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THE WALKER ENDOMORPHISM ALGEBRA OF A MIXED MODULE

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Abstract

Archimedean valuation domains R are characterised in terms of the endomorphism algebras of non-splitting mixed modules of rank 1, in the Walker category.

Introduction

When studying mixed Abelian groups (or more generally mixed modules over some ring R) it is natural to consider the quotient category Walk: this is the category having objects R-modules and morphisms $\mathrm{Hom}_W(G, H) = \mathrm{Hom}_R(G, H)/\mathrm{Hom}_t(G, H)$ where $\mathrm{Hom}_t(G, H)$ consists of the R-homomorphisms of G into H having torsion image. In [5, problems 8 and 9] Warfield asks which algebras occur as endomorphism algebras in Walk of finite rank R-modules when $R = \mathbf{Z}$ or $R = J_p$, the p-adic integers.

Partial answers have been given in [1] and [2], where a full embedding of the category of torsion-free R-modules into Walk is exhibited. In this paper we look at the situation for modules of torsion-free rank 1 over more general valuation domains. In the first section we find results on the Walker endomorphism algebra and construct mixed modules of torsion-free rank 1, finally obtaining a complete characterisation in the case where R is an Archimedean valuation domain. In the final section we look at the global situation for Abelian groups. Our arguments are direct and do not depend on the deep techniques used in [1] or [2]. Our notation is standard, and unexplained terms may be found in [3] and [4].

The local case

We suppose throughout this section that R is a valuation domain, with maximal ideal P and field of quotients Q, and M is a non-splitting mixed R-module of torsion-free rank 1. Note that the condition on M of being non-splitting implies that M does not contain copies of Q, since Q is injective, and that M/tM is not isomorphic to R; in general, M/tM can be isomorphic either to Q or to a non-principal ideal of R; in particular, M/tM is a *uniserial* R-module (see [4]). An element $x \in M$ will be called torsion-free if $x \not\in tM$. The Walker endomorphism algebra $\mathrm{End}_R(M)/\mathrm{Hom}_R(M, tM)$ will be denoted by $\mathrm{End}_W(M)$. It is readily seen that $\mathrm{End}_W(M)$ embeds into $\mathrm{End}_R(M/tM) \subseteq \mathrm{End}_R(Q) = Q$; therefore $\mathrm{End}_W(M) = R_J$ for some prime ideal J of R. If $\varphi \in \mathrm{End}_R(M)$, we shall denote by φ_W the class of $\mathrm{End}_W(M)$ containing φ.

Proposition 1. *Let us suppose that R contains a non-zero minimal prime ideal L; if M is a non-splitting mixed R-module of torsion-free rank 1, then* $\mathrm{End}_W(M) \subseteq R_L$.

Proof. Let us choose a torsion-free element $x \in M$. If $\varphi \in \mathrm{End}_R(M)$, then, since M/tM is a uniserial R-module, either

$$\varphi(x) = \alpha x + t \text{ for some } \alpha \in R,\ t \in tM \tag{1}$$

or

$$\beta\varphi(x) = x + t_1 \text{ for some } \beta \in R,\ t_1 \in tM\,. \tag{2}$$

Since M is of torsion-free rank 1, it follows at once that $\varphi - \alpha \in \mathrm{Hom}(M, tM)$ in case (1), and $\beta\varphi - 1 \in \mathrm{Hom}(M, tM)$ in case (2). We claim that, in case (2), we have necessarily $\beta \not\in L$. By contradiction, let us suppose $\beta \in L$, and let $\psi = \beta\varphi - 1 \in \mathrm{Hom}(M, tM)$. Consider now $\ker \psi$; for all $z \in \ker \psi$ we have

$$z = \beta\varphi(z) = \beta^2\varphi^2(z) = \cdots\,.$$

Moreover, since $\ker \psi$ is clearly invariant under φ, this implies that $\ker \psi$ is divisible by any power of β. Since L is minimal, $\beta \in L$ implies that $\bigcap_n \beta^n R = 0$ so that, for all $r \in R$, there exists $m \in \mathbf{N}$ such that r divides β^m. We deduce that $\ker \psi$ is a divisible submodule of M. By (2) we also get that there exists a $0 \neq s \in R$ such that $sx \in \ker \psi$. But if a divisible submodule contains torsion-free elements, then it must contain a copy of Q, so that $Q \subseteq M$, and this is the required contradiction.

