2010-10-01

Application of the Fractional Diffusion Equation for Predicting Market Behaviour

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Abstract—**Most financial modelling system rely on an underlying hypothesis known as the Efficient Market Hypothesis (EMH) including the famous Black-Scholes formula for placing an option. However, the EMH has a fundamental flaw; it is based on the assumption that economic processes are normally distributed and it has long been known that this is not the case. This fundamental assumption leads to a number of shortcomings associated with using the EMH to analyse financial data which includes failure to predict the future volatility of a market share value. This paper introduces a new financial risk assessment model based on Lévy statistics and considers a financial forecasting system that uses a solution to a non-stationary fractional diffusion equation characterized by the Lévy index. Variation in the Lévy index are considered in order to assess the future volatility of financial data together with the likelihood of the markets become bear or bull dominant thereby providing a solution to securing an investment portfolio. The key hypothesis associated with this approach is that a change in the Lévy index precedes a change in the financial signal from which the index is computed and can therefore be It is shown that there is a quantitative relationship between Lévy's characteristic function and a random scaling fractal signal obtained through a Green's function solution to the fractional diffusion equation. In this sense, the model considered is based on the Fractal Market Hypothesis and a case study is presented to illustrate this hypothesis by predicting the volatility associated with the foreign exchange markets.**

Keywords: Fractional diffusion, Financial signal analysis, Volatility, Non-stationary signal processing, Asset management, Foreign exchange markets

1 Introduction

In 1900, Louis Bachelier concluded that the price of a commodity today is the best estimate of its price in the future. The random behaviour of commodity prices was again noted by Working in 1934 in an analysis of time series data. In the 1950s, Kendall attempted to find periodic cycles in indices of security and commodity prices but did not find any. Prices appeared to be yesterday's price plus some random change and he suggested that price changes were independent and that prices apparently followed random walks. The majority of financial research seemed to agree; asset price changes appeared to be random and independent and so prices were taken to follow random walks. Thus, the first model of price variation was based on the sum of independent random variations often referred to as Brownian motion.

Some time later, it was noticed that the size of price movements depend on the size of the price itself. The Brownian motion model was therefore revised to include this proportionality effect and a new model developed which stated that the log price changes should be Gaussian distributed. This model is the basis for the equation

$$\frac{1}{S}\frac{d}{dt}S(t) = \sigma dX + \mu dt$$

where S is the price at time t, μ is a drift term which reflects the average rate of growth of an asset, σ is the volatility and dX is a sample from a normal distribution. In other words, the relative price change of an asset is equal to some random element plus some underlying trend component. This model is an example of a log-normal random walk and has the following important properties: (i) Statistical stationarity of price increments in which samples of data taken over equal time increments can be superimposed onto each other in a statistical sense; (ii) scaling of price where samples of data corresponding to different time increments can be suitably re-scaled such that they too, can be superimposed onto each other in a statistical sense; (iii) independence of price changes.

It is often stated that asset prices should follow Gaussian random walks because of the Efficient Market Hypothesis (EMH) [1], [2] and [3]. The EMH states that the current price of an asset fully reflects all available information relevant to it and that new information is immediately incorporated into the price. Thus, in an efficient market, models for asset pricing are concerned with the arrival of new information which is taken to be independent and random.

The EMH implies independent price increments but why should they be Gaussian distributed? A Gaussian Probability Density Function (PDF) is chosen because price

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movements are presumed to be an aggregation of smaller ones and sums of independent random contributions have a Gaussian PDF because due to the Central Limit Theorem [4], [5]. This is equivalent to arguing that all financial time series used to construct an 'averaged signal' such as the Dow Jones Industrial Average are statistically independent. However, this argument is not fully justified because it assumes that the reaction of investors to one particular stock market is independent of investors in other stock markets which, in general, will not be the case as each investor may have a common reaction to economic issues that transcend any particular stock. In other words asset management throughout the markets relies on a high degree of connectivity and the arrival of new information sends 'shocks' through the market as people react to it and then to each other's reactions. The EMH assumes that there is a rational and unique way to use available information, that all agents possess this knowledge and that any chain reaction produced by a 'shock' happens instantaneously. This is clearly not physically possible.

In common with other applications of signal analysis, in order to understand the nature of a financial signal, it is necessary to be clear about what assumptions are being made in order to develop a suitable model. It is therefore necessary to introduce some of the issues associated with financial engineering as given in the following section [6], [7] and [8].

2 The Black-Scholes Model

For many years, investment advisers focused on returns with the occasional caveat 'subject to risk'. Modern Portfolio Theory (MPT) is concerned with a trade-off between risk and return. Nearly all MPT assumes the existence of a risk-free investment, e.g. the return from depositing money in a sound financial institute or investing in equities. In order to gain more profit, the investor must accept greater risk. Why should this be so? Suppose the opportunity exists to make a guaranteed return greater than that from a conventional bank deposit say; then, no (rational) investor would invest any money with the bank. Furthermore, if he/she could also borrow money at less than the return on the alternative investment, then the investor would borrow as much money as possible to invest in the higher yielding opportunity. In response to the pressure of supply and demand, the banks would raise their interest rates. This would attract money for investment with the bank and reduce the profit made by investors who have money borrowed from the bank. (Of course, if such opportunities did arise, the banks would probably be the first to invest our savings in them.) There is elasticity in the argument because of various 'friction factors' such as transaction costs, differences in borrowing and lending rates, liquidity laws etc., but on the whole, the principle is sound because the market is saturated with arbitrageurs whose purpose is to seek out and exploit irregularities or miss-pricing.

The concept of successful arbitraging is of great importance in finance. Often loosely stated as, 'there's no such thing as a free lunch', it means that one cannot ever make an instantaneously risk-free profit. More precisely, such opportunities cannot exist for a significant length of time before prices move to eliminate them.

2.1 Financial Derivatives

As markets have grown and evolved, new trading contracts have emerged which use various tricks to manipulate risk. Derivatives are deals, the value of which is derived from (although not the same as) some underlying asset or interest rate. There are many kinds of derivatives traded on the markets today. These special deals increase the number of moves that players of the economy have available to ensure that the better players have more chance of winning. To illustrate some of the implications of the introduction of derivatives to the financial markets we consider the most simple and common derivative, namely, the option.

2.1.1 Options

An option is the right (but not the obligation) to buy (call) or sell (put) a financial instrument (such as a stock or currency, known as the 'underlying') at an agreed date in the future and at an agreed price, called the strike price. For example, consider an investor who 'speculates' that the value of an asset at price S will rise. The investor could buy shares at S, and if appropriate, sell them later at a higher price. Alternatively, the investor might buy a call option, the right to buy a share at a later date. If the asset is worth more than the strike price on expiry, the holder will be content to exercise the option, immediately sell the stock at the higher price and generate an automatic profit from the difference. The catch is that if the price is less, the holder must accept the loss of the premium paid for the option (which must be paid for at the opening of the contract). If C denotes the value of a call option and E is the strike price, the option is worth $C(S,t) = \max(S - E, 0)$.

Conversely, suppose the investor speculates that an asset is going to fall, then the investor can sell shares or buy puts. If the investor speculates by selling shares that he/she does not own (which in certain circumstances is perfectly legal in many markets), then he/she is selling 'short' and will profit from a fall in the asset. (The opposite of a short position is a 'long' position.) The principal question is how much should one pay for an option? Clearly, if the value of the asset rises, then so does the value of a call option and vice versa for put options. But how do we quantify exactly how much this gamble is worth? In previous times (prior to the Black-Scholes model which is discussed later) options were bought and sold for the value that individual traders thought they ought to have. The strike prices of these options were usually the 'forward price', which is just the current price adjusted for interest-rate effects. The value of options rises in active or volatile markets because options are more likely to pay out large amounts of money when they expire if market moves have been large, i.e. potential gains are higher, but loss is always limited to the cost of the premium. This gain through successful 'speculation' is not the only role that options play. Another role is Hedging.

2.1.2 Hedging

Suppose an investor already owns shares as a long-term investment, then he/she may wish to insure against a temporary fall in the share price by buying puts as well. Clearly, the investor would not want to liquidate holdings only to buy them back again later, possibly at a higher price if the estimate of the share price is wrong, and certainly having incurred some transaction costs on the deals. If a temporary fall occurs, the investor has the right to sell his/her holdings for a higher than market price. The investor can then immediately buy them back for less, in this way generating a profit and long-term investment then resumes. If the investor is wrong and a temporary fall does not occur, then the premium is lost for the option but at least the stock is retained, which has continued to rise in value. Since the value of a put option rises when the underlying asset value falls, what happens to a portfolio containing both assets and puts? The answer depends on the ratio. There must exist a ratio at which a small unpredictable movement in the asset does not result in any unpredictable movement in the portfolio. This ratio is instantaneously risk free. The reduction of risk by taking advantage of correlations between the option price and the underlying price is called 'hedging'. If a market maker can sell an option and hedge away all the risk for the rest of the options life, then a risk free profit is guaranteed.

Why write options? Options are usually sold by banks to companies to protect themselves against adverse movements in the underlying price, in the same way as holders do. In fact, writers of options are no different to holders; they expect to make a profit by taking a view of the market. The writers of calls are effectively taking a short position in the underlying behaviour of the markets. Known as 'bears', these agents believe the price will fall and are therefore also potential customers for puts. The agents taking the opposite view are called 'bulls'. There is a near balance of bears and bulls because if everyone expected the value of a particular asset to do the same thing, then its market price would stabilise (if a reasonable price were agreed on) or diverge (if everyone thought it would rise). Clearly, the psychology and dynamics (which must go hand in hand) of the bear/bull cycle play an important role in financial analysis.

The risk associated with individual securities can be hedged through diversification or 'spread betting' and/or various other ways of taking advantage of correlations between different derivatives of the same underlying asset. However, not all risk can be removed by diversification. To some extent, the fortunes of all companies move with the economy. Changes in the money supply, interest rates, exchange rates, taxation, commodity prices, government spending and overseas economies tend to affect all companies in one way or another. This remaining risk is generally referred to as market risk.

2.2 Black-Scholes Analysis

The value of an option can be thought of as a function of the underlying asset price S (a Gaussian random variable) and time t denoted by $V(S,t)$. Here, V can denote a call or a put; indeed, V can be the value of a whole portfolio or different options although for simplicity we can think of it as a simple call or put. Any derivative security whose value depends only on the current value S at time t and which is paid for up front, is taken to satisfy the Black-Scholes equation given by[9]

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

where σ is the volatility and r is the risk. As with other partial differential equations, an equation of this form may have many solutions. The value of an option should be unique; otherwise, again, arbitrage possibilities would arise. Therefore, to identify the appropriate solution, certain initial, final and boundary conditions need to be imposed. Take for example, a call; here the final condition comes from the arbitrage argument. At $t=T$

$$C(S,t)=\max(S-E,0)$$

The spatial or asset-price boundary conditions, applied at $S=0$ and $S\to\infty$ come from the following reasoning: If S is ever zero then dS is zero and will therefore never change. Thus, we have

$$C(0,t)=0$$

As the asset price increases it becomes more and more likely that the option will be exercised, thus we have

$$C(S,t)\propto S, \qquad S\to\infty$$

Observe, that the Black-Sholes equation has a similarity to the diffusion equation but with additional terms. An appropriate way to solve this equation is to transform it into the diffusion equation for which the solution is well known and with appropriate transformations gives the Black-Scholes formula [9]

$$C(S,t)=SN(d_1)-Ee^{r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and N is the cumulative normal distribution defined by

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{\frac{1}{2}s^2} ds.$$

The conceptual leap of the Black-Scholes model is to say that traders are not estimating the future price, but are guessing about how volatile the market may be in the future. The model therefore allows banks to define a fair value of an option, because it assumes that the forward price is the mean of the distribution of future market prices. However, this requires a good estimate of the future volatility σ.

