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Brendan Goldsmith

*Technological University Dublin*, [brendan.goldsmith@tudublin.ie](mailto:brendan.goldsmith@tudublin.ie)

P. Zanardo

*Universita di Padova*

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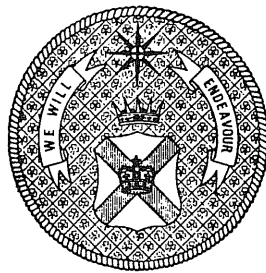
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ROYAL IRISH ACADEMY  
19 DAWSON STREET  
DUBLIN 2, IRELAND

# THE WALKER ENDOMORPHISM ALGEBRA OF A MIXED MODULE

By

B. GOLDSMITH

Dublin Institute of Technology, Kevin Street

and

P. ZANARDO

Dipartimento di Matematica, Università di Padova

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## ABSTRACT

Archimedean valuation domains  $R$  are characterised in terms of the endomorphism algebras of non-splitting mixed modules of rank 1, in the Walker category.

## Introduction

When studying mixed Abelian groups (or more generally mixed modules over some ring  $R$ ) it is natural to consider the quotient category  $\text{Walk}$ : this is the category having objects  $R$ -modules and morphisms  $\text{Hom}_w(G, H) = \text{Hom}_R(G, H)/\text{Hom}_t(G, H)$  where  $\text{Hom}_t(G, H)$  consists of the  $R$ -homomorphisms of  $G$  into  $H$  having torsion image. In [5, problems 8 and 9] Warfield asks which algebras occur as endomorphism algebras in  $\text{Walk}$  of finite rank  $R$ -modules when  $R = \mathbf{Z}$  or  $R = J_p$ , the  $p$ -adic integers.

Partial answers have been given in [1] and [2], where a full embedding of the category of torsion-free  $R$ -modules into  $\text{Walk}$  is exhibited. In this paper we look at the situation for modules of torsion-free rank 1 over more general valuation domains. In the first section we find results on the Walker endomorphism algebra and construct mixed modules of torsion-free rank 1, finally obtaining a complete characterisation in the case where  $R$  is an Archimedean valuation domain. In the final section we look at the global situation for Abelian groups. Our arguments are direct and do not depend on the deep techniques used in [1] or [2]. Our notation is standard, and unexplained terms may be found in [3] and [4].

## The local case

We suppose throughout this section that  $R$  is a valuation domain, with maximal ideal  $P$  and field of quotients  $Q$ , and  $M$  is a non-splitting mixed  $R$ -module of torsion-free rank 1. Note that the condition on  $M$  of being non-splitting implies that  $M$  does not contain copies of  $Q$ , since  $Q$  is injective, and that  $M/tM$  is not isomorphic to  $R$ ; in general,  $M/tM$  can be isomorphic either to  $Q$  or to a non-principal ideal of  $R$ ; in particular,  $M/tM$  is a *uniserial*  $R$ -module

(see [4]). An element  $x \in M$  will be called torsion-free if  $x \notin tM$ . The Walker endomorphism algebra  $\text{End}_R(M)/\text{Hom}_R(M, tM)$  will be denoted by  $\text{End}_w(M)$ . It is readily seen that  $\text{End}_w(M)$  embeds into  $\text{End}_R(M/tM) \subseteq \text{End}_R(Q) = Q$ ; therefore  $\text{End}_w(M) = R_J$  for some prime ideal  $J$  of  $R$ . If  $\varphi \in \text{End}_R(M)$ , we shall denote by  $\varphi_w$  the class of  $\text{End}_w(M)$  containing  $\varphi$ .

**Proposition 1.** *Let us suppose that  $R$  contains a non-zero minimal prime ideal  $L$ ; if  $M$  is a non-splitting mixed  $R$ -module of torsion-free rank 1, then  $\text{End}_w(M) \subseteq R_L$ .*

PROOF. Let us choose a torsion-free element  $x \in M$ . If  $\varphi \in \text{End}_R(M)$ , then, since  $M/tM$  is a uniserial  $R$ -module, either

$$\varphi(x) = \alpha x + t \text{ for some } \alpha \in R, t \in tM \quad (1)$$

or

$$\beta\varphi(x) = x + t_1 \text{ for some } \beta \in R, t_1 \in tM. \quad (2)$$