From (1) and (2) with $\beta \not\in L$ it is easy to conclude that either $\varphi_W = \alpha 1_W$ or $\beta\varphi_W = 1_W$ with $\alpha \in R$, $\beta \in R\backslash L$, i.e. $\mathrm{End}_W(M) \subseteq R_L$. ■

In our next theorem we shall use the following obvious remark on valuation domains:

for every valuation domain R there exists a set of elements $\{r_\alpha : \alpha < \sigma\}$ indexed by an ordinal σ, such that $r_\alpha R > r_\beta R$ if $\alpha < \beta$ and $\bigcap_{\alpha<\sigma} r_\alpha R = 0$. (*)

Theorem 2. *For every valuation domain R there exists a non-splitting mixed R-module M with $M/tM \cong Q$.*

Proof. Choose elements $r_\alpha \in R$, $\alpha < \sigma$, as in (*). Let $A = \{\xi, \eta_\alpha : \alpha < \sigma\}$ be a set of symbols; let F be the free R-module with basis A and let N be the submodule of F generated by the elements of the form: $\xi - r_\alpha \eta_\alpha$, $\alpha < \sigma$. Then let us define $M = F/N$, $x = \xi + N$, $y_\alpha = \eta_\alpha + N$, $\alpha < \sigma$. Straightforward calculations show that x is a torsion-free element of M and that, for every torsion-free $z \in M$, there exists $0 \neq s \in R$ such that $sz \in Rx$, so that M/tM has rank 1; moreover, from $x = r_\alpha y_\alpha$ for all $\alpha < \sigma$, it follows that $M/tM \cong Q$, since the elements r_α^{-1}, $\alpha < \sigma$, generate Q as an R-module. It remains to show that M is non-splitting; it is enough to verify that M does not contain copies of Q. By contradiction, suppose that a torsion-free $z \in M$ is such that $Qz \subseteq M$. We can assume that $z = sx$ for some $0 \neq s \in P$. There exists $z_1 = bx + \Sigma_i b_{\alpha i} y_{\alpha i} \in Qz$ such that

$$sx = s^2 z_1 .$$

Not all the $b_{\alpha i}$ can be zero, otherwise $s(1 - sb)x = 0$: impossible, since x is torsion-free and $1 - sb$ is a unit of R. Moreover, we can assume that $b_{\alpha i} \not\in r_{\alpha i} R$ for all i. Choose now $0 \neq q \in P$ such that $b_{\alpha i} \not\in qR$ for all i. Then there exists $z_2 = cx + \Sigma_j c_{\beta j} y_{\beta j} \in Qz$ such that $z_1 = qz_2$. This equality, read inside the free R-module F, implies that there exists an element $\Sigma_k d_{\gamma k}(\xi - r_{\gamma k} \eta_{\gamma k}) \in N$ such that

$$b\xi + \sum_i b_{\alpha i} \eta_{\alpha i} = qc\xi + \sum_j qc_{\beta j} \eta_{\beta j} + \sum_k d_{\gamma k}(\xi - r_{\gamma k} \eta_{\gamma k}) .$$

The above equality gives the required contradiction, because the first member's summand $b_{\alpha 1} \eta_{\alpha 1}$ cannot appear in the second member, since $b_{\alpha 1} \not\in r_{\alpha 1} R + qR$. The desired conclusion follows. ■

The following corollary will be used in Theorem 4.

Corollary 3. *For every non-zero prime ideal L of R there exists a non-splitting mixed R-module M with $M/tM \cong Q$ and* $\mathrm{End}_W(M) \supseteq R_L$.

PROOF. Let us construct an R_L-module M as in Theorem 2. Consider M as an R-module; it is straightforward to verify that the R-module M satisfies the requirements of the assertion. ■

Before stating our characterisation theorem, we recall that a valuation domain R is said to be *Archimedean* if the maximal ideal P is the unique non-zero prime ideal; equivalently, the value group of R is a subgroup of the real numbers. Of course, a discrete valuation ring is Archimedean.

Theorem 4. *Let R be a valuation domain. The following are equivalent:* (a) *R is Archimedean*; (b) *for every non-splitting mixed R-module M of torsion-free rank 1 we have* $\mathrm{End}_W(M) = R$.

PROOF. (a)$\rightarrow$(b). If R is Archimedean, P is the minimal non-zero prime ideal of R; by Proposition 1 we obtain that $\mathrm{End}_W(M) \subseteq R_P = R$, for every non-splitting mixed R-module M of torsion-free rank 1.

(b)$\rightarrow$(a). If R is not Archimedean, there exists a non-zero prime ideal $L \subset P$. In view of Corollary 3, for a convenient choice of M we have $\mathrm{End}_W(M) \supseteq R_L \supset R$. ■

However, it is possible to construct a non-splitting mixed module M with $M/tM \cong Q$ and $\mathrm{End}_W(M) = R$, even when R does not contain a minimal non-zero prime ideal, as shown in the following example.