The relatively simple and robust way of valuing options using Black-Scholes analysis has rapidly gained in popularity and has universal applications. Black-Scholes analysis for pricing an option is now so closely linked into the markets that the price of an option is usually quoted in option volatilities or 'vols'. However, Black-Scholes analysis is ultimately based on random walk models that assume independent and Gaussian distributed price changes and is thus, based on the EMH.

The theory of modern portfolio management is only valuable if we can be sure that it truly reflects reality for which tests are required. One of the principal issues with regard to this relates to the issue of assuming that the markets are Gaussian distributed. However, it has long been known that financial time series do not adhere to Gaussian statistics. This is the most important of the shortcomings relating to the EMH model (i.e. the failure of the independence and Gaussian distribution of increments assumption) and is fundamental to the inability for EMH-based analysis such as the Black-Scholes equation to explain characteristics of a financial signal such as clustering, flights and failure to explain events such as 'crashes leading to recession.

In this paper, we present an approach to analysing financial signals that is based on a non-stationary fractional diffusion equation derived under the assumption that the data are Lévy distributed. We then consider methods of solving this equation and provide an algorithm for computing the non-stationary Lévy index using a standard moving window principle. We also consider a case study in which the method is use to predict market volatility focusing on foreign currency exchange [10], [11].

3 The Classical and Fractional Diffusion Equations

For diffusivity $D = \sigma^{-1}$, the homogeneous diffusion equation is given by

$$\left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right) u(\mathbf{r}, t) = 0$$

where $\mathbf{r} \equiv (x, y, z)$ is the three-dimensional space vector and ∇^2 is the Laplacian operator given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The field $u(\mathbf{r}, t)$ represents a measurable quantity whose space-time dependence is determined by the random walk of a large ensemble of particles, a strongly scattered wavefield or information flowing through a complex network. We consider an initial value for this field denoted by $u_0 \equiv u(\mathbf{r}, 0) = u(\mathbf{r}, t)$ at $t = 0$. The diffusion equation can be derived using a random walk model for particles undergoing inelastic collisions. It is assumed that the movements of the particles are independent of the movements of all other particles and that the motion of a single particle at some interval of time is independent of its motion at all other times. This derivation (usually attributed to Einstein [12]) is given in the following section for the one-dimensional case. For completeness, the three-dimensional case is considered in Appendix A.

3.1 Derivation of the Classical one-Dimensional Diffusion Equation

Let τ be a small interval of time in which a particle moves some distance between λ and $\lambda + d\lambda$ with a probability $p(\lambda)$ where τ is long enough to assume that the movements of the particle in two separate periods of τ are independent. If n is the total number of particles and we assume that $p(\lambda)$ is constant between λ and $\lambda + d\lambda$, then the number of particles that travel a distance between λ and $\lambda + d\lambda$ in τ is given by

$$dn = np(\lambda)d\lambda$$

If $u(x, t)$ is the concentration (number of particles per unit length) then the concentration at time $t + \tau$ is described by the integral of the concentration of particles which have been displaced by λ in time τ, as described by the equation above, over all possible λ, i.e.

$$u(x, t+\tau) = \int_{-\infty}^{\infty} u(x+\lambda, t)p(\lambda)d\lambda \tag{1}$$

Since, τ is assumed to be small, we can approximate $u(x, t + \tau)$ using the Taylor series and write

$$u(x, t+\tau) \simeq u(x, t) + \tau\frac{\partial}{\partial t}u(x, t)$$

Similarly, using a Taylor series expansion of $u(x+\lambda,t)$, we have

$$u(x+\lambda,t) \simeq u(x,t) + \lambda\frac{\partial}{\partial x}u(x,t) + \frac{\lambda^2}{2!}\frac{\partial^2}{\partial x^2}u(x,t)$$

where the higher order terms are neglected under the assumption that if τ is small, then the distance travelled, λ, must also be small. We can then write

$$u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) = u(x,t)\int_{-\infty}^{\infty} p(\lambda)d\lambda$$

$$+\frac{\partial}{\partial x}u(x,t)\int_{-\infty}^{\infty}\lambda p(\lambda)d\lambda + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t)\int_{-\infty}^{\infty}\lambda^2 p(\lambda)d\lambda$$

For isotropic diffusion, $p(\lambda) = p(-\lambda)$ and so p is an even function with usual normalization condition

$$\int_{-\infty}^{\infty} p(\lambda)d\lambda = 1$$

As λ is an odd function, the product $\lambda p(\lambda)$ is also an odd function which, if integrated over all values of λ, equates to zero. Thus we can write

$$u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) = u(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t)\int_{-\infty}^{\infty}\lambda^2 p(\lambda)d\lambda$$

so that

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)\int_{-\infty}^{\infty}\frac{\lambda^2}{2\tau}p(\lambda)d\lambda$$

Finally, defining the diffusivity as

$$D = \int_{-\infty}^{\infty}\frac{\lambda^2}{2\tau}p(\lambda)d\lambda$$

we obtain the diffusion equation

$$\left(\frac{\partial^2}{\partial x^2} - \sigma\frac{\partial}{\partial t}\right)u(x,t) = 0$$

where $\sigma = 1/D$. Note that this derivation does not depend explicitly on $p(\lambda)$. However, there is another approach to deriving this result that is informative with regard to the discussion given in the following section and is determined by $p(\lambda)$. Under the condition that $p(\lambda)$ is a symmetric function, equation (1) - a correlation integral - is equivalent to a convolution integral. Thus, using the convolution theorem, in Fourier space, equation (1) becomes

$$U(k,t+\tau) = U(k,t)P(k)$$

where U and P are the Fourier transforms of u and p given by

$$U(k,t+\tau) = \int_{-\infty}^{\infty} u(x,t+\tau)\exp(-ikx)dx$$

and

$$P(k) = \int_{-\infty}^{\infty} p(x)\exp(-ikx)dx$$

respectively. Suppose we consider a Probability Density Function (PDF) $p(x)$ that is Gaussian distributed. Then the Characteristic Function $P(k)$ is also Gaussian given by say (ignoring scaling constants)

$$P(k) = \exp(-a \mid k \mid^2) = 1 - a \mid k \mid^2 + ...$$

Let

$$P(k) = 1 - a \mid k \mid^2, \quad a \to 0$$

We can then write

$$\frac{U(k,t+\tau) - U(k,t)}{\tau} = -\frac{a}{\tau} \mid k \mid^2 U(k,t)$$

so that as $\tau \to 0$ we obtain the equation

$$\sigma\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$$

where $\sigma = \tau/a$ and we have used the result

$$\frac{\partial^2}{\partial x^2}u(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k^2 U(k,t)\exp(ikx)dk$$

This approach to deriving the diffusion equation relies on specifying the characteristic function $P(k)$ and upon the conditions that both a and τ approach zero, thereby allowing $\sigma = \tau/a$ to be of arbitrary value. This is the basis for the approach considered in the following section with regard to a derivation of the anomalous or fractional diffusion equation.

3.2 Derivation of the Fractional Diffusion Equation for a Lévy Distributed Process

Lévy processes are random walks whose distribution has infinite moments. The statistics of (conventional) physical systems are usually concerned with stochastic fields that have PDFs where (at least) the first two moments (the mean and variance) are well defined and finite. Lévy statistics is concerned with statistical systems where all the moments (starting with the mean) are infinite. Many distributions exist where the mean and variance are finite but are not representative of the process, e.g. the tail of the distribution is significant, where rare but extreme events occur. These distributions include Lévy distributions [13],[14]. Lévy's original approach to deriving such distributions is based on the following question: Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine. The characteristic function $P(k)$ of such a distribution $p(x)$ was first shown by Lévy to be given by (for symmetric distributions only)

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2 \tag{2}$$

where a is a constant and γ is the Lévy index. For $\gamma \geq 2$, the second moment of the Lévy distribution exists and the sums of large numbers of independent trials are Gaussian distributed. For example, if the result were a random walk with a step length distribution governed by $p(x), \quad \gamma \geq 2$, then the result would be normal (Gaussian) diffusion, i.e. a Brownian random walk process. For $\gamma < 2$ the second moment of this PDF (the mean square), diverges and the characteristic scale of the walk is lost. For values of γ between 0 and 2, Lévy's characteristic function corresponds to a PDF of the form (see Appendix B)

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

Lévy processes are consistent with a fractional diffusion equation as we shall now show [15]. Consider the evolution equation for a random walk process to be given by equation (1) which, in Fourier space, is

$$U(k, t + \tau) = U(k, t)P(k)$$

From equation (2),

$$P(k) = 1 - a \mid k \mid^{\gamma}, \quad a \to 0$$

so that we can write

$$\frac{U(k, t + \tau) - U(k, t)}{\tau} \simeq -\frac{a}{\tau} \mid k \mid^{\gamma} U(k, t)$$

which for $\tau \to 0$ gives the fractional diffusion equation

$$\sigma \frac{\partial}{\partial t} u(x, t) = \frac{\partial^{\gamma}}{\partial x^{\gamma}} u(x, t), \quad \gamma \in (0, 2] \tag{3}$$

where $\sigma = \tau / a$ and we have used the result

$$\frac{\partial^{\gamma}}{\partial x^{\gamma}} u(x, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mid k \mid^{\gamma} U(k, t) \exp(ikx) dk \tag{4}$$

The solution to this equation with the singular initial condition $u(x, 0) = \delta(x)$ is given by

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx - t \mid k \mid^{\gamma} / \sigma) dk$$

which is itself Lévy distributed. This derivation of the fractional diffusion equation reveals its physical origin in terms of Lévy statistics.

