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An Explicit Super-Time-Stepping Scheme for Non-Symmetric Parabolic Problems

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Abstract. Explicit numerical methods for the solution of a system of differential equations may suffer from a time step size that approaches zero in order to satisfy stability conditions. When the differential equations are dominated by a skew-symmetric component, the problem is that the real eigenvalues are dominated by imaginary eigenvalues. We compare results for stable time step limits for the super-time-stepping method of Alexiades, Amiez, and Gremaud (super-time-stepping methods belong to the Runge-Kutta-Chebyshev class) and a new method modeled on a predictor-corrector scheme with multiplicative operator splitting. This new explicit method increases stability of the original super-time-stepping whenever the skew-symmetric term is nonzero, which occurs in particular convection-diffusion problems and more generally when the iteration matrix represents a nonlinear operator. The new method is stable for skew symmetric dominated systems where the regular super-time-stepping scheme fails. This method is second order in time (may be increased by Richardson extrapolation) and the spatial order is determined by the user's choice of discretization scheme. We present a comparison between the two super-time-stepping methods to show the speed up available for any non-symmetric system using the nearly symmetric Black-Scholes equation as an example.

Keywords: explicit method, symmetric, skew-symmetric, multiplicative operator splitting, super-time-stepping, Black-Scholes
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INTRODUCTION

In our problem, space discretization has converted the parabolic equations in $\mathbb{R}^n \times [t_0, T]$ into a system of ordinary differential equations in $[t_0, T]$. This discretization may be chosen to be of any order. We therefore consider the initial value problem

$$\mathbf{Y}'(t) = \mathbf{F}(\mathbf{Y}(t)), \quad \mathbf{Y}(t_0) = \mathbf{Y}_0. \tag{1}$$

We denote the current time as t^ℓ and seek the solution at a later time $t^{\ell+1} = t^\ell + \tau$, where τ is a positive and real time step. In particular we consider a system of ordinary differential equations that may be discretized as

$$\mathbf{Y}^{\ell+1} = (\mathsf{I} - \tau \mathsf{M})\,\mathbf{Y}^\ell. \tag{2}$$

The system (2) may be solved using an explicit scheme by calculating $\mathbf{Y}$ at the time $t^{\ell+1}$ using $\mathbf{Y}(t^\ell)$, while an implicit scheme will calculate $\mathbf{Y}$ by solving an equation involving both $t^{\ell+1}$ and $\mathbf{Y}(t^\ell)$. The solution to implicit schemes will require extra computations and are more difficult to implement, especially if the code is to be solved on a parallel machine. The difficulty with explicit schemes is based on satisfying the Courant-Friedrichs-Lewy (CFL) stability condition [4] which requires a numerical scheme satisfy $\rho(\mathsf{I} - \tau\mathsf{M}) < 1$ where $\rho(\cdot)$ denotes the spectral radius. If M is symmetric then this requires

$$\tau < \frac{2}{\lambda_1(\mathsf{M})}, \tag{3}$$

using the notation that the eigenvalues of an $n \times n$ symmetric matrix M are ordered $\lambda_1(\mathsf{M}) \geq \lambda_2(\mathsf{M}) \geq \ldots \geq \lambda_n(\mathsf{M})$. The CFL stability restriction on the time step forces the stepsize τ to be much smaller than the necessary size to satisfy the accuracy condition for the computation.

The method presented in this paper builds on two approaches, super-time-stepping (STS) [1] and multiplicative operator splitting [2] in order to increase the time-step with an explicit scheme. Super-time-stepping schemes belong to the class of Runge-Kutta-Chebyshev methods which are discussed in [8, 9] as well as many other sources. Multiplicative operator splitting is used frequently in separate convection-diffusion problems into two parts: one to be solved with an explicit method, the other with an implicit method.

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In the following section we present the regular super-time-stepping scheme and our scheme skew dominated super-time-stepping scheme. In [5] we have shown analytically that in a system without a symmetric component, the regular STS method must have a time step that goes to zero, while the skew dominated STS retains a nonzero time step. In this paper we provide a numerical comparison of the two methods using the Black-Scholes equation. The numerical results indicate that the skew dominated super-time-stepping scheme provides benefits even for a slightly nonsymmetric system.

SUPER-TIME-STEPPING SCHEMES

The super-time-stepping method of Alexiades, Amiez, and Gremaud [1] for a symmetric M, uses N intermediate steps

$$\mathbf{Y}^{\ell+1} = \left(\prod_{j=1}^{N}(\mathsf{I}-\tau_j\mathsf{M})\right)\mathbf{Y}^{\ell}. \tag{4}$$

The benefit from the STS method follows by enforcing the CFL condition on the exterior step rather than enforcing the CFL condition on each of the N interior steps resulting in larger time steps.