Since  $M$  is of torsion-free rank 1, it follows at once that  $\varphi - \alpha \in \text{Hom}(M, tM)$  in case (1), and  $\beta\varphi - 1 \in \text{Hom}(M, tM)$  in case (2). We claim that, in case (2), we have necessarily  $\beta \notin L$ . By contradiction, let us suppose  $\beta \in L$ , and let  $\psi = \beta\varphi - 1 \in \text{Hom}(M, tM)$ . Consider now  $\ker \psi$ ; for all  $z \in \ker \psi$  we have

$$z = \beta\varphi(z) = \beta^2\varphi^2(z) = \dots$$

Moreover, since  $\ker \psi$  is clearly invariant under  $\varphi$ , this implies that  $\ker \psi$  is divisible by any power of  $\beta$ . Since  $L$  is minimal,  $\beta \in L$  implies that  $\bigcap_n \beta^n R = 0$  so that, for all  $r \in R$ , there exists  $m \in \mathbb{N}$  such that  $r$  divides  $\beta^m$ . We deduce that  $\ker \psi$  is a divisible submodule of  $M$ . By (2) we also get that there exists a  $0 \neq s \in R$  such that  $sx \in \ker \psi$ . But if a divisible submodule contains torsion-free elements, then it must contain a copy of  $Q$ , so that  $Q \subseteq M$ , and this is the required contradiction.

From (1) and (2) with  $\beta \notin L$  it is easy to conclude that either  $\varphi_w = \alpha 1_w$  or  $\beta\varphi_w = 1_w$  with  $\alpha \in R, \beta \in R \setminus L$ , i.e.  $\text{End}_w(M) \subseteq R_L$ . ■

In our next theorem we shall use the following obvious remark on valuation domains:

for every valuation domain  $R$  there exists a set of elements  $\{r_\alpha : \alpha < \sigma\}$  indexed by an ordinal  $\sigma$ , such that  $r_\alpha R > r_\beta R$  if  $\alpha < \beta$  and  $\bigcap_{\alpha < \sigma} r_\alpha R = 0$ . (\*)

**Theorem 2.** *For every valuation domain  $R$  there exists a non-splitting mixed  $R$ -module  $M$  with  $M/tM \cong Q$ .*

PROOF. Choose elements  $r_\alpha \in R, \alpha < \sigma$ , as in (\*). Let  $A = \{\xi, \eta_\alpha : \alpha < \sigma\}$  be a set of symbols; let  $F$  be the free  $R$ -module with basis  $A$  and let  $N$  be the submodule

of  $F$  generated by the elements of the form:  $\xi - r_\alpha \eta_\alpha$ ,  $\alpha < \sigma$ . Then let us define  $M = F/N$ ,  $x = \xi + N$ ,  $y_\alpha = \eta_\alpha + N$ ,  $\alpha < \sigma$ . Straightforward calculations show that  $x$  is a torsion-free element of  $M$  and that, for every torsion-free  $z \in M$ , there exists  $0 \neq s \in R$  such that  $sz \in Rx$ , so that  $M/tM$  has rank 1; moreover, from  $x = r_\alpha y_\alpha$  for all  $\alpha < \sigma$ , it follows that  $M/tM \cong Q$ , since the elements  $r_\alpha^{-1}$ ,  $\alpha < \sigma$ , generate  $Q$  as an  $R$ -module. It remains to show that  $M$  is non-splitting; it is enough to verify that  $M$  does not contain copies of  $Q$ . By contradiction, suppose that a torsion-free  $z \in M$  is such that  $Qz \subseteq M$ . We can assume that  $z = sx$  for some  $0 \neq s \in P$ . There exists  $z_1 = bx + \sum_i b_{\alpha i} y_{\alpha i} \in Qz$  such that

$$sx = s^2 z_1.$$

Not all the  $b_{\alpha i}$  can be zero, otherwise  $s(1 - sb)x = 0$ : impossible, since  $x$  is torsion-free and  $1 - sb$  is a unit of  $R$ . Moreover, we can assume that  $b_{\alpha i} \notin r_{\alpha i} R$  for all  $i$ . Choose now  $0 \neq q \in P$  such that  $b_{\alpha i} \notin qR$  for all  $i$ . Then there exists  $z_2 = cx + \sum_j c_{\beta j} y_{\beta j} \in Qz$  such that  $z_1 = qz_2$ . This equality, read inside the free  $R$ -module  $F$ , implies that there exists an element  $\sum_k d_{\gamma k} (\xi - r_{\gamma k} \eta_{\gamma k}) \in N$  such that

$$b\xi + \sum_i b_{\alpha i} \eta_{\alpha i} = qc\xi + \sum_j qc_{\beta j} \eta_{\beta j} + \sum_k d_{\gamma k} (\xi - r_{\gamma k} \eta_{\gamma k}).$$

The above equality gives the required contradiction, because the first member's summand  $b_{\alpha 1} \eta_{\alpha 1}$  cannot appear in the second member, since  $b_{\alpha 1} \notin r_{\alpha 1} R + qR$ . The desired conclusion follows. ■

The following corollary will be used in Theorem 4.

