 0.$$

In this sense, the asymptotic solution (ignoring scaling)

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t), \quad q > 0$$

is compatible with the case of a wavelet transform where

$$w_1(t) = \frac{1}{t^{1-q/2}}$$

for the stationary case and where, for the non-stationary case,

$$w_1(t, \tau) = \frac{1}{t^{1-q(\tau)/2}}.$$

C. Macrotrend Analysis

In order to develop a macrotrend signal that has optimal properties with regard the assessment of risk (i.e. the likely future behaviour of an economic signal), it is important that the filter used is: (i) consistent with the properties of a Variation Diminishing Smoothing Kernel (VDSK); (ii) that the last few values of the trend signal are 'data consistent'. VDSKs are convolution kernels with properties that guarantee smoothness around points of discontinuity of a given signal where the smoothed function is composed of a similar succession of concave or convex arcs equal in number to those of signal. VDSKs also have 'geometric properties' that preserve the 'shape' of the signal. There are a range of VDSKs of which the most common is a Gaussian function and, for completeness, Appendix II provides a overview of the principal analytical properties, including fundamental Theorems and Proofs of such kernels including the Gaussian kernel.

In practice, the computation of the smoothing process using a VDSK must be performed in such a way that the initial and final elements of the output data are entirely data consistent with the input array within the locality of any element. Since a VDSK is a non-localised filter which tends to zero at infinity, in order to optimise the numerical efficiency of the smoothing process, filtering is undertaken in Fourier space. However, in order to produce a data consistent macrotrend signal using a Discrete Fourier Transform, wrapping effects must be eliminated. The solution is to apply an 'end point extension' scheme which involves padding the input vector with elements equal to the first and last values of the vector. The length of the 'padding vectors' are taken to be at least half the size of the input vector. The output vector is obtained by deleting the filtered padding vectors.

Figures 6 and 7 show examples of macrotrend analysis applied to the economic time series obtained from [19] and [20] and the signal $q(t)$ using the VDSK filter $\exp(-\beta\omega^2)$. Table 1 provides quantitative information of the statistics of the signal $q(t)$. Figures 6 and 7 include the normalised gradients computed using a 'forward differencing scheme' which clearly illustrate 'phase shifts' associated with the two signals. From Table 1, the mean value of $q(t)$ for the Dow Jones index is slightly lower than the mean for the FTSE and in both cases, the Null Hypothesis test as to whether $q(t)$ is Gaussian distributed is negative, i.e. the 'Composite Normality' is of type 'Reject'.

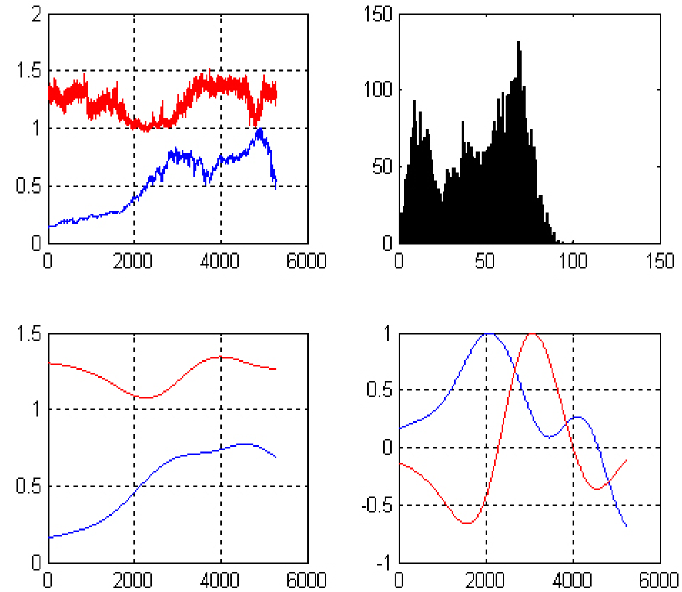

Fig. 6. Analysis of FTSE Close-of-Day data from 25-04-1988 to 20-03-2009. Top-left:FTSE data (blue) and $q(t)$ (red) computed using a 1024 moving window; Top-right: 100 bin histogram; Bottom-left: Macrotrends ($\beta = 0.1$); Bottom-right: Normalised gradients of macrotrends.

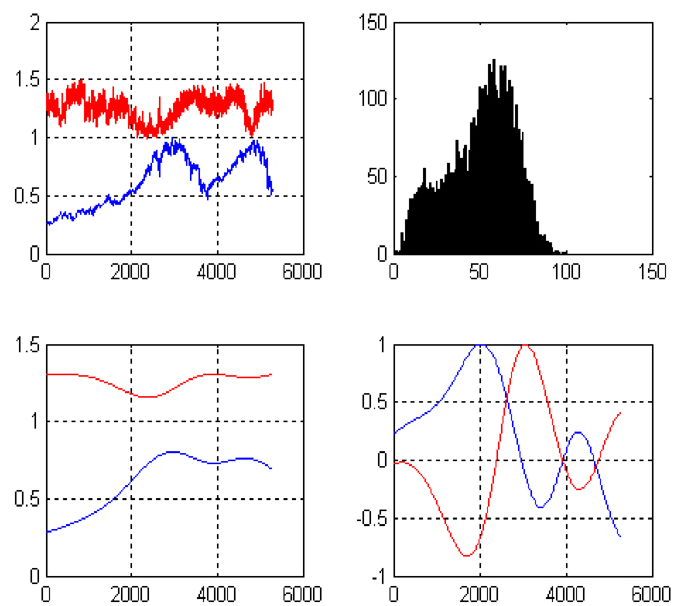

Fig. 7. Analysis of DJ Close-of-Day data from 25-04-1988 to 20-03-2009. Top-left: FTSE data (blue) and $q(t)$ (red) computed using a window of 1024; Top-right: 100 bin histogram; Bottom-left: Macrotrends ($\beta = 0.1$); Bottom-right: Normalised gradients of macrotrends.

VII. Case Study: Analysis of ABX Indices

ABX indices serve as a benchmark of the market for securities backed by home loans issued to borrowers with weak credit. The index is administered by the London-based Markit Group which specialises in credit derivative pricing [52].

A. What is an ABX index?

The index is based on a basket of Credit Default Swap (CDS) contracts for the sub-prime housing equity sector. Credit Default Swaps operarte as a type of insurance policy for banks or other holders of bad mortgages. If the mortgage goes bad, then the seller of the CDS must pay the bank for the

Statistical Parameter	$q(t)$-FTSE	$q(t)$-DJ
Minimum Value	0.9876	0.9752
Maximum value	1.5067	1.5154
Range	0.5190	0.5402
Mean	1.2482	1.2218
Median	1.2639	1.2452
Standard Deviation	0.1017	0.1269
Variance	0.0104	0.0161
Skew	-0.4080	-0.2881
Kertosis	2.3745	1.8233
Composite NormalityN	Reject	Reject

TABLE I
STATISTICAL VALUES ASSOCIATED WITH $q(t)$ COMPUTED FOR FTSE AND DJ CLOSE-OF-DAY DATA FROM 25-04-1988 TO 20-03-2009 GIVEN IN FIGURES 6 AND 7 RESPECTIVELY.

lost mortgage payments. Alternatively, if the mortgage stays good then the seller makes a lot of money. The riskier the bundle of mortgages the lower the rating.