3.3 Generalisation

The approach used to derived the fractional diffusion equation given in the previous section can be generalised further for arbitrary PDFs. Applying the correlation theorem to equation (1) we note that

$$U(k, t + \tau) = U(k, t)P^*(k)$$

where the characteristic function $P(k)$ may be asymmetric. Then

$$U(k, t + \tau) - U(k, t) = U(k, t)[P^*(k) - 1]$$

so that as $\tau \to 0$ we obtained a generalised anomalous diffusion equation given by

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{\tau} \left(\int_{-\infty}^{\infty} u(x + y, t) p(y) dy - u(x, t) \right)$$

3.4 Green's Function for the Fractional Diffusion Equation

Let

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x, \omega) \exp(i\omega t) d\omega$$

where ω is the angular frequency so that equation (3) can be written as

$$\left(\frac{\partial^{\gamma}}{\partial x^{\gamma}} + \Omega^2 \right) U(x, \omega) = 0$$

where $\Omega^2 = -i\omega\sigma$ and we choose the positive root $\Omega = i(i\omega\sigma)^{\frac{1}{2}}$. For an ideal impulse located at x_0, the Green's function g is then defined in terms of the solution of [16]

$$\left(\frac{\partial^{\gamma}}{\partial x^{\gamma}} + \Omega^2 \right) g(x \mid x_0, \omega) = -\delta(x - x_0)$$

Fourier transforming this equation with regard to x and using equation (4), we obtain an expression for g given by (with $X \equiv \mid x - x_0 \mid$)

$$g(X, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ikX) dk}{(k^{\frac{\gamma}{2}} - \Omega)(k^{\frac{\gamma}{2}} + \Omega)}$$

The integral has two roots at $k = \pm\Omega^{\frac{2}{\gamma}}$ and for $k = \Omega^{\frac{2}{\gamma}}$, the Green's function is given by

$$g(x \mid x_0, \omega) = \frac{i}{2\Omega_{\gamma}} \exp(i\Omega_{\gamma} \mid x - x_0 \mid)$$

where

$$\Omega_{\gamma} = i^{\frac{2}{\gamma}} (i\omega\sigma)^{1/\gamma}$$

Suppose we consider the fractional diffusion equation

$$\left(\frac{\partial^2}{\partial x^2} - \sigma^q \frac{\partial^q}{\partial t^q} \right) u(x, t)$$

where we call q the 'Fourier Dimension'. Using the result

$$\frac{\partial^q}{\partial t^q}u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(x,\omega)(i\omega)^q\exp(i\omega t)d\omega$$

the Green's function is defined as the solution to

$$\left(\frac{\partial^2}{\partial x^2}+\Omega_q^2\right)g(x\mid x_0,\omega)=\delta(x-x_0)$$

where (ignoring the negative root)

$$\Omega_q=i(i\omega\sigma)^{\frac{q}{2}}$$

In this case, the Green's function is given by

$$g(x\mid x_0,\omega)=\frac{i}{2\Omega_q}\exp(i\Omega_q\mid x-x_0\mid)$$

This analysis provides a relationship between the Lévy index and the Fourier Dimension given by

$$\frac{1}{\gamma}=\frac{q}{2}$$

Gaussian processes associated with the classical diffusion equation are thus recovered when $\gamma=2$ and $q=1$ and $\gamma\in(0,2]\equiv q\in(\infty,1]$

3.4.1 Green's Function for $q=2$

When $q=2$, we recover the Green's function for the wave equation For $\Omega_2=-\omega\sigma$, we have

$$g(X,\omega)=\frac{1}{2i\omega\sigma}\exp(-i\omega\sigma X)$$

Fourier inverting, using the convolution theorem and noting that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\omega\sigma X)\exp(i\omega t)d\omega=\delta(t-\sigma X)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2i\omega}\exp(i\omega t)d\omega=\frac{1}{4}\text{sgn}(t)$$

where

$$\text{sgn}(t)=\begin{cases}+1, & t<0\\ -1, & t>0\end{cases}$$

we obtain an expression for the time-dependent Green's function given by

$$G(x\mid x_0,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(x\mid x_0,\omega)\exp(i\omega t)d\omega$$

$$=\frac{1}{4\sigma}\text{sgn}(t-\sigma\mid x-x_0\mid)$$

which describes the propagation of a wave travelling at velocity $1/\sigma$ with a wavefront that occurs at $t=\sigma\mid x-x_0\mid$.

3.4.2 Green's Function for $q=1$

For $q=1$, $\Omega_1=i\sqrt{i\omega\sigma}$ and

$$G(X,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-\sqrt{i\omega\sigma}X)}{2\sqrt{i\omega\sigma}}\exp(i\omega t)d\omega$$

Using the result

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(-a\sqrt{p})}{2\sqrt{p}}\exp(pt)dp=\frac{1}{\sqrt{\pi t}}\exp[-a^2/(4t)]$$

where a is a constant, then, with $p=i\omega$ we obtain

$$G(x\mid x_0,t)=\frac{1}{\sqrt{\pi\sigma t}}\exp[-\sigma(x-x_0)^2/(4t)],\quad t>0$$

which is the Green's function for the classical diffusion equation, i.e. a Gaussian function.

4 Green's Function Solution to the Fractional Diffusion Equation for a Seperable Stochastic Source Function

We consider a Green's function solution to the equation

$$\left(\frac{\partial^2}{\partial x^2}-\sigma^q\frac{\partial^q}{\partial t^q}\right)u(x,t)=-F(x,t)$$

when $F(x,t)=f(x)n(t)$ where $f(x)$ and $n(t)$ are stochastic functions described by PDFs $\Pr[f(x)]$ and $\Pr[n(t)]$ respectively. Although a Green's function solution does not require the source function to be separable, utilising a separable function in this way allows a solution to be generated in which the terms affecting the temporal behaviour of $u(x,t)$ are clearly identifiable. Although we consider the fractional diffusion equation for values of $q\in(1,\infty)$, for generality, we consider a solution for $q\in(-\infty,\infty)$. Thus, we require a general solution to the equation

$$\left(\frac{\partial^2}{\partial x^2}-\sigma^q\frac{\partial^q}{\partial t^q}\right)u(x,t)=-f(x)n(t),\quad -\infty\le q\le\infty$$

Let

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(x,\omega)\exp(i\omega t)d\omega$$

and

$$n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}N(\omega)\exp(i\omega t)d\omega$$

Then, using the result

$$\frac{\partial^q}{\partial t^q}u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(x,\omega)(i\omega)^q\exp(i\omega t)d\omega$$

we can transform the fractional diffusion equation to the form

$$\left(\frac{\partial^2}{\partial x^2}+\Omega_q^2\right)U(x,\omega)=-f(x)N(\omega)$$

The Green's function solution is then given by

$$U(x_0,\omega)=N(\omega)\int\limits_{-\infty}^{\infty}g(x\mid x_0,\omega)f(x)dx \qquad (5)$$

under the assumption that u and $\partial u/\partial x\to 0$ as $x\to\pm\infty$.

4.1 General Series Solution

The evaluation of $u(x_0,t)$ via direct Fourier inversion for arbitrary values of q is not possible because of the irrational nature of the Green's function with respect to ω. To obtain a general solution, we use the series representation of the exponential function and write

$$U(x_0,\omega)=\frac{iM_0N(\omega)}{2\Omega_q}\left[1+\sum_{m=1}^{\infty}\frac{(i\Omega_q)^m}{m!}\frac{M_m(x_0)}{M_0}\right] \qquad (6)$$

where

$$M_m(x_0)=\int\limits_{-\infty}^{\infty}f(x)\mid x-x_0\mid^m dx$$

We can now Fourier invert term by term to develop a series solution. Given that we consider $-\infty<q<\infty$, this requires us to consider three distinct cases.

4.1.1 Solution for $q=0$

Evaluation of $u(x_0,t)$ in this case is trivial since, from equation (5)

$$U(x_0,\omega)=\frac{M(x_0)}{2}N(\omega)\quad\text{or}\quad u(x_0,t)=\frac{M(x_0)}{2}n(t)$$

where

$$M(x_0)=\int\limits_{-\infty}^{\infty}\exp(-\mid x-x_0\mid)f(x)dx$$

4.1.2 Solution for $q>0$

Fourier inverting, the first term in equation (6) becomes

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{iN(\omega)M_0}{2\Omega_q}\exp(i\omega t)d\omega=$$

$$\frac{M_0}{2\sigma^{\frac{q}{2}}}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{N(\omega)}{(i\omega)^{\frac{q}{2}}}\exp(i\omega t)d\omega$$

$$=\frac{M_0}{2\sigma^{\frac{q}{2}}}\frac{1}{(2i)^q\sqrt{\pi}}\frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\int\limits_{-\infty}^{\infty}\frac{n(\tau)}{(t-\tau)^{1-(q/2)}}d\tau$$

The second term is

$$-\frac{M_1}{2}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}N(\omega)\exp(i\omega t)d\omega=-\frac{M_1}{2}n(t)$$

The third term is

$$-\frac{iM_2}{2.2!}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}N(\omega)i(i\omega\sigma)^{\frac{q}{2}}\exp(i\omega t)d\omega=\frac{M_2\sigma^{\frac{q}{2}}}{2.2!}\frac{d^{\frac{q}{2}}}{dt^{\frac{q}{2}}}n(t)$$

and the fourth and fifth terms become

$$\frac{M_3}{2.3!}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}N(\omega)i^2(i\omega\sigma)^q\exp(i\omega t)d\omega=-\frac{M_3\sigma^q}{2.3!}\frac{d^q}{dt^q}n(t)$$

and

$$i\frac{M_4}{2.4!}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}N(\omega)i^3(i\omega\sigma)^{\frac{3q}{2}}\exp(i\omega t)d\omega=\frac{M_4\sigma^{\frac{3q}{2}}}{2.4!}\frac{d^{\frac{3q}{2}}}{dt^{\frac{3q}{2}}}n(t)$$

respectively with similar results for all other terms. Thus, through induction, we can write $u(x_0,t)$ as a series of the form

$$u(x_0,t)=$$

$$\frac{M_0(x_0)}{2\sigma^{q/2}}\frac{1}{(2i)^q\sqrt{\pi}}\frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\int\limits_{-\infty}^{\infty}\frac{n(\tau)}{(t-\tau)^{1-(q/2)}}d\tau$$

$$-\frac{M_1(x_0)}{2}n(t)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}M_{k+1}(x_0)\sigma^{kq/2}\frac{d^{kq/2}}{dt^{kq/2}}n(t)$$

Observe that the first term involves a fractional integral (the Riemann-Liouville integral), the second term is composed of the source function $n(t)$ alone (apart from scaling) and the third term is an infinite series composed of fractional differentials of increasing order $kq/2$. Also note that the first term is scaled by a factor involving $\sigma^{-q/2}$ whereas the third term is scaled by a factor that includes $\sigma^{kq/2}$ so that

$$\lim_{\sigma\to 0}u(x_0,t)=-\frac{M_1(x_0)}{2}n(t)$$

$$+\frac{M_0(x_0)}{2\sigma^{q/2}}\frac{1}{(2i)^q\sqrt{\pi}}\frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\int\limits_{-\infty}^{\infty}\frac{n(\tau)}{(t-\tau)^{1-(q/2)}}d\tau$$

4.1.3 Solution for $q < 0$

In this case, the first term becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{iN(\omega)M_0}{2\Omega_q}\exp(i\omega t)d\omega$$

$$= \frac{M_0}{2}\sigma^{\frac{q}{2}}\frac{1}{2\pi}\int_{-\infty}^{\infty}N(\omega)(i\omega)^{\frac{q}{2}}\exp(i\omega t)d\omega = \frac{M_0}{2}\sigma^{\frac{q}{2}}\frac{d^{\frac{q}{2}}}{dt^{\frac{q}{2}}}n(t)$$

The second term is the same is in the previous case (for $q > 0$) and the third term is

$$-\frac{iM_2}{2.2!}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N(\omega)i}{(i\omega\sigma)^{\frac{q}{2}}}\exp(i\omega t)d\omega$$

$$= \frac{M_2}{2.2!}\frac{1}{\sigma^{q/2}}\frac{1}{(2i)^q\sqrt{\pi}}\frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\int_{-\infty}^{\infty}\frac{n(\tau)}{(t-\tau)^{1-(q/2)}}d\tau$$

Evaluating the other terms, by induction we obtain

$$u(x_0,t) = \frac{M_0(x_0)\sigma^{q/2}}{2}\frac{d^{q/2}}{dt^{q/2}}n(t) - \frac{M_1(x_0)}{2}n(t)$$

$$+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}\frac{M_{k+1}(x_0)}{\sigma^{kq/2}}\frac{1}{(2i)^{kq}\sqrt{\pi}}\frac{\Gamma\left(\frac{1-kq}{2}\right)}{\Gamma\left(\frac{kq}{2}\right)}$$

$$\times\int_{-\infty}^{\infty}\frac{n(\tau)}{(t-\tau)^{1-(kq/2)}}d\tau$$

Here, the solution is composed of three terms: a fractional differential, the source term and an infinite series of fractional integrals of order $kq/2$. Thus, the roles of fractional differentiation and fractional integration are reversed as q changes from being greater than to less than zero.