Example 6. Let R be a valuation domain satisfying the following condition: there exists a countable strictly decreasing sequence of non-zero prime ideals of R

$$P_0 = P > P_1 > \cdots > P_n > \cdots$$

such that $\cap_{n=0}^{\infty} P_n = 0$. The existence of such an R is ensured, for instance, by proposition 5 of [6]. For all $n \in \mathbf{N}$, choose $p_n \in P_{n-1} \backslash P_n$ (here we set $P_{-1} = R$, $p_0 = 1$); clearly $\cap_n p_n R = 0$. Let $T = \oplus_{n=0}^{\infty} R/P_n$. For all $k \in \mathbf{N}$, we set $T_k = \oplus_{n=0}^{k} R/P_n$; let us note that, given $\varphi \in \mathrm{End}(T)$, we have $\varphi(T_k) \subseteq T_k$ for all k: in fact $\mathrm{Hom}_R(R/P_m, R/P_n) = 0$ if $n < m$, because the P_i are prime ideals. We embed T into the R-module $T' = \Pi_{n=0}^{\infty} R/P_n$. We denote the elements of T' by formal series $\Sigma_{n=0}^{\infty} (a_n + P_n)$. Let us consider the following mixed R-submodule of T',

$$M = \langle T, z_j \colon j \in \mathbf{N} \rangle ,$$

where $z_j = \Sigma_{n=0}^{\infty} (b_n + P_n) \in T'$ is defined as follows: $b_n = 0$ if $n < j$, $b_n = p_n p_j^{-1}$ if $n \geqslant j$. Note that, since $p_0 = 1$, we have $z_0 = \Sigma_{n=0}^{\infty} (p_n + P_n)$. It is easy to check that: (a) $T = tM$; (b) for all j, we have $z_0 - p_j z_j \in T_{j-1}$, so that $M/tM \cong Q$; (c) if $x = \Sigma_{n=0}^{\infty} (c_n + P_n)$, $y = \Sigma_{n=0}^{\infty} (d_n + P_n) \in M$ are such that $x - y \in T$, there exists a k such that $c_n + P_n = d_n + P_n$ for all $n \geqslant k$.

Let us now prove that $\mathrm{End}_W(M) = R$. By contradiction, let us assume that $s \in R$, not a unit, is such that $s^{-1}1_W \in \mathrm{End}_W(M)$; then there exists $\varphi \in \mathrm{End}(M)$ such that $s\varphi - 1 \in \mathrm{Hom}(M, tM)$. Let us now consider the element z_0; let $\varphi(z_0) = \Sigma_{n=0}^{\infty} (c_n + P_n)$. Since $s\varphi(z_0) - z_0 \in T$, by property (c) there exists a $k \in \mathbf{N}$ such that $sc_n + P_n = p_n + P_n$ for all $n \geqslant k$. We also know, by property (b), that $z_0 = p_k z_k + t$, for a suitable $t \in T_{k-1}$, from which

$$s\varphi(z_0) = sp_k \varphi(z_k) + \varphi(st)$$

where $\varphi(st) \in T_{k-1}$, since $\varphi(T_{k-1}) \subseteq T_{k-1}$. Therefore, if $\varphi(z_k) = \Sigma_{n=0}^{\infty} (d_n + P_n)$, we have, for all $n \geqslant k$,

$$sc_n + P_n = sp_k d_n + P_n .$$

In particular,

$$p_k + P_k = sc_k + P_k = sp_k d_k + P_k$$

implies $p_k(1 - sd_k) \in P_k$, from which $p_k \in P_k$, because $1 - sd_k$ is a unit, against our assumption on the p_n. From the contradiction we conclude that $\mathrm{End}_W(M) = R$, as desired.

We close this section by remarking that Fuchs' divisible module ∂ (see [4, ch. VI]) is mixed and of torsion-free rank 1, and is non-splitting *if and only if the projective dimension of* Q *is* $\geqslant 2$, by [4, ch. VI, lemma 3.6]; hence a result similar to our Theorem 2 was already available, but with restrictions on R. It is also worth noting that the modules constructed in Theorem 2 have a much simpler structure than that of ∂; of course, they are reminiscent of the classical construction of the Prüfer group $H_{\omega+1}$ (see [3, 35]). Some relations can also be found between Example 6 and the mixed group in [3, example 3.100].

The global case

Our results in the first section have a natural global generalisation to certain classes of Prüfer domains. It is not possible, however, to give a generalisation without the imposition of some additional properties on the Prüfer domain that essentially make it very similar to $\mathbf{Z}$. Thus we restrict our attention here to Abelian groups.