The original goal of the index was to create visibility and transparency but it was not clear at the time of its inception that the index would be so closely followed. As subprime securities have become increasingly uncertain, the ABX index has become a key point of reference for investors navigating risky mortgage debt on an international basis. Hence, in light of the current financial crisis (i.e. from 2008-date), and given that most economist agree that the subprime mortgage was a primary catalyst for the crisis, analysis of the ABX index has become a key point of reference for investors navigating the world of risky mortgage debt.

On asset-backed securities such as home equity loans the CDS provides an insurance against the default of a specific security. The index enables users to trade in a security without being limited to the physical outstanding amount of that security thereby given investors liquid access to the most frequently traded home equity tranches in a basket form. The ABX uses five indices that range from triple-A to triple-B minus. Each index chooses deals from 20 of the largest sub-prime home equity shelves by issuance amount from the previous six months. The minimum deal size is $500 million and each tranche referenced must have an average life of between four and six years, except for the triple-A tranche, which must have a weighted average life greater than five years. Each of the indices is referenced to by different rated tranches, i.e. AAA, AA, A, BBB and BBB-. They are selected through identification of the most recently issued deals that meet the specific size and diversity criteria. The principal 'market-makers' in the index were/are: Bank of America, Bear Stearns, Citigroup, Credit Suisse, Deutsche Bank, Goldman Sachs, J P Morgan, Lehman Brothers, Merrill Lynch (now Bank of America), Morgan Stanley, Nomura International, RBS Greenwich Capital, UBS and Wachovia. However, during the financial crisis that developed in 2008, a number of changes have taken place. For example, on September 15, 2008, Lehman Brothers filed for bankruptcy protection following a massive exodus of most of its clients, drastic losses in its stock, and devaluation of its assets by credit rating agencies and in 2008 Merrill Lynch was acquired by Bank of America at which point Bank of America merged its global banking and wealth management division with the newly acquired firm. The Bear Stearns Companies, Inc. was a global investment bank and securities trading and brokerage, until its collapse and fire sale to J P Morgan Chase in 2008.

ABX contracts are commonly used by investors to speculate on or to hedge against the risk that the underling mortgage securities are not repaid as expected. The ABX swaps offer protection if the securities are not repaid as expected, in return for regular insurance-like premiums. A decline in the ABX index signifies investor sentiment that subprime mortgage holders will suffer increased financial losses from those investments. Likewise, an increase in the ABX index signifies investor sentiment looking for subprime mortgage holdings to perform better as investments.

B. *ABX and the Sub-prime Market*

Prime loans are often packaged into securities and sold to investors to help lenders reduce risk. More than $500B of such securities were issued in the US in 2006. The problem for investors who bought 2006's crop of high-risk mortgage originations, was that as the US housing market slowed as did mortgage applications. To prop up the market, mortgage lenders relaxed their underwriting standards lending to ever-riskier borrowers at ever more favourable terms.

In the last few weeks of 2006, the poor credit quality of the 2006 vintage subprime mortgage origination started to become apparent. Delinquencies and foreclosures among high-risk borrowers increased at a dramatic rate, weakening the performance of the mortgage pools. In one security backed by subprime mortgages issued in March 2006, foreclosure rates were already 6.09% by December that year, while 5.52% of borrowers were late on their payments by more than 30 days. Lenders also began shutting their doors, sending shock waves through the high-risk mortgage markets throughout 2007. The problem kept new investor money at bay, and dramatically weakened a key derivative index tied to the performance of 2006 high-risk mortgages, i.e. the ABX index. As a result the ABX suffered a major plummet of the index starting in December 2006 when BBB- fell below 100 for the first time. The most heavily traded subindex, representing loans rated BBB-, fell as hedge funds flocked to bet on the downturn and pushed up the cost of insuring against default. This led to a knock-on effect as lenders withdrew from the ABX market

In early 2007 the issues were seen as: (i) Which investors were bearing the losses from having bought sub-prime mortgage backed securities? (ii) How large and concentrated were these losses? (iii) Had this sub-prime securitization distributed their risk among many players in the financial system or were the positions and losses concentrated among a few players? (iv) What were the potential systemic risk effects of these losses? We now know that the systemic risk had a devastating affect on the global economy and became known as the 'Credit Crunch'. One of the catalysts for the problem was a US bill allowing bankruptcy judges to alter loan balances which nobody dealing in CDS had considered. The second key factor was the speed of deterioration of the ABX Indices in 2007 which shocked investors and left them waiting to see the bottom of the market before getting back in - they are still waiting. The third key factor was the failure of the US Treasury to provide foreclosure relief for distressed home owners which congress had approved. The following series of reactions (denoted by $\rightarrow$) were triggered as a result: The treasury said it won't take steps to prevent home foreclosures, so that prices of mortgage securities collapsed $\rightarrow$ bank equity was wiped out $\rightarrow$ banks, with shrunken equity capital, were forced to cut back on all types of credit $\rightarrow$ financing for anything, especially residential mortgage loans, dried up $\rightarrow$ market values of homes declined further $\rightarrow$ mortgage securities declined further, and the downward spiral becomes self perpetuating.

C. *Effect of ABX on Bank Equities*

At the end of February 2007 a price of 92.5 meant that a protection buyer will need to pay the protection seller 7.5% upfront and then 0.64% per year. At the time, this kind of mortgage yield was about 6.5%, so the upfront charge was more than the yield per year. By April 2009 the A grade index had fallen to 8 meaning that the protection seller would want 92% upfront which meant that the sub-prime market 'died'. In July 2007 AAA mortgage securities started trading at prices materially below par, or below 100. Until then, many banks had bulked up mortgage securities that were rated AAA at the time of issue. This was because they believed that AAA bonds could always be traded at prices close to par, and consequently the bonds' value would have a very small impact on the earnings and equity capital. The mystique about AAA ratings dated back more than 80 years. From 1920 onward, the default experience on AAA rated bonds, even during the Great Depression, was nominal.

The way the securities are structured is that different classes of creditors, or different tranches, all hold ownership interests in the same pool of mortgages. However, the tranches with the lower ratings - BBB, A, AA - take the first credit losses and they are supposed to be destroyed before the AAA bondholders lose anything. Typically, AAA bondholders represent about 75-80% of the entire mortgage pool. During the Great Depression (1929-1933), national average home prices held their value far better than they have since 2007. The assumptions that a highly liquid trading market and gradual price declines, have proved to be wrong. Beginning in the last half of 2007, the price declines of AAA bonds was steep, and the trading market suddenly became very illiquid. Under standard accounting rules, those securities must be marked to market every fiscal quarter, and the banks' equity capital shrank beyond all expectations. Hundreds of billions of dollars have been lost as a result. However, the losses in mortgage securities, and from financial institutions such Lehman that were undone by mortgage securities, dwarf everything else. Before the end of each fiscal quarter, bank managements must also budget for losses associated with mortgage securities. But since they cannot control market prices at a future date, they compensate by adjusting what they can control, which is all discretionary extensions of credit. Banks cannot legally lend beyond a certain multiple of their capital.