**Corollary 3.** *For every non-zero prime ideal  $L$  of  $R$  there exists a non-splitting mixed  $R$ -module  $M$  with  $M/tM \cong Q$  and  $\text{End}_w(M) \supseteq R_L$ .*

PROOF. Let us construct an  $R_L$ -module  $M$  as in Theorem 2. Consider  $M$  as an  $R$ -module; it is straightforward to verify that the  $R$ -module  $M$  satisfies the requirements of the assertion. ■

Before stating our characterisation theorem, we recall that a valuation domain  $R$  is said to be *Archimedean* if the maximal ideal  $P$  is the unique non-zero prime ideal; equivalently, the value group of  $R$  is a subgroup of the real numbers. Of course, a discrete valuation ring is Archimedean.

**Theorem 4.** *Let  $R$  be a valuation domain. The following are equivalent: (a)  $R$  is Archimedean; (b) for every non-splitting mixed  $R$ -module  $M$  of torsion-free rank 1 we have  $\text{End}_w(M) = R$ .*

PROOF. (a)  $\rightarrow$  (b). If  $R$  is Archimedean,  $P$  is the minimal non-zero prime ideal of  $R$ ; by Proposition 1 we obtain that  $\text{End}_w(M) \subseteq R_P = R$ , for every non-splitting mixed  $R$ -module  $M$  of torsion-free rank 1.

(b) → (a). If  $R$  is not Archimedean, there exists a non-zero prime ideal  $L \subset P$ . In view of Corollary 3, for a convenient choice of  $M$  we have  $\text{End}_w(M) \supseteq R_L \supset R$ . ■

However, it is possible to construct a non-splitting mixed module  $M$  with  $M/tM \cong Q$  and  $\text{End}_w(M) = R$ , even when  $R$  does not contain a minimal non-zero prime ideal, as shown in the following example.

*Example 6.* Let  $R$  be a valuation domain satisfying the following condition: there exists a countable strictly decreasing sequence of non-zero prime ideals of  $R$

$$P_0 = P > P_1 > \dots > P_n > \dots$$

such that  $\bigcap_{n=0}^{\infty} P_n = 0$ . The existence of such an  $R$  is ensured, for instance, by proposition 5 of [6]. For all  $n \in \mathbb{N}$ , choose  $p_n \in P_{n-1} \setminus P_n$  (here we set  $P_{-1} = R, p_0 = 1$ ); clearly  $\bigcap_n p_n R = 0$ . Let  $T = \bigoplus_{n=0}^{\infty} R/P_n$ . For all  $k \in \mathbb{N}$ , we set  $T_k = \bigoplus_{n=0}^k R/P_n$ ; let us note that, given  $\varphi \in \text{End}(T)$ , we have  $\varphi(T_k) \subseteq T_k$  for all  $k$ : in fact  $\text{Hom}_R(R/P_m, R/P_n) = 0$  if  $n < m$ , because the  $P_i$  are prime ideals. We embed  $T$  into the  $R$ -module  $T' = \prod_{n=0}^{\infty} R/P_n$ . We denote the elements of  $T'$  by formal series  $\sum_{n=0}^{\infty} (a_n + P_n)$ . Let us consider the following mixed  $R$ -submodule of  $T'$ ,

$$M = \langle T, z_j : j \in \mathbb{N} \rangle,$$

where  $z_j = \sum_{n=0}^{\infty} (b_n + P_n) \in T'$  is defined as follows:  $b_n = 0$  if  $n < j$ ,  $b_n = p_n p_j^{-1}$  if  $n \geq j$ . Note that, since  $p_0 = 1$ , we have  $z_0 = \sum_{n=0}^{\infty} (p_n + P_n)$ . It is easy to check that: (a)  $T = tM$ ; (b) for all  $j$ , we have  $z_0 - p_j z_j \in T_{j-1}$ , so that  $M/tM \cong Q$ ; (c) if  $x = \sum_{n=0}^{\infty} (c_n + P_n), y = \sum_{n=0}^{\infty} (d_n + P_n) \in M$  are such that  $x - y \in T$ , there exists a  $k$  such that  $c_n + P_n = d_n + P_n$  for all  $n \geq k$ .