Recall that if G is a mixed Abelian group and $x \in G \backslash tG$, then we can associate with x a height matrix $\mathbf{H}(x)$ defined as follows: let $\mathbf{P} = \{p_n : n \in \mathbf{N}\}$ be the set of all primes; then $\mathbf{H}(x) = (\sigma_{nk})$ where σ_{nk} is the ordinal or symbol ∞ that occurs as the generalised p_n-height in G of $p_n^k x$ for all n and k. We say that two such height matrices (σ_{nk}) and (ρ_{nk}) are equivalent if, for almost all n, the nth rows of the two matrices are identical and for the remaining n there exist non-negative integers h, m (depending on n) such that $\sigma_{n,k+h} = \rho_{n,k+m}$ for all k. It is well known (see [3, 103]) that if G is of torsion-free rank 1 and x, y are torsion-free elements of G, then $\mathbf{H}(x)$ and $\mathbf{H}(y)$ are equivalent. Thus, in this case, associated with G we have an invariant that is the equivalence class of height matrices of any torsion-free element. Let $\mathbf{U} = \mathbf{U}(G)$ denote this invariant. With this invariant of G we associate a set $\mathbf{P}(\mathbf{U})$ of primes as follows: $\mathbf{P}(\mathbf{U}) = \{p_n \in \mathbf{P} : \sigma_{nk} < \infty \forall k \in \mathbf{N}\}$ where $(\sigma_{nk}) \in \mathbf{U}$. With this notation we can state our global result.

Theorem 7. *Let G be a mixed Abelian group of torsion-free rank 1; then* $\mathrm{End}_W(G) = \mathbf{Z}_{\mathbf{P}(\mathbf{U})}$, *the localisation of* $\mathbf{Z}$ *at* $\mathbf{P}(\mathbf{U})$.

PROOF. Since $\mathrm{End}_W(G)$ is a subring of the rational field $\mathbf{Q}$, it will suffice to compute the characteristic of the identity in $\mathrm{End}_W(G)$. If $1_G \in p\,\mathrm{End}_W(G)$ for some $p \in \mathbf{P}$, then $\psi = 1_G - p\varphi \in \mathrm{Hom}(G, tG)$ for some endomorphism φ. Since $\ker \psi$ must then contain a torsion-free element y, we have that

$$y = p\varphi(y) = p^2\varphi^2(y) = \cdots$$

and so $ht_p(y) = \infty$, implying that $p \notin \mathbf{P}(\mathbf{U})$. Thus if $q \in \mathbf{P}(\mathbf{U})$ then $\chi(1_G)$, the characteristic of 1_G, is zero at the place corresponding to q. Equivalently we have $\mathrm{End}_W(M) \subseteq \mathbf{Z}_{\mathbf{P}(\mathbf{U})}$.

To prove the reverse inequality we make use of an observation of Warfield [5, lemma 1], which he refers to as the Abelian group theorist's 'Hasse principle'; see Lemma 8 below. Suppose that $q \in \mathbf{P} \backslash \mathbf{P}(\mathbf{U})$ and x is a torsion-free element of G such that x has infinite q-height in G. Let $x = qy$ for some $y \in G$ and take $S = \langle x \rangle$, $T = \langle y \rangle$. Then $f: S \rightarrow T$ with $f(x) = y$ is a homomorphism. Now if p is a prime distinct from q, $G_p = G \otimes \mathbf{Z}_p$ is a $\mathbf{Z}_p$-module and division by q is a homomorphism of G_p that extends the induced map f_p. Moreover, since $q \not\in \mathbf{P}(\mathbf{U})$, G_q is a split extension of tG_p by $\mathbf{Q}$, so that

$$G_q = \mathbf{Q}x \oplus tG_q = \mathbf{Q}y \oplus tG_q ,$$

and then f_q lifts to an endomorphism of G_q. Hence, by Lemma 8, there is an endomorphism g of G with $g|_S = f$. However, we now have that $(qg - 1_G)(x) = (qf - 1_G)(x) = qy - x = 0$, so that $qg - 1_G \in \mathrm{Hom}(G, tG)$. Thus 1_G is divisible in $\mathrm{End}_W(G)$ by q and so $q^{-1} \in \mathrm{End}_W(G)$; it follows that $\mathbf{Z}_{\mathbf{P}(\mathbf{U})} \subseteq \mathrm{End}_W(G)$. ■

Lemma 8 [5]. *Let G, H be groups and S and T subgroups such that G/S and H/T are torsion, and for each prime p identify $S_p = S \otimes \mathbf{Z}_p$, $T_p = T \otimes \mathbf{Z}_p$ with submodules of $G_p = G \otimes \mathbf{Z}_p$, $H_p = H \otimes \mathbf{Z}_p$, respectively. Let $f: S \rightarrow T$ be a homomorphism and $f_p: S_p \rightarrow T_p$ the induced map. Then f extends to a homomorphism from G to H if and only if, for each prime p, f_p extends to a homomorphism from G_p to H_p.*

Proof. The proof is identical to [5, lemma 1], where it is shown for localisations at the p-adic integers J_p. However, the proof by Warfield makes no use of the completion. ■

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