D. *Credit Default Swap Index*

This index is used to hedge credit risk or to take a position on a basket of credit entities. Unlike a credit default swap, a credit default swap index is a completely standardised credit security and may therefore be more liquid and trade at a smaller bid-offer spread. This means that it can be cheaper to hedge a portfolio of credit default swaps or bonds with a CDS index than to buy many CDS to achieve a similar effect. Credit-default swap indexes are benchmarks for protecting investors owning bonds against default, and traders use them to speculate on changes in credit quality. There are currently two main families of CDS indices: CDX and iTraxx. CDX indices contain North American and Emerging Market companies and are administered by CDS Index Company and marketed by Markit Group Limited, and iTraxx contain companies from the rest of the world and are managed by the International Index Company (IIC). A new series of CDS indices is issued every six months by Markit Group and IIC. Running up to the announcement of each series, a group of investment banks is polled to determine the credit entities that will form the constituents of the new issue. This process is intended to ensure that the index does not become 'cluttered with instruments that no longer exist, or which trade illiquidly. On the day of issue a fixed coupon is decided for the whole index based on the credit spread of the entities in the index. Once this has been decided the index constituents and the fixed coupon is published and the indices can be actively traded.

E. *Analysis of Sub-Prime CDS Market ABX Indices using the FMH*

The US Sub-Prime Housing Market is widely viewed as the source of the current economic crisis. The reason that it has had such a devastating effect on the global economy is that investment grade bonds were purchased by many substantial international financial institutions but in reality the method used to designate the relatively low risk required for investment grade securities was seriously flawed. This resulted in the investment grade bonds becoming virtually worthless very quickly when systemic risks that wrongly had been ignored undermined the entire market. About 80% of the market was designated investment grade (AAA - highest, AA and A - lowest) with protection provided by a high risk grades (BBB- and BBB). The flawed risk model was based on an assumption that the investment grades would always be protected by the higher risk grades that would take all of the first 20% of defaults. Once defaults exceeded 20% the 'house of cards' was demolished. It is therefore of interest to see if a FMH based analysis of the ABX indices could have been used a predictive tool in order to develop a superior risk model.

Figure 8 shows the ABX index for each grade using data supplied by the Systemic Risk Assessment Division of the Bank of England. During the second week of December 2006 the BBB- index slipped to 99.76 for a couple of days but then recovered. In March 2007 the index for BBB- slipped just below 90 and seemed to be recovering and by mid-May was above 90 again. In June 2007 the BBB- really began to slide and this time it never recovered and was closely followed by

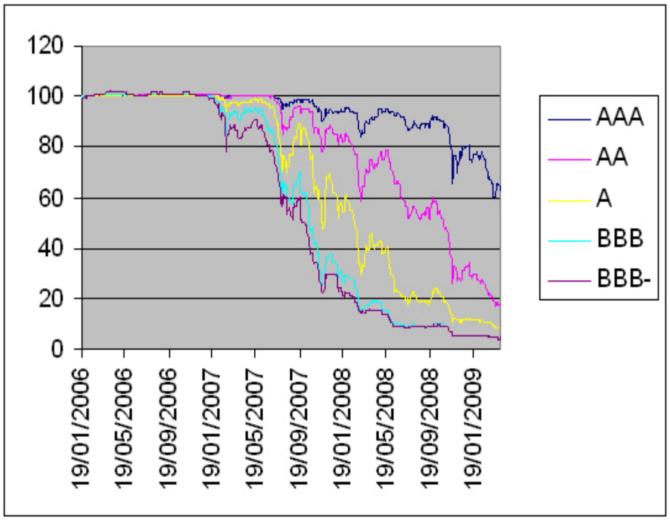

Fig. 8. Grades for the ABX Indices from 19 January 2006 to 2 April 2009 based on Close-of-Day prices.

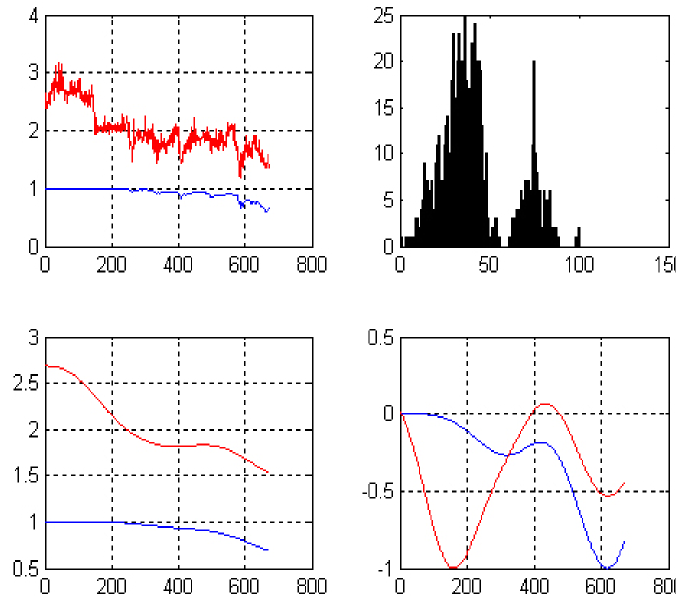

Fig. 9. Analysis of AAA ABX.HE indices (2006 H1 vintage) by rating (closing prices) from 24-07-2006 to 02-04-2009. Top-left: AAA data (blue) and $q(t)$ (red); Top-right: 100 bin histogram; Bottom-left: Macotrends for $\beta = 0.1$; Bottom-right: Normalised gradients of macrotrends.

the collapse of the BBB index after which there was no further protection for the investment grades. The default swaps work like an insurance so that if the cost of insuring against risk becomes greater than the annual return from the loan then the market is effectively dead. By February 2008 the AAA grade was below this viable level.

The results of applying the FMH based on the algorithms discussed in Section 6 is given in Figures 9-13. Table 2 provides a list of the statistical variables associated with $q(t)$ for each case. In each case, $q(t)$ initially has values > 2 but this falls rapidly prior to a change of the index. Also, in each case, the turning point of the normalised gradient of the Gaussian filtered signal (i.e. point in time of the minimum value) is an accurate reflection of the point in time prior to when the index falls rapidly relatively to the prior data. This turning point occurs before the equivalent characteristic associated with the smoothed index. The model consistently 'signals' the coming meltdown with sufficient notice for orderly withdrawal from the market. For example, the data used for Figure 9 reflects the highest Investment Grade and would be regarded as particularly safe. The normalised gradient of the output data provides a very early signal of a change in trend, in this case, at around approximately 180 days from the start of the run, which is equivalent to early April 2007 at which point the index was just above 100. In fact the AAA index appears to be viable as an investment right up to early November 2008 after which is falls dramatically. In Figure 11, a trend change is again observed in the normalised gradient at approximately 190 days which is equivalent mid April 2007. It is not until the second week of July 2007 that this index begins to fall rapidly. In Figure 13 the normalised gradient signals a trend change at around 170 for the highest risk grade. This is equivalent to the third week of March 2007. At this stage the index was only just below 90 and appeared to be recovering.