Let us now prove that  $\text{End}_w(M) = R$ . By contradiction, let us assume that  $s \in R$ , not a unit, is such that  $s^{-1}1_w \in \text{End}_w(M)$ ; then there exists  $\varphi \in \text{End}(M)$  such that  $s\varphi - 1 \in \text{Hom}(M, tM)$ . Let us now consider the element  $z_0$ ; let  $\varphi(z_0) = \sum_{n=0}^{\infty} (c_n + P_n)$ . Since  $s\varphi(z_0) - z_0 \in T$ , by property (c) there exists a  $k \in \mathbb{N}$  such that  $sc_n + P_n = p_n + P_n$  for all  $n \geq k$ . We also know, by property (b), that  $z_0 = p_k z_k + t$ , for a suitable  $t \in T_{k-1}$ , from which

$$s\varphi(z_0) = sp_k \varphi(z_k) + \varphi(st)$$

where  $\varphi(st) \in T_{k-1}$ , since  $\varphi(T_{k-1}) \subseteq T_{k-1}$ . Therefore, if  $\varphi(z_k) = \sum_{n=0}^{\infty} (d_n + P_n)$ , we have, for all  $n \geq k$ ,

$$sc_n + P_n = sp_k d_n + P_n.$$

In particular,

$$p_k + P_k = sc_k + P_k = sp_k d_k + P_k$$

implies  $p_k(1 - sd_k) \in P_k$ , from which  $p_k \in P_k$ , because  $1 - sd_k$  is a unit, against

our assumption on the  $p_n$ . From the contradiction we conclude that  $\text{End}_w(M) = R$ , as desired.

We close this section by remarking that Fuchs' divisible module  $\partial$  (see [4, ch. VI]) is mixed and of torsion-free rank 1, and is non-splitting *if and only if the projective dimension of  $Q$  is  $\geq 2$* , by [4, ch. VI, lemma 3.6]; hence a result similar to our Theorem 2 was already available, but with restrictions on  $R$ . It is also worth noting that the modules constructed in Theorem 2 have a much simpler structure than that of  $\partial$ ; of course, they are reminiscent of the classical construction of the Prüfer group  $H_{\omega+1}$  (see [3, 35]). Some relations can also be found between Example 6 and the mixed group in [3, example 3.100].

### The global case

Our results in the first section have a natural global generalisation to certain classes of Prüfer domains. It is not possible, however, to give a generalisation without the imposition of some additional properties on the Prüfer domain that essentially make it very similar to  $\mathbf{Z}$ . Thus we restrict our attention here to Abelian groups.

Recall that if  $G$  is a mixed Abelian group and  $x \in G \setminus tG$ , then we can associate with  $x$  a height matrix  $\mathbf{H}(x)$  defined as follows: let  $\mathbf{P} = \{p_n : n \in \mathbf{N}\}$  be the set of all primes; then  $\mathbf{H}(x) = (\sigma_{nk})$  where  $\sigma_{nk}$  is the ordinal or symbol  $\infty$  that occurs as the generalised  $p_n$ -height in  $G$  of  $p_n^k x$  for all  $n$  and  $k$ . We say that two such height matrices  $(\sigma_{nk})$  and  $(\rho_{nk})$  are equivalent if, for almost all  $n$ , the  $n$ th rows of the two matrices are identical and for the remaining  $n$  there exist non-negative integers  $h, m$  (depending on  $n$ ) such that  $\sigma_{n,k+h} = \rho_{n,k+m}$  for all  $k$ . It is well known (see [3, 103]) that if  $G$  is of torsion-free rank 1 and  $x, y$  are torsion-free elements of  $G$ , then  $\mathbf{H}(x)$  and  $\mathbf{H}(y)$  are equivalent. Thus, in this case, associated with  $G$  we have an invariant that is the equivalence class of height matrices of any torsion-free element. Let  $\mathbf{U} = \mathbf{U}(G)$  denote this invariant. With this invariant of  $G$  we associate a set  $\mathbf{P}(\mathbf{U})$  of primes as follows:  $\mathbf{P}(\mathbf{U}) = \{p_n \in \mathbf{P} : \sigma_{nk} < \infty \forall k \in \mathbf{N}\}$  where  $(\sigma_{nk}) \in \mathbf{U}$ . With this notation we can state our global result.