4.2 Fractional Differentials

Fractional differentials $\hat{D}^q$ of any order q need to be considered in terms of the definition for a fractional differential given by

$$\hat{D}^q f(t) = \frac{d^n}{dt^n}[\hat{I}^{n-q}f(t)], \quad n - q > 0$$

where n is an integer and $\hat{I}$ is the fractional integral operator (the Riemann-Liouville transform),

$$\hat{I}^p f(t) = \frac{1}{\Gamma(p)}\int_{-\infty}^{t}\frac{f(\tau)}{(t-\tau)^{1-p}}d\tau, \quad p > 0$$

The reason for this is that direct fractional differentiation can lead to divergent integrals. However, there is a deeper interpretation of this result that has a synergy with the issue over whether a fractionally diffusive system has 'memory' and is based on observing that the evaluation of a fractional differential operator depends on the history of the function in question. Thus, unlike an integer differential operator of order n, a fractional differential operator of order q has 'memory' because the value of $\hat{I}^{q-n}f(t)$ at a time t depends on the behaviour of $f(t)$ from $-\infty$ to t via the convolution with $t^{(n-q)-1}/\Gamma(n-q)$. The convolution process is of course dependent on the history of a function $f(t)$ for a given kernel and thus, in this context, we can consider a fractional derivative defined via the result above to have memory. In this sense, the operator

$$\frac{\partial^2}{\partial x^2} - \sigma^q\frac{\partial^q}{\partial t^q}$$

describes a process that has a memory association with the temporal characteristics of the system it is attempting to model. This is not an intrinsic characteristic of systems that are purely diffusive $q = 1$ or propagative $q = 2$.

4.3 Asymptotic Solutions for an Impulse

We consider a special case in which the source function $f(x)$ is an impulse so that

$$M_m(x_0) = \int_{-\infty}^{\infty}\delta(x) \mid x - x_0 \mid^m dx = \mid x_0 \mid^m$$

This result immediately suggests a study of the asymptotic solution

$$u(t) = \lim_{x_0\to 0}u(x_0,t)$$

$$= \begin{cases} \frac{1}{2\sigma^{q/2}}\frac{1}{(2i)^q\sqrt{\pi}}\frac{\Gamma\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\int_{-\infty}^{\infty}\frac{n(\tau)}{(t-\tau)^{1-(q/2)}}d\tau, & q > 0; \\ \frac{n(t)}{2}, & q = 0; \\ \frac{\sigma^{q/2}}{2}\frac{d^{q/2}}{dt^{q/2}}n(t), & q < 0. \end{cases}$$

The solution for the time variations of the stochastic field u for $q > 0$ are then given by a fractional integral alone and for $q < 0$ by a fractional differential alone. In particular, for $q > 0$, we see that the solution is based on the convolution integral (ignoring scaling)

$$u(t) = \frac{1}{t^{1-q/2}} \otimes n(t), \quad q > 0 \tag{7}$$

where $\otimes$ denotes convolution integral over t which in ω-space is given by (ignoring scaling)

$$U(\omega) = \frac{N(\omega)}{(i\omega)^{q/2}}$$

This result is the conventional random fractal noise model for Fourier Dimension q [17], [18] and [19]. Table 1 quantifies the results for different values of q with conventional name associations. Note that Brown noise conventionally refers to the integration of white noise but that Brownian motion is a form of pink noise because it classifies diffusive processes identified by the case when $q = 1$. The field u has the following fundamental property for $q > 0$:

$$\lambda^{q/2}\Pr[u(t)] = \Pr[u(\lambda t)]$$

This property describes the statistical self-affinity of u. Thus, the asymptotic solution considered here, yields a result that describes a random scaling fractal field characterized by a Power Spectral Density Function (PSDF) of the form $1/\mid\omega\mid^q$ which is a measure of the time correlations in the signal.

q-value	t-space	ω-space (PSDF)	Name
$q = 0$	$\frac{1}{t}\otimes n(t)$	1	White noise
$q = 1$	$\frac{1}{\sqrt{t}}\otimes n(t)$	$\frac{1}{\lvert\omega\rvert}$	Pink noise
$q = 2$	$\int^t n(t)dt$	$\frac{1}{\omega^2}$	Brown noise
$q > 2$	$t^{(q/2)-1}\otimes n(t)$	$\frac{1}{\lvert\omega\rvert^q}$	Black noise

Table 1: Noise characteristics for different values of q. Note that the results given above ignore scaling factors.

Note that $q = 0$ defines the Hilbert transform of $n(t)$ whose spectral properties in the positive half space are identical to $n(t)$ because

$$\frac{1}{t}\otimes n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[-i\pi\text{sign}(\omega)]N(\omega)\exp(i\omega t)d\omega$$

where

$$\text{sgn}(\omega) = \begin{cases} 1, & \omega > 0; \\ -1, & \omega < 0. \end{cases}$$

The statistical properties of the Hilbert transform of $n(t)$ are therefore the same as $n(t)$ so that

$$\Pr[t^{-1}\otimes n(t)] = \Pr[n(t)]$$

Hence, as $q \to 0$, the statistical properties of $u(t)$ 'reflect' those of n, i.e.

$$\Pr\left[\frac{1}{t^{1-q/2}}\otimes n(t)\right] = \Pr[n(t)], \quad q \to 0$$

However, as $q \to 2$ we can expect the statistical properties of $u(t)$ to be such that the width of the PDF of $u(t)$ is reduced. This reflects the greater level of coherence (persistence in time) associated with the stochastic field $u(t)$ for $q \to 2$.

The asymptotic solution compounded in equation (7) is equivalent to considering a model for u based on the equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^q}{\partial t^q}\right)u(x,t) = \delta(x)n(t), \quad q > 0, \quad x \to 0 \quad (8)$$

where $n(t)$ is 'white noise'. However, we note that for $\mid x \mid > 0$ the function $\exp(i\Omega_q \mid x \mid)$ does not affect the ω^{-q} scaling law of the power spectrum, i.e. $\forall x$,

$$\mid U(x,\omega)\mid^2 = \frac{\mid N(\omega)\mid^2}{4\sigma^q\omega^q}, \quad \omega > 0$$

Thus for a 'white noise' spectrum $N(\omega)$ the Power Spectrum Density Function of U is determined by ω^{-q} and any algorithm for computing an estimate of q given $\mid U(\omega)\mid^2$ therefore applies $\forall x$ and not just for the case when $x \to 0$. Since we can write

$$U(x,\omega) = N(\omega)\frac{i}{2\Omega_q}\exp(i\Omega_q \mid x \mid) = N(\omega)\frac{1}{2(i\omega\sigma)^{q/2}}$$

$$\times\left(1 + i(i\omega\sigma)^{q/2}\mid x \mid - \frac{1}{2!}(i\omega\sigma)^q \mid x \mid^2 + ...\right)$$

unconditionally, by inverse Fourier transforming, we obtain the following expression for $u(x,t)$ (ignoring scaling factors including σ):

$$u(x,t) = n(t)\otimes\frac{1}{t^{1-q/2}} + i\mid x \mid n(t)$$

$$+\sum_{k=1}^{\infty}\frac{i^{k+1}}{(k+1)!}\mid x \mid^{2k}\frac{d^{kq/2}}{dt^{kq/2}}n(t)$$

Here, the solution is composed of three terms composed of (i) a fractional integral, (ii) the source term $n(t)$; (iii) an infinite series of fractional differentials of order $kq/2$.

4.4 Non-stationary Model

Real financial signals are non-stationary. An example of this is illustrated in Figure 1 which shows a non-stationary walk in the complex plane obtained by taking the Hilbert transform of the corresponding signal, i.e. computing the analytic signal

$$s(t) = u(t) + \frac{i}{\pi t}\otimes u(t)$$

and plotting the real and imaginary component of this signal in the complex plane.

The fractional diffusion operator in equation (8) is appropriate for modelling fractional diffusive processes that are stationary. Since financial signals are highly non-stationary, the model must therefore be extended to the non-stationary case if it is to be of value for financial signal analysis. In developing a non-stationary fractional

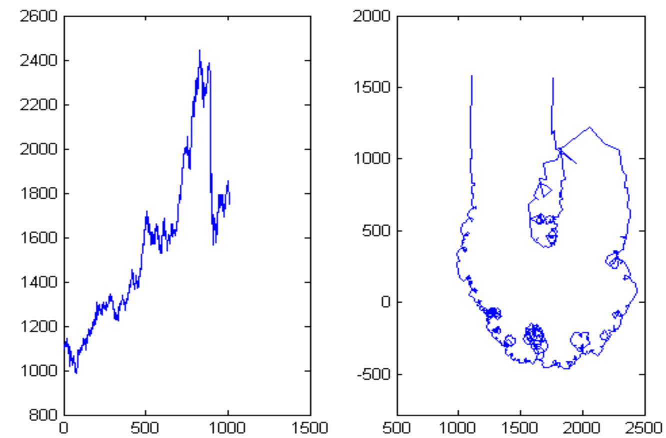

Figure 1: Non-stationary behaviour in the complex plane (right) obtained by computig the Hilbert transform of the signal (left) - FTSE Close-of-Day from 02-04-1984 to 24-12-1987.

diffusive model, we could consider the case where the diffusivity is time variant as defined by the function $\sigma(t)$. However, a more interesting case arises when the characteristics of the diffusion process itself change over time becoming more or less diffusive as $q \to 1$ and $q \to 2$ respectively. In thise sense we consider q to be a function of time τ and introduce a non-stationary fractional diffusion operator given by

$$\frac{\partial^2}{\partial x^2} - \sigma^{q(\tau)} \frac{\partial^{q(\tau)}}{\partial t^{q(\tau)}}$$

This operator is the theoretical basis for the results presented in this paper. In terms of using this model to develop an asset management metric that is predictive of future behaviour we consider the following hypothesis: a change in $q(\tau)$ precedes an associated change in a financial signal. The goal of this model/hypothesis is to be able to compute a function - namely $q(\tau)$ - which is a measure of the non-stationary behaviour of an economic signal with regard to its future behaviour. This is because, in principle, the value of $q(\tau)$ should reflect the early stages of a change in the behaviour of $u(t)$ provided an accurate enough numerical algorithm for computing $q(\tau)$ is available. This is discussed in the following section.