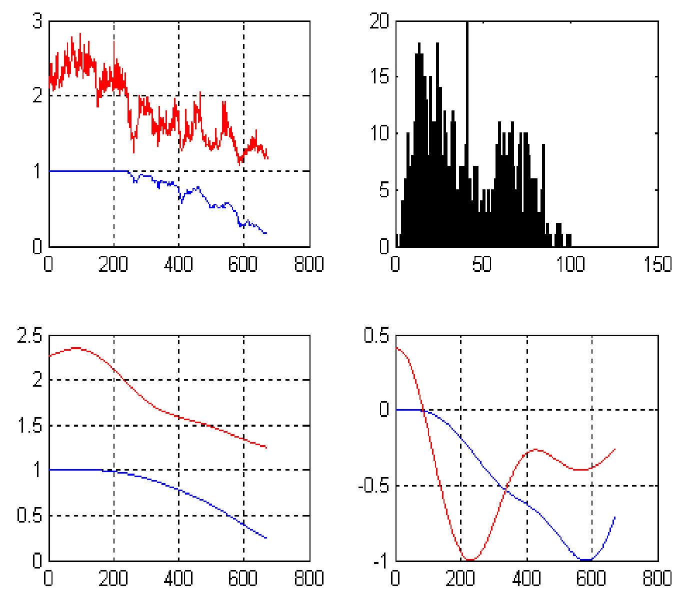

Fig. 10. Analysis of AA ABX.HE indices (2006 H1 vintage) by rating (closing prices) from 24-07-2006 to 02-04-2009 for a 128 moving window. Top-left: AA data (blue) and $q(t)$ (red); Top-right: 100 bin histogram; Bottom-left: Macotrends for $\beta = 0.1$; Bottom-right: Normalised gradients of macrotrends.

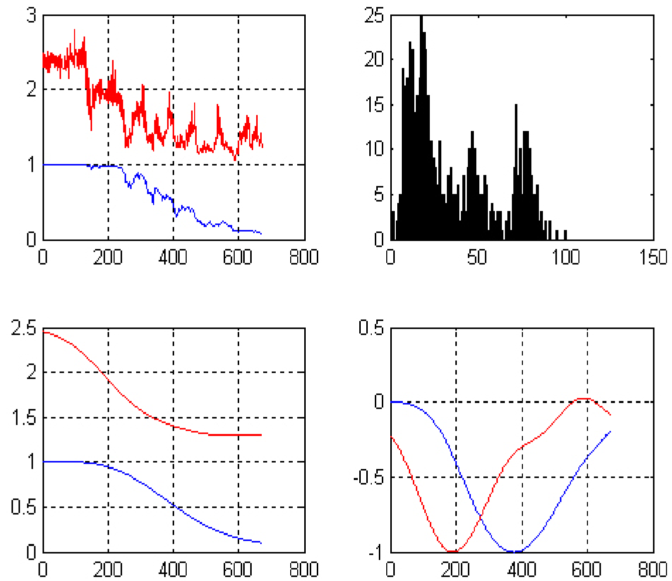

Fig. 11. Analysis of A ABX.HE indices (2006 H1 vintage) by rating (closing prices) from 24-07-2006 to 02-04-2009 for a 128 size moving window. Top-left: AA data (blue) and $q(t)$ (red); Top-right: 100 bin histogram; Bottom-left: Macotrends for $\beta = 0.1$; Bottom-right: Normalised gradients of macrotrends.

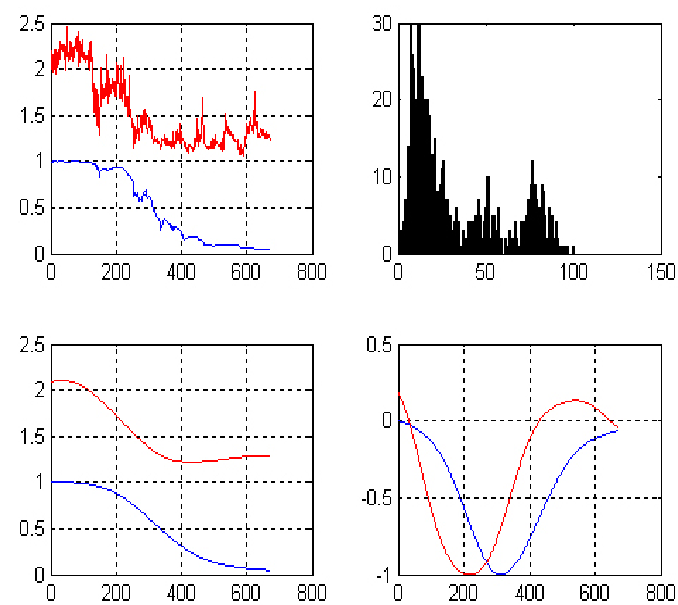

Fig. 12. Analysis of BBB ABX.HE indices (2006 H1 vintage) by rating (closing prices) from 24-07-2006 to 02-04-2009 for a moving window with 128 elements. Top-left: AA data (blue) and $q(t)$ (red); Top-right: 100 bin histogram; Bottom-left: Macotrends for $\beta = 0.1$; Bottom-right: Normalised gradients of macrotrends.

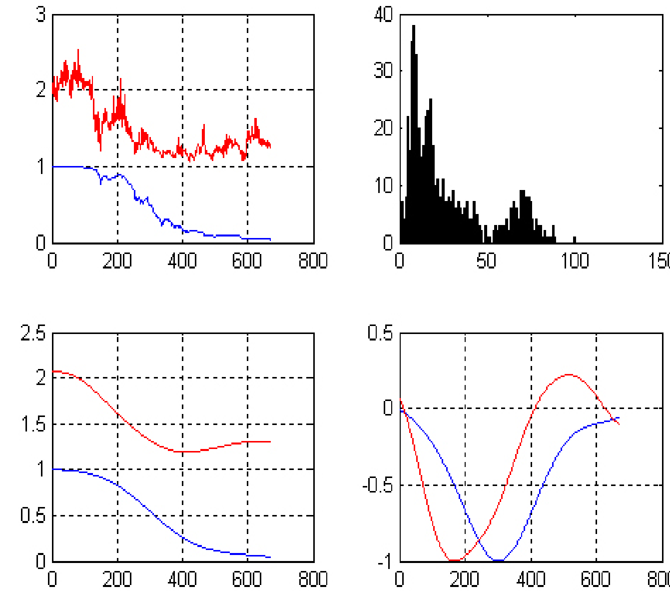

Fig. 13. Analysis of BBB- ABX.HE indices (2006 H1 vintage) by rating (closing prices) from 24-07-2006 to 02-04-2009 for a moving window of size 128 element. Top-left:AA data (blue) and $q(t)$ (red); Top-right: 100 bin histogram; Bottom-left: Macotrends for $\beta = 0.1$; Bottom-right: Normalised gradients of macrotrends.

Statistical Parameter	AAA	AA	A	BBB	BBB-
Min.	1.1834	1.0752	1.0522	1.0610	1.0646
Max.	3.1637	2.8250	2.7941	2.4476	2.5371
Range	1.9803	1.7499	1.7420	1.3867	1.4726
Mean	2.0113	1.7869	1.6663	1.5141	1.4722
Median	1.9254	1.7001	1.4923	1.3425	1.3243
SD	0.3928	0.4244	0.4384	0.3746	0.3476
Variance	0.1543	0.1801	0.1922	0.1404	0.1208
Skew	0.7173	0.3397	0.6614	0.8359	1.0345
Kertosis	2.7117	1.8479	2.0809	2.2480	2.7467
CN	Reject	Reject	Reject	Reject	Reject

TABLE II

STATISTICAL VALUES ASSOCIATED WITH $q(t)$ COMPUTED FOR ABX.HE INDICES (2006 H1 VINTAGE) BY RATING (CLOSING PRICES) FROM 24-07-2006 TO 02-04-2009. NOTE THAT THE ACRONYMS SD AND CN STAND FOR 'STANDARD DEVIATION' AND 'COMPOSITE NORMALITY' REPECTIVELY.