We wish to consider a real, non-symmetric M split into symmetric and skew-symmetric components. This decomposition into symmetric and skew-symmetric components is $\mathsf{P}=\frac{1}{2}(\mathsf{M}+\mathsf{M}^T)$ and $\mathsf{S}=\frac{1}{2}(\mathsf{M}-\mathsf{M}^T)$ respectively. Thus, the system is

$$Y^{\ell+1} = (\mathsf{I}-\tau\mathsf{P}-\tau\mathsf{S})\,Y^{\ell}. \tag{5}$$

We refer to the regular super-time-stepping scheme as

$$\mathsf{T} = \prod_{k=1}^{N}\left[\mathsf{I}-\tau_k\left(\mathsf{P}+\mathsf{S}\right)\right]. \tag{6}$$

The analytic results for the STS method been extended in [5] from symmetric M to nonsymmetric M. The full analysis shows that if $\bar{\tau}_{\text{explicit}}$ is the time step for one time step of equation (4), then

$$\tau_T = \frac{\bar{\tau}_{\text{explicit}}}{2}\sum_{k=1}^{N}\left[1+\sqrt{1+\frac{(\nu-1)\cos\left(\frac{(2k-1)\pi}{2N}\right)+1+\nu}{\lambda_1(\mathsf{P})\,\bar{\tau}_{\text{explicit}}}}\right], \tag{7}$$

where $\lambda_1(\mathsf{P})>0$ and ν is a practical implementation of the damping factor that can be manipulated directly to modify the time-step size.

Our new scheme improves the stability of the scheme by allowing a much larger time-step for the instances when skew-symmetric term dominates the symmetric term. This scheme incorporates multiplicative operator splitting [2] with a predictor-corrector scheme.

$$\mathsf{H} = \prod_{k=1}^{N}(\mathsf{I}-\tau_k\mathsf{P})\times(\mathsf{I}-\tau_k\mathsf{S}+\tau_k^2\mathsf{S}^2),$$

which is equivalent to a super-time-step version of the three step scheme:

$$\begin{aligned}
Y^{\ell+1} &= (\mathsf{I}-\tau\mathsf{P})\tilde{Y}^{\ell+1},\\
\tilde{Y}^{\ell+1} &= Y^{\ell}-\tau\mathsf{S}\bar{Y}^{\ell+1},\\
\bar{Y}^{\ell+1} &= (\mathsf{I}-\tau\mathsf{S})Y^{\ell}.
\end{aligned}$$

Since even powers of a skew operator are symmetric this correction adds a pure symmetric component back into the scheme and stabilizes it allowing a time step of τ_H for a stable scheme H for a $n\times n$ matrix $\mathsf{M}=\mathsf{P}+\mathsf{S}$ and $\lambda_1(\mathsf{P})>0$ to be bounded above by

$$\tau_H = \frac{\bar{\tau}_{\text{explicit}}}{2}\sum_{k=1}^{N}\left[1+\sqrt{1+\frac{(\nu-1)\cos\left(\frac{(2k-1)\pi}{12N}\right)+1+\nu}{\lambda_1(\mathsf{P})\,\bar{\tau}_{\text{explicit}}}}\right]. \tag{8}$$

BLACK-SCHOLES EQUATION

A modified Black-Scholes model [3] for the European option pricing in the form of a finite domain initial boundary value problem [10] can be written as

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 w}{\partial s^2} + (r-d)s\frac{\partial w}{\partial s} - rw, \quad (s,t) \in (0,S_{\max}) \times (O,T) \\
&w(s,0) = \max(s-E,0), \quad 0 \le s \le S_{\max}, \\
&w(0,t) = 0, \quad 0 \le t \le T, \\
&w(S_{\max},t) = S_{\max}\exp(-dt) - E\exp(-rt), \quad 0 \le t \le T,
\end{aligned} \tag{9}$$

where $w = w(s,t)$ is the call option price at the underlying asset price s at the expiration time t up to the expiration date T, E is the strike price, r is the interest rate, d is the dividend yield, $\sigma = \sigma(s)$ is the volatility of the underlying asset.

The analytic solution for the infinite domain problem is given by the Black-Scholes formula [3]

$$w(s,t) = \frac{s}{2}\exp(-dt)\left(\mathrm{erf}(\frac{d_1}{\sqrt{2}})+1\right) - \frac{E}{2}\exp(-rt)\left(\mathrm{erf}(\frac{d_2}{\sqrt{2}})+1\right),$$

where $\mathrm{erf}(s) = (2/\sqrt{\pi})\int_0^s e^{-u^2}\,du$, $d_1 = \left(\log(s/E)+(r-d+\sigma^2/2)t\right)/(\sigma\sqrt{t})$ and $d_2 = d_1 - \sigma\sqrt{t}$.