5 Financial Signal Analysis

If we consider the case where $q(\tau)$ is a relatively slowly varying function of time so that

$$\left| \frac{\partial q}{\partial \tau} \right| << \left| \frac{\partial u}{\partial t} \right|$$

then we can consider $q(\tau)$ to be constant over a 'window of time' τ. Ignoring scaling, for a quasi-stationary segment of a financial signal,

$$u(t,\tau) = \frac{1}{t^{1-q(\tau)/2}} \otimes n(t), \quad q > 0$$

which has characteristic spectrum

$$U(\omega,\tau) = \frac{N(\omega)}{(i\omega)^{q(\tau)/2}}$$

The PSDF is characterised by $\omega^{-q(\tau)}, \ \ \omega > 0$ and our problem is thus, to compute $q(\tau)$ from the data

$$P(\omega,\tau) = \mid U(\omega,\tau) \mid^2 = \frac{\mid N(\omega) \mid^2}{\omega^{q(\tau)}}, \quad > 0$$

. Consider the PSDF

$$\hat{P}(\omega,\tau) = \frac{c}{\omega^{q(\tau)}}$$

with logarithmic transformation

$$\ln \hat{P}(\omega,\tau) = C + q(\tau) \ln \omega$$

where $C = \ln c$. The problem is then reduced to implementing an appropriate method to compute q (and C) by finding a best fit of the line $\ln \hat{P}(\omega)$ to the data $\ln P(\omega)$. Application of the least squares method for computing q for a given τ, which is based on minimizing the error

$$e(q,C) = \| \ln P(\omega) - \ln \hat{P}(\omega,q,C) \|_2^2$$

with regard to q and C, leads to errors in the estimates for q. The reason for this is that relative errors at the start and end of the data $\ln P$ may vary significantly especially because any errors inherent in the data P will be 'amplified' through application of the logarithmic transform required to linearise the problem. In general, application of a least squares approach is very sensitive to statistical heterogeneity [20] and may provide values of q that are not compatible with the rationale associated with the model (i.e. values of $1 < q < 2$ that are intermediate between diffusive and propagative processes). For this reason, an alternative approach is considered which, in this paper, is based on Orthogonal Linear Regression (OLR) [21].

Applying a standard moving window, $q(\tau)$ is computed by repeated application of OLR based on the m-code available from [22]. This provides a numerical estimate of the function $q(\tau)$ whose behaviour 'reflects' the 'state' of the financial signal. Since q is, in effect, a statistic, its computation is only as good as the quantity (and quality) of data that is available for its computation. For this reason, a relatively large window is required whose length is compatible with the number of samples available.

5.1 Numerical Algorithm

The principal numerical algorithm associated with the application of the model as follows:

Step 1: Read data (financial time series) from file into operating array $a[i], i = 1, 2, ..., N$.

Step 2: Set length $L < N$ of moving window w to be used.

Step 3: For $j = 1$ assign $L + j - 1$ elements of $a[i]$ to array $w[i], i = 1, 2, ..., L$.

Step 4: Compute the power spectrum $P[i]$ of $w[i]$ using a Discrete Fourier Transform (DFT).

Step 5: Compute the logarithm of the spectrum excluding the DC, i.e. compute $\log(P[i]) \forall i \in [2, L/2]$.

Step 6: Compute $q[j]$ using the OLR algorithm.

Step 7: For $j = j + 1$ repeat Step 3 - Step 5 stopping when $j = N - L$.

Step 8: Write the signal $q[j]$ to file for further analysis and post processing.

The following points should be noted:

(i) The DFT is taken to generate an output in standard form where the zero frequency component of the power spectrum is taken to be $P[1]$.

(ii) With $L = 2^m$ for integer m, a Fast Fourier Transform can be used

(iii) The minimum window size that should be used in order provide statistically significant values of $q[j]$ is $L = 64$ when q can be computed accurate to 2 decimal places.

An example of the output generated by this algorithm for a 1024 element 'look-back' window is given in Figure 2 using Dow Jones Industrial Average Close-of-Day data obtained from [23]. Table 2 provides some basic statistical information with regard to $q(\tau)$ for this data whose mean value is 1.27. Application of the Null Hypothesis test with regard to whether or not $q(\tau)$ is Gaussian distributed is negative, i.e. the 'Composite Normality' is of type 'Reject'. Thus, $q(\tau)$ is not normally distributed.

A closer inspection of data such as that given in Figure 2 reveals a qualitative relationship between trends in the financial signal $u(t)$ and $q(\tau)$ in accordance with the theoretical model considered. In particular, over periods of time in which $q(\tau)$ increases in value, the amplitude of the financial signal $u(t)$ decreases. Moreover, and more importantly, an upward trend in $q(\tau)$ appears to be a precursor to a downward trend in $u(t)$, a correlation that

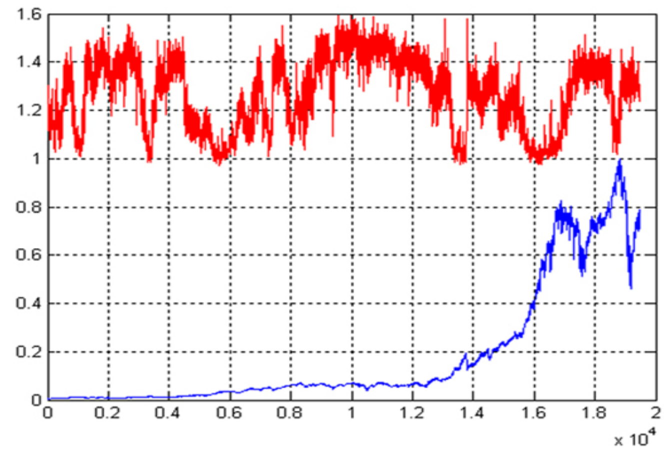

Figure 2: Computation of $q(\tau)$ (red) using a 1024 element window obtained from a time series composed of Dow Jones Industrial Average Close-of-Day data $u(t)$ from from 02-11-1932 to 24-05-2010 after normalisation (blue).

is compatible with the idea that a rise in the value of $q(\tau)$ relates to the 'system' becoming more propagative, which in stock market terms, indicates the likelihood for the markets becoming 'bear' dominant in the future.

Results of the type given in Figure 2 not only provides for a general appraisal of different macroeconomic financial time series, but, with regard to the size of the selected window used, an analysis of data at any point in time. The output can be interpreted in terms of 'persistence' and 'anti-persistence' and in terms of the existence or absence of after-effects (macroeconomic memory effects). For those periods in time when $q(\tau)$ is relatively constant, the existing market tendencies usually remain. Changes in the existing trends tend to occur just after relatively sharp changes in $q(\tau)$ have developed. This behaviour indicates the possibility of using the time series $q(\tau)$ for identifying the behaviour of a macroeconomic financial system in terms of both inter-market and between-market analysis. These results support the possibility of using $q(\tau)$ as an independent volatility predictor to give a risk assessment associated with the likely future behaviour of different time series. Further, because this analysis is based on a self-affine model, the interpretation of a financial signal via $q(\tau)$ should, in principle, be scale invariant.

5.2 Macrotrend Analysis

In order to develop a macrotrend signal that has optimal properties with regard the assessment of risk in terms of the likely long-term future behaviour of a financial signal, it is important that the filter used is: (i) consistent with the properties of a Variation Diminishing Smoothing Kernel (VDSK); (ii) that the last few values of the trend signal are 'data consistent'. VDSKs are convolu-

Statistical Parameter	Value for $q(\tau)$
Minimum Value	0.9702
Maximum value	1.5947
Range	0.6245
Mean	1.2662
Median	1.2936
Standard Deviation	0.1444
Variance	0.0209
Skew	-0.2422
Kertosis	1.9043
Composite Normality	Reject

Table 2: Statistical parameters associated with $q(\tau)$ computed for the Dow Jone Industrial Average Close-of-Day data from 02-11-1928 to 24-05-2010 given in Figures 2

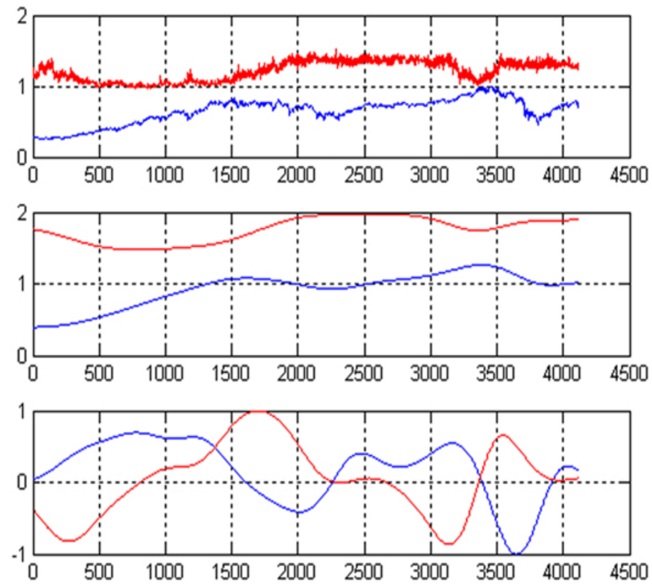

Figure 3: Analysis of Dow Jones Industrial Average Close-of-Day data from 18-01-1994 to 25-05-2010. Top: Data $u(t)$ (blue) and $q(\tau)$ (red) computed using a window of 1024; Centre: Associated macrotrends obtained for $\alpha = 0.01$; Bottom: Normalised gradients of macrotrends computed using a forward differencing scheme.

tion kernels with properties that guarantee smoothness around points of discontinuity of a given signal where the smoothed function is composed of a similar succession of concave or convex arcs equal in number to those of the signal. VDSKs also have 'geometric properties' that preserve the 'shape' of the signal. There are a range of VDSKs including the Gaussian function and, for completeness, Appendix C provides an overview of the principal analytical properties, including some basic Theorem's and Proof's relating to VDSKs.

In practice, the computation of the smoothing process using a VDSK must be performed in such a way that the initial and final elements of the output data are entirely data consistent with the input array in the locality of any element. Since a VDSK is a non-localised filter which tends to zero at infinity (see Appendix C), in order to optimise the numerical efficiency of the smoothing process, filtering is undertaken in Fourier space. However, in order to produce a data consistent macrotrend signal using a Discrete Fourier Transform, wrapping effects must be eliminated. The solution is to apply an 'end point extension' scheme which involves padding the input vector with elements equal to the first and last values of the vector. The length of the 'padding vectors' is taken to be at least half the size of the input vector. The output vector is obtained by deleting the filtered padding vectors.

Figures 3 is an example of a macrotrend analysis obtained using the VDSK filter $\exp(-\alpha\omega^2)$ which include the normalised gradients computed using a 'forward differencing scheme' illustrating 'phase shifts' associated with the two signals.

6 Case Study: Volatility Predictions for Foreign Exchange Markets

Since the publication of the preference-free option by Black and Sholes in 1973 [9], option pricing theory has developed into a standard tool for designing, pricing and hedging derivative securities of all types. For an ideal market, six inputs are required: the current stock price, the strike price, the time to expiry, the risk-free interest rate, the dividend and the volatility. The first three inputs are known from the outset but the last three must be estimated. The problem with this is that the correct values for these parameters are only known when the option expires which means that the future values of these quantities need to be known or accurately estimated if an option is to be priced correctly. The most important of these parameters is the volatility. This is because a change in volatility has the biggest impact on the price of an option. Volatility measures variability or dispersion about a central tendency and is a measure of the degree of price movement in a stock, a futures contract or any other market.

Fundamental to the Black-Scholes model is that financial asset prices are random variables that are log-normally distributed. Therefore, the relative price changes are usually measured by taking the differences between the logarithmic prices which are taken to be normally distributed. This is consistent with the log-normal model discussed in Section 1 where the volatility is proportional to $d(\ln S)/dt$. Thus, for a stock price signal $(S_1, S_2, ...S_n)$, using a forward differencing scheme, we let

$$v_i = \ln\left(\frac{S_{i+1}}{S_i}\right)$$

and define the volatility as

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(v_i - \bar{v})^2}$$

where $\bar{v}$ is the mean given by

$$\bar{v} = \frac{1}{n}\sum_{i=1}^{n} v_i$$

This result defines the volatility in terms of the standard deviation of the sample signal v_i which is a measure of the dispersion of a stock price signal whose variations in time are computed using a moving window.