VIII. CONCLUSION

In terms of the non-stationary fractional diffusion model considered in this paper, the time varying Fourier dimension $q(t)$ can be interpreted in terms of a 'gauge' on the characteristics of a dynamical system. This includes the management processes from which all modern economies may be assumed to be derived. In this sense, the FMH is based on three principal considerations: (i) the non-stationary behaviour associated with any system undergoing continuous change that is driven by a management infrastructure; (ii) the cause and effect that is inherent at all scales (i.e. all levels of management hierarchy); (iii) the self-affine nature of outcomes relating to points (i) and (ii).

In a modern economy, the principal issue associated with any form of financial management is based on the flow of information and the assessment of this information at different points connecting a large network. In this sense, a macroeconomy can be assessed in terms of its information network which consists of a distribution of nodes from which information can flow in and out. The 'efficiency' of the system is determined by the level of randomness associated with the direction of flow of information to and from each node. The nodes of the system are taken to be individuals or small groups of individuals whose assessment of the information they acquire together with their remit, responsibilities and initiative, determines the direction of the information flow from one node to the next. The determination of the efficiency of a system in terms of randomness is the most critical in terms of the model developed. It suggests that the performance of a business is related to how well information flows through an organisation.

The FMH has a number of fundamental differences with regard to the EMH which are tabulated in Table 3.

EMH	FMH
Gaussian Statistics	Non-Gaussian Statistics
Stationary Process	Non-stationary Process
No memory - no historical correlations	Memory - historical correlations
No repeating patterns at any scale	Many repeating patterns at all scales - 'Elliot waves'
Continuously stable at all scales	Continuously unstable at any scale - 'Lévy Flights'

TABLE III
PRINCIPAL DIFFERENCES BETWEEN THE EFFICIENT MARKET HYPOTHESIS (EMH) AND THE FRACTAL MARKET HYPOTHESIS (FMH).

The non-stationary nature of the model presented in this paper is taken to account for stochastic processes that can vary in time and are intermediate between diffusive and propagative or persistent behaviour. Application of Orthogonal Linear Regression to macroeconomic time series data provides an accurate and robust method to compute $q(t)$ when compared to other statistical estimation techniques such as the least squares method. As a result of the physical interpretation associated with the fractional diffusion equation and the 'meaning' of $q(t)$, we can, in principal, use the signal $q(t)$ as a predictive measure in the sense that as the value of $q(t)$ continues to increases, there is a greater likelihood for volatile behaviour of the markets. This is reflected in the data analysis based on the examples given in which a Gaussian lowpass filter $\exp(-\beta\omega^2)$ has been used to smooth both $u(t)$ and $q(t)$ to produce the associated macrotrends in which the value of β deterimes the level of detail they contain. From the examples provided, it is clear that the turning points of the gradients of a macrotend in $q(t)$ flag a future change in the trend of the economic signal $u(t)$. This is compounded in the phase shifts that exist in the normalised gradients of $u(t)$ and $q(t)$ over frequency bands determined by the value of β. Although the interpretation of these phase shifts requires further study, from the results presented in this paper, it is clear that they provide an assessment of the risk associated with investing in a particular economic time series provided the series in question is a random scaling fractal. The 'case study' on the ABX Close-of-Day indices clearly illustrates the ability for the model to flag a point in time after which the indices change rapidly. The ABX indices exhibit a clear transition between a period when $q(t) > 2$ and when $1 < q(t) < 2$ - Figures 9-13 - which preceeds the 'collapse' of the indices in 2008 are thereby the onset of the 'Credit Crunch'

In a statistical sense, $q(t)$ is just another measure that may, or otherwise, be of value to market traders. In comparison with other statistical measures, this can only be assessed through its practical application in a live trading environment. However, in terms of its relationship to a stochastic model for macroeconomic data, $q(t)$ does provide a measure that is consistent with the physical principles associated with a random walk that includes a directional bias, i.e. fractional Brownian motion. The model considered, and the signal processing algorithm proposed, has a close association with re-scaled range analysis for computing the Hurst exponent H [35]. In this sense, the principal contribution of this paper has been to consider a model that is quantified in terms of a physically significant (but phenomenological) model that is compounded in a specific (fractional) partial differential equation. As with other financial time series, their derivatives, transforms etc., a range of statistical measures can be used to characterise $q(t)$ examples of which have been provided in this paper. It should be noted that in all cases studied to date, the composite normality of the signal $q(t)$ is of type 'Reject'. In other words, the statistics of $q(t)$ are non-Gaussian. Further, assuming that a financial time series is statistically self-affine, the computation of $q(t)$ can be applied over any time scale provided there is sufficient data for the computation of $q(t)$ to be statistically significant. Thus, the results associated with the Close-of-Day data studied in this paper are, in principle, applicable to economic time series associated with tick data over a range of time scales.

APPENDIX I
M-CODE FOR THE ORTHOGONAL LINEAR REGRESSION ALGORITHM

The following m-code is used to compute the Fourier dimension q from the power spectrum of a random fractal signal and is based on the code given in [51].

```
function x=linortfit(xdata,ydata)
% Input arrays are
%
%xdata: 2,3,...,L/2
%ydata: P[2], P[3], ..., P(L/2)
%
% Output value is x which gives the Fourier
% dimension q for input data P[i].
```

```
%
fun=inline('sum((p(1)+p(2)*xdata-ydata...
...).^2)/(1+p(2)^2)','p','xdata','ydata');
x0=flipdim(polyfit(xdata,ydata,1),2);
options=optimset('TolX',1e-6,...
...'TolFun',1e-6);
x=fminsearch(fun,x0,options,xdata,ydata);
```

APPENDIX II
VARIATION DIMINISHING SMOOTHING KERNELS

Variation Diminishing Smoothing Kernels (VDSK) are convolution kernels with properties that guarantee smoothness and thereby, eliminate Gibbs' effect around points of discontinuity of a given function. Further the smoothed function can be shown to be made up of a similar succession of concave or convex arcs equal in number to those of the function. Thus, we consider the following question: let there be given a continuous or discontinuous function f whose graph is composed of a succession of alternating concave or convex arcs. Is there a smoothing kernel (or a set of them) which produces a smoothed function whose graph is also made up of a similar succession of concave or convex arcs equal in number to those of f?.