We scale the problem to solve on $[0,1]$ rather than $[0,S_{\max}]$ with m nodes. Using a central difference scheme to approximate the derivatives on the asset price domain gives us a local error of order 2. A more sophisticated scheme will give a higher order, but to illustrate the effect of the effect of the super-time-stepping time derivative scheme, a scheme of 2nd order is sufficient. For central differences we have the tridiagonal matrices P and S

$$P = \begin{pmatrix} b_1 & \frac{1}{2}(a_2+c_1) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2}(a_{m+1}+c_m) & b_m \end{pmatrix} \quad S = \begin{pmatrix} 0 & -\frac{1}{2}(a_2-c_1) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2}(a_{m+1}-c_m) & 0 \end{pmatrix} \tag{10}$$

where

$$a_n = \frac{1}{2}(r-d)\frac{s_n}{h} - \frac{1}{2}\sigma^2\frac{s_n^2}{h^2}, \quad b_n = r + \sigma^2\frac{s_n^2}{h^2}, \quad c_n = -\frac{1}{2}(r-d)\frac{s_n}{h} - \frac{1}{2}\sigma^2\frac{s_n^2}{h^2}, \tag{11}$$

and $h = \Delta s = 1/m$.

A real square matrix is positive definite if and only if its symmetric part is positive definite [7]. Furthermore, by Sylvester's criterion, a matrix is positive definite if and only if the determinants of all upper-left sub-matrices are positive. Therefore it suffices for us to prove the Sylvester criterion holds for P. The determinant, D_j , for the upper-left $j \times j$ sub-matrix of the tridiagonal matrix are easily expressed by means of a simple three-term recurrence relation [6]

$$D_j = b_j D_{j-1} - a_j c_{j-1} D_{j-2}, \tag{12}$$

with $D_1 \equiv 0$ and $D_0 \equiv 1$. It is straightforward to show that we can express the above expression as a simple product

$$D_j = \prod_{k=1}^{j}\left|\frac{1}{2}(k+1)(\sigma^2+r)\right|. \tag{13}$$

Under the condition $r > 0$, P is positive-definite and therefore M is necessarily positive definite. Note that this result does not depend on uniform $\sigma(s)$and $r(s)$.

NUMERICAL RESULTS

We choose $r = 0.05$, $d = 0.01$, $\sigma = 0.35$, $E = 70$, $m = 1000$, and $S_{\max} = 200$. We make two choices for the damping factor ν. The closer ν is to zero, the larger differential in the number of time steps for the schemes T and H. N represents the number of interior steps in the super-time-stepping-scheme. The results are shown in Table 1. With this

TABLE 1. Comparison of super time stepping schemes: T and H.

	N, Number of Internal Steps	ν	Number of Super Steps	Step Ratio T/H	Error $\lvert H-T\rvert$
T: Regular STS	1	0.002	1233368	3.39	4.16E-09
H: Skew STS	1	0.002	363452		
T: Regular STS	5	0.002	177584	3.09	2.21E-07
H: Skew STS	5	0.002	57491		
T: Regular STS	10	0.002	81219	2.83	2.65E-07
H: Skew STS	10	0.002	28706		
T: Regular STS	50	0.002	15537	2.71	3.71E-07
H: Skew STS	50	0.002	5743		
T: Regular STS	100	0.002	7769	2.71	8.94E-07
H: Skew STS	100	0.002	2871		
T: Regular STS	1	0.0	1242641	3.59	2.10E-08
H: Skew STS	1	0.0	346370		
T: Regular STS	5	0.0	174987	4.28	1.99E-07
H: Skew STS	5	0.0	40885		
T: Regular STS	10	0.0	77524	4.48	2.67E-07
H: Skew STS	10	0.0	17323		
T: Regular STS	50	0.0	12261	4.79	4.73E-07
H: Skew STS	50	0.0	2559		
T: Regular STS	100	0.0	5623	4.89	9.11E-07
H: Skew STS	100	0.0	1149		

choice for r, d, and σ, the eigenvalues $\lambda_1(\mathsf{P})$ are much larger than $|\lambda(\mathsf{S}|_1$. For example, if $m=5$, $\lambda_1(\mathsf{P}) = 3.98759$ and $|\lambda(\mathsf{S}|_1 = 0.24526$. If $m=9$, $\lambda_1(\mathsf{P}) = 14.1723$ and $|\lambda(\mathsf{S}|_1 = 0.518488$. As m increases, $\lambda_1(\mathsf{P})$ will continue growing faster than $|\lambda(\mathsf{S}|_1$.

Consequently, this problem is nearly symmetric, and the effect of S should be small compared to P. However, that the skew-symmetric scheme H completes the calculation in fewer time steps than the regular super-time-stepping scheme T.

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