In order to compare volatilities for different interval lengths, it is common to express volatility in annual terms by scaling the estimate with an annualisation factor (normalising constant) h which is the number of intervals per annum such that $\sigma_{\text{annual}} = \sigma\sqrt{h}$ where $h = 252$ for daily data (252 being approximately, the number of trading days per annum), $h = 52$ for weekly data and $h = 12$ for monthly data. Defining volatility in terms of variations in the standard deviation of an assets returns from the mean implies that large values of volatility fluctuate over a wide range leading to high risk. This means that if we assume that the mean of the log-relative returns is zero, then, a 10% volatility represents the following (according to a normal distribution): in one year, returns will be within [-10%, +10%] with a probability of 68.3% (1 standard deviation from the mean); within [-20% , +20%] with a probability of 95.4% (2 standard deviations), and within [-30% , +30%] with a probability of 99.7% (3 standard deviations). For this reason volatility is usually presented in terms of a percentage.

In this case study we use the value of $q(\tau)$ to predict the volatility associated with foreign exchange markets. This is based on observing the positions in $q(\tau)$ where the derivative is zero (i.e. when $q'(\tau) \equiv dq(\tau)/d\tau = 0$) after generating a macrotrend (as discussed in Section 4.2) and applying a threshold. Above this threshold, the volatility is predicted to reduce and below the threshold, it is predicted to increase. The rationale for this approach relates to the idea that as q approaches 1 the 'economy' becomes diffusive and as q approaches 2, it becomes propagative. This is equivalent to the Lévy index approaching 2 and 1, respectively. The problem is to determine optimum values for the length of the moving window L, the smoothing parameter α and the threshold T which is achieved by undertaking a range of tests using historical data. Figure 4 shows an example of this approach based on the US to Australian dollar exchange rate from 03/08/2003 to 16/04/2010, close-of-day. The positions at which $q'(\tau) = 0$ coupled with the applied threshold, define the 'critical points' at which a prediction is made on the future volatility that, in this example, is computed using a 30 day 'look-back' window. This depends upon whether the value of q at which $q'(\tau) = 0$ is above (volatility predicted to decrease) or below (volatility predicted to increase) the threshold.

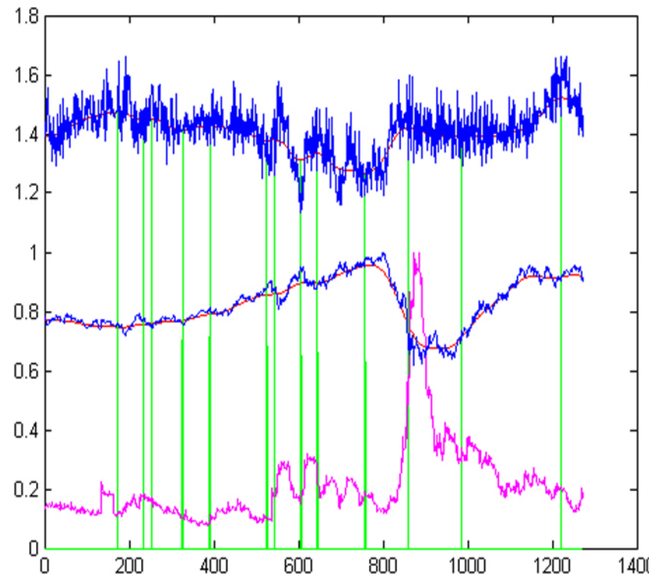

Figure 4: US/Australian dollar exchange rate from 03/08/2003 to 16/04/2010 for $L = 512$, $\alpha = 0.002$ and $T = 1.4$. The signals are as follows: On the scale of 1 to 2 are $q(\tau)$ (blue) and the associated macrotrend (red); on the scale of 0 to 1 are the normalised exchange data (blue) and associated macrotrend (red) and the normalised volatility (cyan). The positions in time at which $q' = 0$ is given in green.

Table 3 provides a quantification of the method. Here, the date at which $q'(t) = 0$ is provide together with a prediction on whether the volatility will go up or down, the volatility at the date given (i.e. the date at which the prediction is made) and the volatility 30 days and then 60 days after the date of the prediction. Also provided is a *True/False* criterion associated with each prediction that, for this example, yields a prediction accuracy of 69% .

Figure 5 shows the same analysis applied to the Japanese Yen to US Dollar exchange from 03/08/2003 to 18/04/2010 with an assessment of the predictions being given in Table 4 that, in this example, provides 75% accuracy. As a final example, Table 5 shows the results obtained for the Euro/US Dollar exchange rate from 28/10/2002 to 09/04/2010.

Date at which $q'(t) = 0$	Prediction: Up $\Uparrow$ or Down $\Downarrow$	30 day volatility at date	30 day volatility after date	60 day volatility after date	True/False $\surd/\times$
2 Mar 2006	$\Downarrow$	7.34%	9.58%	10.70%	$\times$
29 May 2006	$\Downarrow$	11.58%	9.95%	9.08%	$\surd$
23 Jun 2006	$\Downarrow$	10.94%	7.93%	8.15%	$\surd$
4 Oct 2006	$\Uparrow$	6.59%	5.95%	5.58%	$\times$
3 Jan 2007	$\Uparrow$	5.58%	8.52%	8.36%	$\surd$
11 Jul 2007	$\Uparrow$	6.51%	16.71%	15.41%	$\surd$
7 Aug 2007	$\Uparrow$	11.42%	18.26%	14.77%	$\surd$
1 Nov 2008	$\Uparrow$	10.00%	18.94%	16.97%	$\surd$
26 Feb 2008	$\Uparrow$	15.58%	14.52%	12.82%	$\times$
20 May 2008	$\Uparrow$	11.57%	8.07%	9.88%	$\times$
9 Oct 2009	$\Uparrow$	34.55%	59.36%	45.11%	$\surd$
2 Apr 2010	$\Uparrow$	18.69%	21.54%	20.38%	$\surd$
1 Mar 2010	$\Downarrow$	13.23%	8.27%	10.74%	$\surd$

Table 3: Assessment of volatility predictions obtained for the US dollar to Australian dollar exchange rate from 03/08/2003 to 16/04/2010.

Date at which $q'(t) = 0$	Prediction: Up $\Uparrow$ or Down $\Downarrow$	30 day volatility at date	30 day volatility after date	True/False $\surd/\times$
25 Jul 2005	$\Downarrow$	9.53%	7.07%	$\surd$
27 Oct 2005	$\Uparrow$	7.59%	10.29%	$\surd$
19 May 2006	$\Downarrow$	9.55%	8.33%	$\surd$
16 Jan 2007	$\Uparrow$	5.08%	10.27%	$\surd$
7 Aug 2007	$\Uparrow$	8.03%	11.24%	$\surd$
5 Oct 2007	$\Uparrow$	11.49%	11.44%	$\times$
17 Dec 2007	$\Downarrow$	14.23%	10.48%	$\surd$
5 Mar 2008	$\Uparrow$	8.66%	14.13%	$\surd$
9 Jun 2008	$\Downarrow$	10.90%	9.30%	$\surd$
26 Dec 2008	$\Uparrow$	17.56%	15.17%	$\times$
20 Apr 2009	$\Uparrow$	16.57%	13.66%	$\times$
28 May 2009	$\Uparrow$	12.77%	13.24%	$\surd$
16 Jul 2009	$\Uparrow$	13.80%	10.33%	$\times$
26 Oct 2009	$\Uparrow$	9.94%	13.41%	$\surd$
13 Apr 2010	$\Uparrow$	8.29%	14.20%	$\surd$
6 May 2010	$\Uparrow$	11.49%	14.42%	$\surd$

Table 4: Assessment of volatility predictions associated with Japanese Yen to US dollar exchange rate from 03/08/2003 to 18/04/2010.

7 Discussion

In terms of the non-stationary fractional diffusion model considered in this paper, the time varying Fourier Dimension $q(\tau)$ can be interpreted in terms of a 'gauge' on the characteristics of dynamical system characterised by fractional diffusive processes. In a statistical sense, $q(\tau)$ is just another measure that may, or otherwise, be of value to market traders. In comparison with other statistical measures, this can only be assessed through its practical application in a live trading environment. However, in terms of its relationship to a stochastic model for financial time series data, $q(\tau)$ appears to provide a measure that is consistent with the physical principles associated with a random walk that includes a directional bias, i.e. fractional Brownian motion. The model considered, and the signal processing algorithm proposed, has a close association with re-scaled range analysis for computing the Hurst exponent H [24] [25] and [26]. In this sense, the principal contribution of this paper has been to consider an approach that is quantifiable in terms of a model that has been cast in terms of a specific fractional partial differential equation. As with other financial time series, their derivatives, transforms etc., a range of statistical measures can be used to characterise $q(\tau)$ and it is noted that in all cases studied to date, the composite normality of the signal $q(\tau)$ is of type 'Reject'. In other words, the statistics of $q(\tau)$ are non-Gaussian. Further, assuming that a financial time series is statistically self-affine, the computation of $q(\tau)$ can be applied over any time scale

Date at which $q'(t) = 0$	Prediction: Up $\Uparrow$ or Down $\Downarrow$	30 day volatility at date	60 day mean volatility after date	True/False $\surd/\times$
17 Dec 2003	$\Uparrow$	9.82%	11.09%	$\surd$
17 Jun 2004	$\Downarrow$	12.17%	9.09%	$\surd$
3 Nov 2004	$\Uparrow$	7.70%	8.56%	$\surd$
20 Jul 2005	$\Downarrow$	10.43%	9.03%	$\surd$
27 Dec 2005	$\Uparrow$	8.62%	8.90%	$\surd$
26 Jun 2006	$\Downarrow$	9.42%	7.97%	$\surd$
7 Feb 2007	$\Uparrow$	5.65%	5.42%	$\times$
28 May 2007	$\Uparrow$	4.56%	5.26	$\surd$
12 Mar 2008	$\Uparrow$	7.64%	8.95%	$\surd$
19 Nov 2008	$\Downarrow$	22.16%	20.64%	$\surd$
12 Mar 2009	$\Uparrow$	14.21%	20.64%	$\surd$
29 Jun 2009	$\Downarrow$	18.98%	10.92%	$\surd$
30 Sep 2009	$\Uparrow$	7.93%	8.66%	$\surd$

Table 5: Assessment of volatility predictions associated with Euro to US dollar exchange rate from 28/10/2002 to 09/04/2010

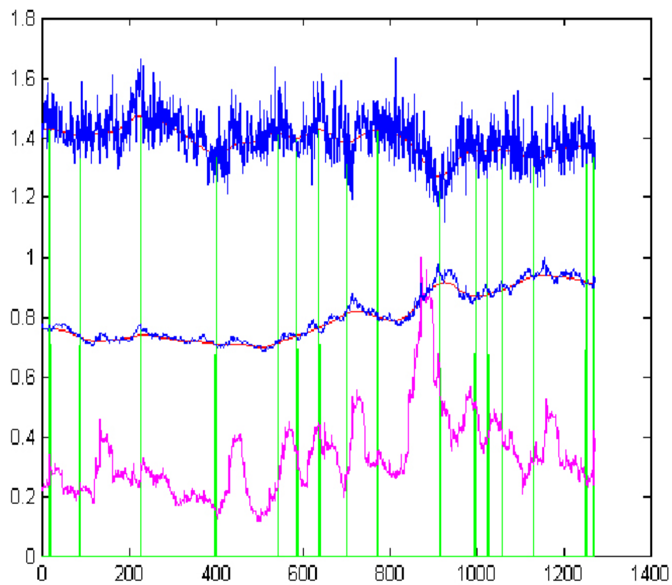

Figure 5: Japanese Yen to US dollar exchange rate from 03/08/2003 to 18/04/2010 for $L = 512$, $\alpha = 0.002$ and $T = 1.4$. The signals are as follows: On the scale of 1 to 2 are $q(\tau)$ (blue) and the associated macrotrend (red); on the scale of 0 to 1 are the normalised exchange data (blue) and associated macrotrend (red) and the normalised volatility (cyan). The positions in time when $q' = 0$ is given in green.

provided there is sufficient data for the computation of $q(\tau)$ to be statistically significant. Thus, the results associated with the Close-of-Day data studied in this paper are, in principle, applicable to signals associated with data over a range of time scales.