B.1 Laguerre-Pôlya Class Entire Functions

The class of kernels which relate to this question are a class of entire functions which shall be called class E originally studied earlier by E Laguerre and G Pôlya. An entire function $E(z), z\in\mathbf{C}$ belongs to the class E

$\Longleftrightarrow$

$$E(z)=\exp(bz-cz^2)\prod_{\ell=1}^{\infty}\Big(1-\frac{z}{a(\ell)}\Big)\exp[z/a(\ell)], \quad (B.1.1)$$

where $b,c,a(\ell)\in\mathbf{R}$, $c\geq 0$, and

$$\sum_{\ell=1}^{\infty}a^{-2}(\ell)<\infty. \quad (B.1.2)$$

where $\Longleftrightarrow$ is taken to denote 'if and only if' - *iff*. The convergence of the series $(B.1.2)$ guarantees that the product in $(B.1.1)$ converges and represents an entire function. Laguerre proved, and Pôlya added a refinement, that a sequence of polynomials, having real roots only, which converge uniformly in every compact set of the complex plane $\mathbf{C}$, approaches a function of class E in the uniform limit of such a sequence. For example,

$$\exp(-z^2)=\lim_{\ell\to\infty}\Big(1-\frac{z^2}{\ell^2}\Big)^{\ell^2},$$

and the polynomials $(1-z^2/\ell^2)$ have real roots only. In this definition, it is not assumed that the $a(\ell)$ are distinct. To include the case in which the product has a finite number of factors or reduces to 1 without additional notation, it is assumed that certain points on all the $a(\ell)$ may be ∞. Furthermore, it is assumed, without loss of generality, that the roots $a(\ell)$ are arranged in an order of increasing absolute values,

$$0<|a(1)|\leq|a(2)|\leq|a(3)|\leq\ldots$$

Examples of functions belonging to class E are

$$1,\ 1-z,\ \exp(z),\ \exp(z^2),\ \cos z$$

$$\frac{\sin z}{z},\ \Gamma^{-1}(1-z),\ \Gamma^{-1}(z)$$

Note that the product of two functions of this class produce a new function of the same class.

B.2 Variation Diminishing Smoothing Kernels (VDSKs)

A function k is variation diminishing iff it is of the form

$$k(x)=(2\pi i)^{-1}\int\limits_{-i\infty}^{i\infty}[E(z)]^{-1}\exp(zx)\,dz, \quad (B.2.1)$$

where $E(z)\in E$ is given by

$$E(z)=\exp(bz-cz^2)\prod_{\ell=1}^{\infty}\Big(1-\frac{z}{a(\ell)}\Big)\exp[z/a(\ell)], \quad (B.2.2)$$

with $b,c,a(\ell)\in\mathbf{R}$, $c\geq 0$, and

$$\sum_{\ell=1}^{\infty}a^{-2}(\ell)<\infty$$

In other words, a frequency function k is variation diminishing iff its bilateral Laplace transform equals $[E(z)]^{-1}$:

$$[E(z)]^{-1}=\int\limits_{-\infty}^{\infty}k(x)\exp(-zx)\,dx. \quad (B.2.3)$$

In order to define a smoothing kernel, the function k given in $(B.2.1)$ must be an even function. For, if $k(x)$ is even, then the corresponding bilateral Laplace transform $[E(z)]^{-1}$ is also even. This fact follows readily from

$$[E(z)]^{-1}$$

$$=\int\limits_{-\infty}^{\infty}k(x)\exp(-zx)\,dx=\int\limits_{-\infty}^{\infty}k(-x)\exp(-zx)\,dx$$

$$=\int\limits_{-\infty}^{\infty}k(x)\exp(zx)\,dx=[E(-z)]^{-1}$$

Conversely, if $[E(z)]^{-1}$ is even, then its inverse bilateral transform is even since a component of convergence of $(B.2.3)$ contains the imaginary axis. This follows from the fact that the component of convergence of each one of the functions which compose $E(z)$ contains completely the imaginary axis. Further, it follows that

$$[E(iu)]^{-1}=K(u), \quad (B.2.4)$$

where $K(u)$ is the FT of k. From the evenness of $[E(z)]^{(-1)}$ it follows that $K(u)$ is real, hence k is even. But $E(z)$ is even iff $b = 0$ and $a(2\ell-1) = -a(2\ell)$, $\ell = 1, 2, \ldots$. Therefore $E(z)$ is taken to be

$$E(z) = \exp(-cz^2)\prod_{\ell=1}^{\infty}\Big(1 - \frac{z^2}{a^2(\ell)}\Big), \qquad (B.2.5)$$

with $c, a(\ell) \in \mathbf{R}$, $c \geq 0$, and

$$\sum_{\ell=1}^{\infty} a^{-2}(\ell) < \infty.$$

Equation $(B.2.4)$ establishes the relationship between the bilateral Laplace transform and the Fourier transform of k. Thus, any analysis associated with use of the bilateral Laplace transform can be undertaken in terms of the Fourier transform.

Using equation $(B.2.4)$ the Fourier transform of $(B.2.1)$ is given by

$$k(x) \leftrightarrow K(u) = [E(iu)]^{-1} = \exp(-cu^2)\prod_{\ell=1}^{\infty}\Big(\frac{a^2(\ell)}{a^2(\ell)+u^2}\Big), \qquad (B.2.6)$$

where $\leftrightarrow$ denotes transformation from real to Fourier space, $c, a(\ell) \in \mathbf{R}$, $c \geq 0$, and $\sum_{\ell=1}^{\infty} a^{-2}(\ell) < \infty$.

Because equation $(B.2.6)$ is a variation diminishing function by construction and $|K(0)| \leq 1$, then the following result holds.

Theorem B.2.1 (VDSKs)

k defined as in equation $(B.2.6)$

$\Longrightarrow$

1. k is a smoothing kernel belonging to SK_1,

2. k is variation diminishing,

3. $k(x) \geq 0$, $x \in \mathbf{R}$.

In order to make a complete study of the VDSKs, such kernels will be divided in three classes: *The Finite VDSKs*, *The Non-Finite VDSKs*, and *The Gaussian VDSK*.

B.3 The Finite VDSKs

The finite and the non-finite VDSKs are kernels which can be synthesized from the following basic function:

$$\mathrm{e}(x) = \frac{1}{2}\exp(-|x|), \ x \in \mathbf{R}. \qquad (B.3.1)$$

The finite VDSKs are made up by a finite number of convolutions of functions $a(\ell)\,\mathrm{e}[a(\ell)x]$, $\ell = 1, 2, \ldots$. Clearly $\mathrm{e}(x)$ is a VDSK with mean $\nu = 0$ and variance $\sigma^2 = 2$ and its Fourier transform is given by

$$\mathrm{e}(x) \leftrightarrow \frac{1}{1+u^2}. \qquad (B.3.2)$$

Note that if $a > 0$, then $a\,\mathrm{e}(ax)$ is again a VDSK. Using the similarity property of the Fourier transform and equation $(B.3.2)$, its Fourier transform is given by

$$a\,\mathrm{e}(ax) \leftrightarrow \frac{a^2}{a^2+u^2}. \qquad (B.3.3)$$

Its mean ν again vanishes and its variance takes the value $\sigma^2 = 2/a^2$.

Let $a(1), a(2), \ldots, a(n) > 0$ be constants, some or all of which may be coincident. The following VDSKs are introduced

$$k_\ell(x) = a(\ell)\,\mathrm{e}[a(\ell)x], \ \ell = 1, 2, \ldots, n. \qquad (B.3.4)$$

The combination of these functions by convolution gives a new VDSKs with properties quantified in the following theorem.