**Theorem 7.** *Let  $G$  be a mixed Abelian group of torsion-free rank 1; then  $\text{End}_w(G) = \mathbf{Z}_{\mathbf{P}(\mathbf{U})}$ , the localisation of  $\mathbf{Z}$  at  $\mathbf{P}(\mathbf{U})$ .*

PROOF. Since  $\text{End}_w(G)$  is a subring of the rational field  $\mathbf{Q}$ , it will suffice to compute the characteristic of the identity in  $\text{End}_w(G)$ . If  $1_G \in p \text{End}_w(G)$  for some  $p \in \mathbf{P}$ , then  $\psi = 1_G - p\varphi \in \text{Hom}(G, tG)$  for some endomorphism  $\varphi$ . Since  $\ker \psi$  must then contain a torsion-free element  $y$ , we have that

$$y = p\varphi(y) = p^2\varphi^2(y) = \dots$$

and so  $ht_p(y) = \infty$ , implying that  $p \notin \mathbf{P}(\mathbf{U})$ . Thus if  $q \in \mathbf{P}(\mathbf{U})$  then  $\chi(1_G)$ , the characteristic of  $1_G$ , is zero at the place corresponding to  $q$ . Equivalently we have  $\text{End}_w(M) \subseteq \mathbf{Z}_{\mathbf{P}(\mathbf{U})}$ .

To prove the reverse inequality we make use of an observation of Warfield [5, lemma 1], which he refers to as the Abelian group theorist's 'Hasse principle'; see Lemma 8 below. Suppose that  $q \in \mathbf{PP}(\mathbf{U})$  and  $x$  is a torsion-free element of  $G$  such that  $x$  has infinite  $q$ -height in  $G$ . Let  $x = qy$  for some  $y \in G$  and take  $S = \langle x \rangle$ ,  $T = \langle y \rangle$ . Then  $f: S \rightarrow T$  with  $f(x) = y$  is a homomorphism. Now if  $p$  is a prime distinct from  $q$ ,  $G_p = G \otimes \mathbf{Z}_p$  is a  $\mathbf{Z}_p$ -module and division by  $q$  is a homomorphism of  $G_p$  that extends the induced map  $f_p$ . Moreover, since  $q \notin \mathbf{P}(\mathbf{U})$ ,  $G_q$  is a split extension of  $tG_p$  by  $\mathbf{Q}$ , so that

$$G_q = \mathbf{Q}x \oplus tG_q = \mathbf{Q}y \oplus tG_q,$$

and then  $f_q$  lifts to an endomorphism of  $G_q$ . Hence, by Lemma 8, there is an endomorphism  $g$  of  $G$  with  $g|_S = f$ . However, we now have that  $(qg - 1_G)(x) = (qf - 1_G)(x) = qy - x = 0$ , so that  $qg - 1_G \in \text{Hom}(G, tG)$ . Thus  $1_G$  is divisible in  $\text{End}_w(G)$  by  $q$  and so  $q^{-1} \in \text{End}_w(G)$ ; it follows that  $\mathbf{Z}_{\mathbf{P}(\mathbf{U})} \subseteq \text{End}_w(G)$ . ■

**Lemma 8** [5]. *Let  $G, H$  be groups and  $S$  and  $T$  subgroups such that  $G/S$  and  $H/T$  are torsion, and for each prime  $p$  identify  $S_p = S \otimes \mathbf{Z}_p$ ,  $T_p = T \otimes \mathbf{Z}_p$  with submodules of  $G_p = G \otimes \mathbf{Z}_p$ ,  $H_p = H \otimes \mathbf{Z}_p$ , respectively. Let  $f: S \rightarrow T$  be a homomorphism and  $f_p: S_p \rightarrow T_p$  the induced map. Then  $f$  extends to a homomorphism from  $G$  to  $H$  if and only if, for each prime  $p$ ,  $f_p$  extends to a homomorphism from  $G_p$  to  $H_p$ .*

**PROOF.** The proof is identical to [5, lemma 1], where it is shown for localisations at the  $p$ -adic integers  $J_p$ . However, the proof by Warfield makes no use of the completion. ■

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