As shown in Section 2.4 we obtain a simple algebraic relationship between Fourier Dimension q and the Lévy index γ given by $\gamma^{-1} = q/2$. Adhering to the numerical range of the Lévy index, we note that for $\gamma \in (0, 2]$ then $q \in (\infty, 1]$. However, q is related to other parameters including the Hurst exponent and the Fractal Dimension as shown in Appendix D. For a fractal signal where $D_F \in (1, 2)$, then $q \in (2, 1)$ and $\gamma \in (1, 2)$. Thus, the Fourier Dimension is simply related to the Fractal Dimension and Lévy index. The approach used is therefore consistent with assuming that financial signals are fractal signals and we may therefore consider equation (7) to be a classification of the 'Fractal Market Hypothesis' (FMH) based on Lévy statistics in contrast to the Efficient Market Hypothesis that is based on Gaussian statistics. In the context of the ideas presented in this paper, the FMH has a number of fundamental differences with regard to the EMH which are tabulated in Table 6.

The material presented in this paper has been exclusively concerned with a one-dimensional model for fractional diffusion. However, we note that for the three-dimensional case, equation (1) becomes

$$u(\mathbf{r}, t + \tau) = \int_{-\infty}^{\infty} u(\mathbf{r} + \lambda, t) p(\lambda) d\lambda$$

where $\mathbf{r}$ denotes the three-dimensional space vector $\mathbf{r} \equiv (x, y, z)$ and that in (three-dimensional) Fourier space

$$U(\mathbf{k}, t + \tau) = U(\mathbf{k}, t) P(k)$$

where $\mathbf{k}$ is the spatial frequency vector (k_x, k_y, k_z) and $k \equiv\mid \mathbf{k} \mid$. Thus, for the Characteristic Function

$$P(k) = \exp(-a \mid k \mid^\gamma) = 1 - a \mid k \mid^\gamma, \quad a \to 0,$$

$$\frac{U(\mathbf{k}, t+\tau) - U(\mathbf{k}, t)}{\tau} = -\frac{a}{\tau} U(\mathbf{k}, t) \mid k \mid^\gamma$$

and using the Reisz definition of a fractional n-dimensional laplacian given by

$$\nabla^\gamma \equiv -\frac{1}{(2\pi)^n} \int d^n\mathbf{k} \mid k \mid^\gamma \exp(i\mathbf{k} \cdot \mathbf{r})$$

we obtain (for $\tau \to 0$) the three dimensional homogeneous fractional diffusion equation

$$\left(\nabla^\gamma - \sigma \frac{\partial}{\partial t}\right) u(\mathbf{r}, t) = 0$$

However, the Green's function for the three-dimensional case is given by [16]

$$g(r, \omega) = \frac{\exp(i\Omega_\gamma r)}{4\pi r}$$

where $r \equiv\mid \mathbf{r} \mid$ which is not characterised by a $\omega^{-\frac{1}{\gamma}}$-type scaling law. In two-dimensions, the Green's function is [16] (ignoring scaling)

$$g(r, \omega) \sim \frac{\exp(i\Omega_\gamma r)}{\sqrt{\Omega_\gamma r}}$$

and it is clear that

$$\mid g(r, \omega) \mid \sim \frac{1}{\sqrt{r(\omega\sigma)^{\frac{1}{\gamma}}}}, \quad \sigma \to 0$$

which is characterised by a $\omega^{-\frac{1}{2\gamma}}$ scaling law. The inability for the three-dimensional fractional diffusion equation to yield $\omega^{-\frac{1}{\gamma}}$-type fractal noise is overcome if we consider a model based on the separable case [27]. We consider the equation

$$\left(\frac{\partial^\gamma}{\partial x^\gamma} + \frac{\partial^\gamma}{\partial y^\gamma} + \frac{\partial^\gamma}{\partial z^\gamma} - \sigma \frac{\partial}{\partial t}\right) u(x, y, z, t) = n(x, y, z, t)$$

and a solution where

$$n(x, y, z, t) = \delta(x) n_x(t) + \delta(y) n_y(t) + \delta(z) n_z(t)$$

and

$$u(x, y, z, t) = u_x(x, t) + u_y(y, t) + u_z(x, t)$$

The source function n is then taken to model a system characterised by a separable spatial impulse with separable white noise function (n_x, n_y, n_z). This model is used for the morphological analysis of Hyphal growth rates, for example, where the fractal dimension provides an estimate of the metabolic production of filamentous fungi [28].

8 Conclusion

The value of $q(\tau)$ characterises stochastic processes that can vary in time and are intermediate between fully diffusive and propagative or persistent behaviour. Application of Orthogonal Linear Regression to financial time series data provides an accurate and robust method to compute $q(\tau)$ when compared to other statistical estimation techniques such as the least squares method. As a result of the physical interpretation associated with the fractional diffusion equation derived in Section 2.2, we can, in principle, use the signal $q(\tau)$ as a predictive measure in the sense that as the value of $q(\tau)$ continues to increases, there is a greater likelihood for propagative behaviour of the markets. This is reflected in the data analysis based on the examples given in which the Gaussian VDSK $\exp(-\alpha\omega^2)$ has been used to smooth both $u(t)$ and $q(\tau)$ to obtain macrotrends in which the value of α determines the level of detail in the output. From the example provided in Figure 3 and other trials that have been undertaken (details of which lie beyond the scope of this paper), it is clear that the 'turning point' (i.e. positions in time where $q'(t) = 0$) appear to 'flag' a future change in the trend of the signal $u(t)$. This feature is reflected in the 'cross-over points' of the normalised gradients for u' and $q'(t)$ illustrated in Figure 3, i.e. points in time when $u'(t), \ \ \|u'\|_\infty = 1$ and $q'(\tau), \ \ \|u'\|_\infty = 1$ are approximately the same and whose gradients are of opposite polarity. These characteristics are a consequence of the phase shifts that exist in the gradients of $u(t)$ and $q(\tau)$ over different frequency bands. Although the interpretation of these phase shifts requires further study, from the results obtained to date, it is clear that they provide an assessment of the risk associated with a particular investment portfolio and has been used to develop a model for forecasting volatility as discussed in Section 5. However, application of this model for predicting market behavour is dependent on optimising two parameters, namely, the length of the moving window L and the filter parameter α. Evaluation of historical data for a given time series (with a specific sampling rate) is required in order to optimise the values of these parameters.

The model used is predicated on the assumption that financial time series are Lévy distributed and one of the key results of this paper has been to provide connectivity between Lévy distributed processes used to derive the fractional diffusion equation and random scaling fractal signals in terms of a Green's function solution to this equation. The value of the Lévy index $\gamma = 2/q$ is then used to gauge the likely future behavior of the signal under the FMH: a change in γ precedes a change in the signal.

EMH	FMH
Gaussian Statistics	Non-Gaussian Statistics
Stationary Process	Non-stationary Process
No memory - no historical correlations	Memory - historical correlations
No repeating patterns at any scale	Many repeating patterns at all scales - 'Elliot waves'
Continuously stable at all scales	Continuously unstable at any scale - 'Lévy Flights'

Table 6: Principal differences between the Efficient Market Hypothesis (EMH) and the Fractal Market Hypothesis (FMH).

Appendix A: Einstein's Derivation of the Diffusion Equation

Let τ be a small interval of time in which a particle moves between λ and $\lambda + d\lambda$ with probability $P(\lambda)$ where $\lambda = \sqrt{\lambda_x^2 + \lambda_y^2 + \lambda_z^2}$ and τ is small enough to assume that the movements of the particle are τ-independent. If $u(\mathbf{r}, t)$ is the concentration (i.e. the number of particles per unit volume) then the concentration at time $t + \tau$ generated by a source function $F(\mathbf{r})$ is given by

$$(\mathbf{r}, t+\tau) = \int_{-\infty}^{\infty} u(\mathbf{r}+\lambda, t) P(\lambda) d\lambda + F(\mathbf{r}, t) \qquad (A.1)$$

Since $\tau << 1$, we may approximate $u(\mathbf{r}, t + \tau)$ as

$$u(\mathbf{r}, t+\tau) = u(\mathbf{r}, t) + \tau \frac{\partial}{\partial t} u(\mathbf{r}, t)$$

and write $u(\mathbf{r} + \lambda, t)$ in terms of the Taylor series

$$\begin{aligned} u(\mathbf{r}+\lambda, t) =& u + \lambda_x \frac{\partial u}{\partial x} + \lambda_y \frac{\partial u}{\partial y} + \lambda_z \frac{\partial u}{\partial z} \\ &+ \frac{\lambda_x^2}{2!} \frac{\partial^2 u}{\partial x^2} + \frac{\lambda_y^2}{2!} \frac{\partial^2 u}{\partial y^2} + \frac{\lambda_z^2}{2!} \frac{\partial^2 u}{\partial z^2} \\ &+ \lambda_x \lambda_y \frac{\partial^2 u}{\partial x \partial y} + \lambda_x \lambda_z \frac{\partial^2 u}{\partial x \partial z} + \lambda_y \lambda_z \frac{\partial^2 u}{\partial y \partial z} + ... \end{aligned}$$

However, higher order terms can be neglected since, if $\tau << 1$, then the distance travelled, λ, must also be small. Equation (A.1) may then be written as

$$\begin{aligned} u + \tau \frac{\partial u}{\partial t} &= F(\mathbf{r}, t) + \int_{-\infty}^{\infty} u P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \left(\lambda_x \frac{\partial u}{\partial x} + \lambda_y \frac{\partial u}{\partial y} + \lambda_z \frac{\partial u}{\partial z} \right) P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \left(\frac{\lambda_x^2}{2!} \frac{\partial^2 u}{\partial x^2} + \frac{\lambda_y^2}{2!} \frac{\partial^2 u}{\partial y^2} + \frac{\lambda_z^2}{2!} \frac{\partial^2 u}{\partial z^2} \right) P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \left(\lambda_x \lambda_y \frac{\partial^2 u}{\partial x \partial y} + \lambda_x \lambda_z \frac{\partial^2 u}{\partial x \partial z} + \lambda_y \lambda_z \frac{\partial^2 u}{\partial y \partial z} \right) P(\lambda) d\lambda \end{aligned}$$