Theorem B.3.1 (Properties of The Finite VDSKs)

1. $a(\ell) > 0$, $\ell = 1, 2, \ldots$,

2. $k_\ell(x) = a(\ell)\,\mathrm{e}[a(\ell)x]$,

3. $k = k_1 \otimes k_2 \otimes \cdots \otimes k_n$,

4. $K(u) = \prod_{\ell=1}^{n}\big(a^2(\ell)/(a^2(\ell)+u^2)\big)$

$\Longrightarrow$

A. k is a VDSK,

B. $k(x) \leftrightarrow K(u)$,

C. k has mean $\nu = 0$,

D. k has variance $\sigma^2 = \sum_{\ell=1}^{n}\big(2/a^2(\ell)\big) < \infty$.

Proof. A. The assertion follows from mathematical induction.

B. It follows from Convolution Theorem and mathematical induction.

C. Let $k_\ell(x) \leftrightarrow K_\ell(u)$. Then because each k_ℓ is a VDSK, it follows that the respective mean, ν_ℓ, is given by

$$\nu_\ell = iK_\ell'(0) = 0, \ \ell = 1, 2, \ldots, n.$$

Moreover, if $n = 2$, then the mean ν of k is given by

$$\nu = iK'(0) = i(K_1K_2)'(0) = i(K_1{K_2}'+{K_1}'K_2)(0) = i(0) = 0.$$

The assertion follows from this result and mathematical induction.

D. Let $k_\ell(x) \leftrightarrow K_\ell(u)$. Then because k_ℓ is a VDSK, it follows that the respective variance, σ_ℓ^2, is given by

$$\sigma_\ell^2 = -K''(0) = \frac{2}{a^2\ell}, \ \ell = 1, 2, \ldots, n.$$

Furthermore, from the result given in C above, if $n = 2$, then the mean σ^2 of k is given by

$$\sigma^2 = -K''(0) = -(K_1K_2)''(0)$$

$$= (-K_1{K_2}'' - 2{K_1}'{K_2}' - {K_1}''K_2)(0) = \frac{2}{a^2(1)} + \frac{2}{a^2(2)}.$$

The assertion follows from this result and mathematical induction. From the explicit expression of $K(u)$ given in Theorem B.3.1. it follows that

$$K(u) = \prod_{\ell=1}^{n}\left(\frac{a^2(\ell)}{a^2(\ell)+u^2}\right)$$

$$= \prod_{\ell=1}^{n}\left(\frac{a(\ell)}{a(\ell)-iu}\right)\left(\frac{-a(\ell)}{-a(\ell)-iu}\right)$$

$$= \prod_{\ell=1}^{n}\left(\frac{a(\ell)}{a(\ell)-iu}\right)\prod_{\ell=1}^{n}\left(\frac{-a(\ell)}{-a(\ell)-iu}\right)$$

$$= \prod_{\ell=1}^{2n}\left(\frac{d(\ell)}{d(\ell)-iu}\right)$$

where $d(\ell) = a(\ell)$ for $\ell = 1, 2, \ldots, n$ and $d(\ell) = -a(\ell)$ for $\ell = n+1, n+2, \ldots, 2n$. Thus k is of degree $2n$ and the following theorem holds.

Theorem B.3.2 (Degree of Differentiability of The Finite VDSKs)

k a finite VDSK,
$\Longrightarrow$
1. $k \in C^{2n-2}(\mathbf{R}, \mathbf{R})$,
2. $k \in C^{2n-1}(\mathbf{R}, \mathbf{R})$ except at $x = 0$, where

$$k^{2n-1}(0^+),\ k^{2n-1}(0^-)$$

both exist.

The asymptotic behaviour of k and its Fourier transform, K, will be now studied.

Theorem B.3.3 (Asymptotic Behaviour of The Fourier transform of The Finite VDSKs)

1. k a finite VDSK,
2. $k(x) \leftrightarrow K(u)$
$\Longrightarrow$

$$|K(u)| = O(|u|^{-2n}),\ |u| \to \infty.$$

Proof. k is made up of a finite convolution operations of functions $k_\ell(x) = a(\ell)\,\mathrm{e}[a(\ell)x]$, where $a(\ell) > 0$, $\ell = 1, 2, \ldots, n$; and whose FT, $K_\ell(u)$, satisfy the inequality

$$|K_\ell(u)| = \left|\frac{a^2(\ell)}{a^2(\ell) + u^2}\right| \leq \frac{a^2(\ell)}{|u|^2},\ \ell = 1, 2, \ldots, n.$$

Thus

$$|K(u)| = \left|\prod_{\ell=1}^{n} K_\ell(u)\right| \leq \prod_{\ell=1}^{n}\left(\frac{a^2(\ell)}{|u|^2}\right) = |u|^{-2n}\prod_{\ell=1}^{n} a^2(\ell). \tag{B.3.5}$$

From the above theorem we construct the following corollarys.

Corollary B.3.4 (Absolute and Quadratic Integrability of The Fourier transform of The Finite VDSKs)

1. k a finite VDSK,
2. $k(x) \leftrightarrow K(u)$
$\Longrightarrow$
$K(u) \in L(\mathbf{R}, \mathbf{R}) \cap L^2(\mathbf{R}, \mathbf{R})$.

Corollary B.3.5 (Absolute and Quadratic Integrability of The Finite VDSKs)

k a finite VDSK,
$\Longrightarrow$
$k(x) \in L(\mathbf{R}, \mathbf{R}) \cap L^2(\mathbf{R}, \mathbf{R})$.

The Fourier transform $K(u)$ of the Fourier transform of k is given by

$$K(u) \leftrightarrow 2\pi k(-x).$$

Since k is a even function then

$$K(u) \leftrightarrow 2\pi k(x).$$

This result in conjunction with Corollary B.3.4. and Riemann-Lebesgue Lemma proves the following theorem.

Theorem B.3.6 (Asymptotic Behaviour of The Finite VDSKs)

k a finite VDSK
$\Longrightarrow$
$k(x) \to 0$ *as* $|x| \to \infty$.

B.4 The Non-Finite VDSKs

We now study kernels k holding the property

$$k(x) \leftrightarrow K(u) = \prod_{\ell=1}^{\infty}\left(\frac{a^2(\ell)}{a^2(\ell) + u^2}\right) \tag{B.4.1}$$

which are non-finite kernels. In particular, the infinite product in equation $(B.4.1)$ may have only a finite number of factors, so that the finite VDSKs of the last section are included. Kernels holding equation $(B.4.1)$ can be synthesized from the basic kernel

$$\mathrm{e}(x) = \frac{1}{2}\exp(-|x|),\ x \in \mathbf{R}.$$

The non-finite VDSKs are composed of a non-finite number of functions $a(\ell)\,\mathrm{e}[a(\ell)x]$, $\ell = 1, 2, \ldots$. The properties of such kernels are given in the following theorem.

Theorem B.4.1 (Properties of The Non-Finite VDSKs)

1. $a(\ell) > 0$, $\ell = 1, 2, \ldots$,
2. $k_\ell(x) = a(\ell)\,\mathrm{e}[a(\ell)x]$,
3. $k = k_1 \otimes k_2 \otimes \cdots \otimes k_n \ldots$,
4. $K(u) = \prod_{\ell=1}^{\infty}\left(a^2(\ell)/(a^2(\ell) + u^2)\right)$
$\Longrightarrow$
A. k is a VDSK,
B. $k(x) \leftrightarrow K(u)$,
C. k has mean $\nu = 0$,
D. k has variance $\sigma^2 = \sum_{\ell=1}^{\infty}\left(2/a^2(\ell)\right) < \infty$.

Since k (Theorem B.4.1) is made up by a non-finite number of convolution operationw, then it is of degree infinity, which leads to the following.

Theorem B.4.2 (Degree of Differentiability of The Non-Finite VDSKs)

k a non-finite VDSK
$\Longrightarrow$
$k \in C^\infty(\mathbf{R}, \mathbf{R})$.

The asymptotic behaviour of the Fourier transform of a non-finite kernel is established in the following theorem.

Theorem B.4.3 (Asymptotic Behaviour of The Fourier transform of The Non-Finite VDSKs)

1. k a non-finite VDSK,
2. $k(x) \leftrightarrow K(u)$,
3. $R, p > 0$
$\Longrightarrow$
$|K(u)| = O(|u|^{-2p})$, $|u| \to \infty$.

Proof. Choose $N > p$ and so large that $|a(\ell)| \geq R$ when $\ell > N$ which is possible since $|a(\ell)| \to \infty$ as $\ell \to \infty$. Set

$$K_N(u) = \prod_{\ell=N+1}^{\infty}\left(\frac{a^2(\ell)}{a^2(\ell) + u^2}\right).$$

By equation $(B.3.5)$, it follows that

$$|K(u)| \leq \frac{|K_N(u)|}{|u|^{2N}}\prod_{\ell=1}^{N} a^2(\ell).$$

Because $|K_N(u)|$ never vanishes and is continuous for all $u \in \mathbf{R}$, then it has a positive lower bound. Hence, for a suitable constant M

$$|K(u)| \leq \frac{M}{|u|^{2N}}.$$

In particular, if $p = 1$ in the above theorem and because k is a variation diminishing function, the following corollary results.

Corollary B.4.4 (Absolute Integrability of The Non-Finite Kernels and Their FT)

1. k a non-finite VDSK,
2. $k(x) \leftrightarrow K(u)$
$\Longrightarrow$
$k, K \in L(\mathbf{R}, \mathbf{R})$.

Application of the symmetry property of the Fourier transform, the Riemann-Lebesgue Lemma and the above corollary proves the following theorem.

Theorem B.4.5 (Asymptotic Behaviour of The Non-Finite VDSKs)

k a non-finite VDSK
$\Longrightarrow$
$k(x) \to 0$ *as* $|x| \to \infty$.

Some examples of non-finite VDSKs are:

$$\frac{\pi}{4}\operatorname{sech}^2(\frac{\pi x}{2}) \leftrightarrow u \operatorname{csch} u = \prod_{\ell=1}^{\infty}\left(\frac{\ell^2\pi^2}{\ell^2\pi^2+u^2}\right), \qquad (B.4.2)$$

$$\frac{1}{2}\operatorname{sech}(\frac{\pi x}{2}) \leftrightarrow \operatorname{sech} u = \prod_{\ell=1}^{\infty}\left(\frac{(2\ell-1)^2\pi^2}{(2\ell-1)^2\pi^2+u^2}\right). \qquad (B.4.3)$$

Note that a non-finite VDSK does not necessarily belongs to $L^2(\mathbf{R}, \mathbf{R})$, e.g. the kernel given by equation $(B.4.3)$.

B.5 The Gaussian VDSK

The Gaussian VDSK, k, is defined by the relation

$$k(x) \leftrightarrow K(u) = \exp(-cu^2),\ c > 0. \qquad (B.5.1)$$

With $c \to 1/4c^2$, the Gaussian VDSK is now defined as

$$k(x) \leftrightarrow K(u) = \exp(-u^2/4c^2),\ c > 0. \qquad (B.5.2)$$

The basic properties of the above kernel follow directly and are collated together in the following theorem.

Theorem B.5.1 (Basic Properties of The Gaussian VDSK)

1. $k(x) = c\,\mathrm{gauss}(cx)$, $c > 0$,
2. $K(u) = \exp(-u^2/4c^2)$, $c > 0$,
3. $p > 0$
$\Longrightarrow$
A. k is a VDSK,
B. $k(x) \leftrightarrow K(u)$,
C. k has mean $\nu = 0$,
D. k has variance $\sigma^2 = 1/2c^2$,
E. $k, K \in L(\mathbf{R}, \mathbf{R}) \cap L^2(\mathbf{R}, \mathbf{R})$,
F. $k, K \in C^\infty(\mathbf{R}, \mathbf{R})$,
G. $|k(x)| = o(|x|^{-p})$,
H. $|K(u)| = o(|u|^{-p})$.

If in equation $(B.5.1)$, c is considered as a variable, say t, then after taking the inverse Fourier transform with respect to x we obtain a real valued function of two variables, i.e.

$$k(x,t) = \frac{1}{\sqrt{4\pi t}}\exp(-x^2/4t). \qquad (B.5.3)$$

This new function is the familiar *source solution* of the diffusion equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)k(x,t) = 0 \qquad (B.5.4)$$

B.6 Geometric Properties of The VDSKs

We consider the general geometric properties shared by the finite, non-finite and the Gaussian VDSKs where k denotes either a finite, non-finite or Gaussian VDSK throughout.

Theorem B.6.1 (Geometric Properties of The VDSKs)

1. k a VDSK,
2. $f\colon \mathbf{R} \to \mathbf{R}$ bounded and convex (concave)
$\Longrightarrow$
A. For $a, b \in \mathbf{R}$

$$V[k(x) \otimes f(x) - a - bx] \le V[f(x) - a - bx], \qquad (B.6.1)$$

B. $(k \otimes f)(x)$ is convex (concave).

Proof. A. Inequality $(B.6.1)$ follows by a direct application of the variation diminishing property of k.

B. It is well known that f is convex iff

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) - f(x) \ge 0,$$

for all $x \in \mathbf{R}$, $h > 0$. Because k is a non-negative function, then

$$\Delta_h^2[(k \otimes f)(x)] = \Delta_h^2\left[\int_{-\infty}^{\infty} k(y)f(x-y)\,dy\right]$$

$$= \int_{-\infty}^{\infty} k(y)\Delta_h^2 f(x-y)\,dy \ge 0.$$

Thus the inequality follows. The case for which f is concave follows using a similar argument but $\Delta_h^2 f(x) \le 0$, for all $x \in \mathbf{R}$, $h > 0$.

The geometric significance of inequality (B.6.1) is that the number of intersections of the straight line $y = a + bx$, $a, b \in \mathbf{R}$, with $(k \otimes f)(x)$ does not exceed the number of intersections of $y = a + bx$ with $y = f(x)$. As a special instance of such an inequality, it follows that $(k \otimes f)(x)$ is non-negative if f is non-negative.

Corollary B.6.2 (Non-Negativity of $k \otimes f$)

1. k a VDSK,
2. $f\colon \mathbf{R} \to \mathbf{R}$, $f \ge 0$, and bounded
$\Longrightarrow$
$(k \otimes f)(x) \ge 0$, $x \in \mathbf{R}$.

From the above results, it is clear that if f is composed of a succession of alternating convex or concave arcs, then $k \otimes f$ is also made up of a similar succession of convex or concave arcs equal in number to those of f. Thus, a VDSK is *shape preserving*.

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