Assuming that $P(\lambda)$ is normalized we have

$$\int_{-\infty}^{\infty} P(\lambda) d\lambda = 1$$

so that

$$\begin{aligned} \tau \frac{\partial}{\partial t} u &= \int_{-\infty}^{\infty} \frac{\lambda_x^2}{2} \frac{\partial^2 u}{\partial x^2} P(\lambda) d\lambda + \int_{-\infty}^{\infty} \frac{\lambda_x \lambda_y}{2} \frac{\partial^2 u}{\partial x \partial y} P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \frac{\lambda_x \lambda_z}{2} \frac{\partial^2 u}{\partial x \partial z} P(\lambda) d\lambda + \int_{-\infty}^{\infty} \frac{\lambda_y \lambda_x}{2} \frac{\partial^2 u}{\partial y \partial x} P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \frac{\lambda_y^2}{2} \frac{\partial^2 u}{\partial y^2} P(\lambda) d\lambda + \int_{-\infty}^{\infty} \frac{\lambda_y \lambda_z}{2} \frac{\partial^2 u}{\partial y \partial z} P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \frac{\lambda_z \lambda_x}{2} \frac{\partial^2 u}{\partial z \partial x} P(\lambda) d\lambda + \int_{-\infty}^{\infty} \frac{\lambda_z \lambda_y}{2} \frac{\partial^2 u}{\partial z \partial y} P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \frac{\lambda_z^2}{2} \frac{\partial^2 u}{\partial z^2} P(\lambda) d\lambda + \int_{-\infty}^{\infty} \lambda_x \frac{\partial u}{\partial x} P(\lambda) d\lambda \\ &+ \int_{-\infty}^{\infty} \lambda_y \frac{\partial u}{\partial y} P(\lambda) d\lambda + \int_{-\infty}^{\infty} \lambda_z \frac{\partial u}{\partial z} P(\lambda) d\lambda + F(\mathbf{r}, t) \end{aligned}$$

which may be written as a matrix equation of the following form

$$\frac{\partial}{\partial t} u(\mathbf{r}, t) = \nabla \cdot \mathbf{D} \nabla u(\mathbf{r}, t) + \mathbf{V} \cdot \nabla u(\mathbf{r}, t) + F(\mathbf{r}, t)$$

where $\mathbf{D}$ is the diffusion tensor given by

$$\mathbf{D} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}$$

where

$$D_{ij} = \int_{-\infty}^{\infty} \frac{\lambda_i \lambda_j}{2\tau} P(\lambda) d\lambda = \frac{1}{2\tau} \langle \lambda_i \lambda_j \rangle$$

and $\mathbf{V}$ is a flow vector which describes any drift velocity that the particle ensamble may have and is given by

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad V_i = \int_{-\infty}^{\infty} \frac{\lambda_i}{\tau} P(\lambda) d\lambda = \frac{1}{\tau}\langle \lambda_i \rangle$$

Note that as $\lambda_i \lambda_j = \lambda_j \lambda_i$, the diffusion tensor is diagonally symmetric (i.e. $D_{ij} = D_{ji}$). For isotropic diffusion where $\langle \lambda_i \lambda_j \rangle = 0$ for $i \neq j$ and $\langle \lambda_i \lambda_j \rangle = \langle \lambda^2 \rangle$ for $i = j$ and with no drift velocity so that $\mathbf{V} = \mathbf{0}$, then

$$\frac{\partial}{\partial t} u(\mathbf{r}, t) = \nabla \cdot \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \nabla u(\mathbf{r}, t) + F(\mathbf{r}, t)$$

$$= D \nabla^2 u(\mathbf{r}, t) + F(\mathbf{r}, t)$$

where

$$D = \int_{-\infty}^{\infty} \frac{\lambda^2}{2\tau} P(\lambda) d\lambda$$

Appendix B: Evaluation of the Lévy Distribution

We wish to show that the Characteristic Function $P(k) = \exp(-a \mid k \mid^\gamma)$, $0 < \gamma \leq 2$ is equivalent to a Probability Density Function given by $p(x) \sim x^{-(1+\gamma)}$, $x \to \infty$., i.e. we wish to prove the following:

Theorem

$$\frac{1}{x^{1+\gamma}} \leftrightarrow \exp(-a \mid k \mid^\gamma), \quad 0 < \gamma \leq 2, \quad x \to \infty$$

where $\leftrightarrow$ denotes transformation from real to Fourier space[1].

Proof of Theorem

For $0 < \gamma < 1$, and since the characteristic function is symmetric, we have

$$p(x) = \mathrm{Re}[f(x)]$$

where

$$f(x) = \frac{1}{\pi} \int_0^\infty e^{ikx} e^{-k^\gamma} dk$$

$$= \frac{1}{\pi} \left(\left[\frac{1}{ix} e^{ikx} e^{-k^\gamma} \right]_{k=0}^\infty - \frac{1}{ix} \int_0^\infty e^{ikx} (-\gamma k^{\gamma-1} e^{-k^\gamma}) dk \right)$$

$$= \frac{\gamma}{2\pi i x} \int_{-\infty}^{\infty} dk H(k) k^{\gamma-1} e^{-k^\gamma} e^{ikx}, \quad x \to \infty$$

[1] The author acknowledges Dr K I Hopcraft, School of Mathematical Sciences, Nottingham University, England, for his advice in respect of this result.

where

$$H(k) = \begin{cases} 1, & k > 0 \\ 0, & k < 0 \end{cases}$$

For $0 < \gamma < 1$, $f(x)$ is singular at $k = 0$ and the greatest contribution to this integral is the inverse Fourier transform of $H(k) k^{\gamma-1}$. Noting that

$$\mathcal{F}^{-1} \left[\frac{1}{(ik)^\gamma} \right] \sim \frac{1}{x^{1-\gamma}}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and that

$$H(k) \leftrightarrow \delta(x) + \frac{i}{\pi x} \sim \delta(x), \quad x \to \infty$$

then, using the convolution theorem, we have

$$f(x) \sim \frac{\gamma}{i\pi x} \frac{i^{1-\gamma}}{x^\gamma}$$

and thus

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

For $1 < \gamma < 2$, we can integrate by parts twice to obtain

$$f(x) = \frac{\gamma}{i\pi x} \int_0^\infty dk k^{\gamma-1} e^{-k^\gamma} e^{ikx}$$

$$= \frac{\gamma}{i\pi x} \left[\frac{1}{ix} k^{\gamma-1} e^{-k^\gamma} e^{ikx} \right]_{k=0}^\infty$$

$$+ \frac{\gamma}{\pi x^2} \int_0^\infty dk e^{ikx} [(\gamma-1) k^{\gamma-2} e^{-k^\gamma} - \gamma (k^{\gamma-1})^2 e^{-k^\gamma}]$$

$$= \frac{\gamma}{\pi x^2} \int_0^\infty dk e^{ikx} [(\gamma-1) k^{\gamma-2} e^{-k^\gamma} - \gamma (k^{\gamma-1})^2 e^{-k^\gamma}], \quad x \to \infty$$

The first term of this result is singular and therefore provides the greatest contribution and thus we can write,

$$f(x) \simeq \frac{\gamma(\gamma-1)}{2\pi x^2} \int_{-\infty}^{\infty} H(k) e^{ikx} (k^{\gamma-2} e^{-k^\gamma}) dk$$

In this case, for $1 < \gamma < 2$, the greatest contribution to this integral is the inverse Fourier transform of $k^{\gamma-2}$ and hence,

$$f(x) \sim \frac{\gamma(\gamma-1)}{\pi x^2} \frac{i^{2-\gamma}}{x^{\gamma-1}}$$

so that

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty$$

which maps onto the previous asymptotic as $\gamma \to 1$ from the above.

Appendix C: Variation Diminishing Smoothing Kernels

Variation Diminishing Smoothing Kernels (VDSK) are convolution kernels with properties that guarantee smoothness and thereby, eliminate Gibbs' effect around points of discontinuity of a given function. Further the smoothed function can be shown to be made up of a similar succession of concave or convex arcs equal in number to those of the function. Thus, we consider the following question: let there be given a continuous or discontinuous function f whose graph is composed of a succession of alternating concave or convex arcs. Is there a smoothing kernel (or a set of them) which produces a smoothed function whose graph is also made up of a similar succession of concave or convex arcs equal in number to those of f?.

C.1 Laguerre-Pôlya Class Entire Functions

The class of kernels which relate to this question are a class of entire functions which shall be called class E originally studied earlier by E Laguerre and G Pôlya. An entire function $E(z), z \in \mathbf{C}$ belongs to the class E

$\iff$

$$E(z) = \exp(bz - cz^2)\prod_{\lambda=1}^{\infty}\Big(1 - \frac{z}{a(\lambda)}\Big)\exp[z/a(\lambda)], \quad (C.1.1)$$

where $b, c, a(\lambda) \in \mathbf{R}$, $c \geq 0$, *and*

$$\sum_{\lambda=1}^{\infty} a^{-2}(\lambda) < \infty. \quad (C.1.2)$$

where $\iff$ is taken to denote 'if and only if' - *iff*. The convergence of the series $(C.1.2)$ guarantees that the product in $(C.1.1)$ converges and represents an entire function. Laguerre proved, and Pôlya added a refinement, that a sequence of polynomials, having real roots only, which converge uniformly in every compact set of the complex plane $\mathbf{C}$, approaches a function of class E in the uniform limit of such a sequence. For example,

$$\exp(-z^2) = \lim_{\lambda\to\infty}\Big(1 - \frac{z^2}{\lambda^2}\Big)^{\lambda^2},$$

and the polynomials $(1 - z^2/\lambda^2)$ have real roots only. In this definition, it is not assumed that the $a(\lambda)$ are distinct. To include the case in which the product has a finite number of factors or reduces to 1 without additional notation, it is assumed that certain points on all the $a(\lambda)$ may be ∞. Furthermore, it is assumed, without loss of generality, that the roots $a(\lambda)$ are arranged in an order of increasing absolute values,

$$0 < |a(1)| \leq |a(2)| \leq |a(3)| \leq \dots$$

Examples of functions belonging to class E are

$$1,\ 1 - z,\ \exp(z),\ \exp(z^2),\ \cos z$$

$$\frac{\sin z}{z},\ \Gamma^{-1}(1 - z),\ \Gamma^{-1}(z)$$

Note that the product of two functions of this class produce a new function of the same class.

C.2 Variation Diminishing Smoothing Kernels (VDSKs)

A function k is variation diminishing iff it is of the form

$$k(x) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty}[E(z)]^{-1}\exp(zx)\,dz, \quad (C.2.1)$$

where $E(z) \in E$ is given by

$$E(z) = \exp(bz - cz^2)\prod_{\lambda=1}^{\infty}\Big(1 - \frac{z}{a(\lambda)}\Big)\exp[z/a(\lambda)], \quad (C.2.2)$$

with $b, c, a(\lambda) \in \mathbf{R}$, $c \geq 0$, and

$$\sum_{\lambda=1}^{\infty} a^{-2}(\lambda) < \infty$$

In other words, a frequency function k is variation diminishing iff its bilateral Laplace transform equals $[E(z)]^{-1}$:

$$[E(z)]^{-1} = \int_{-\infty}^{\infty} k(x)\exp(-zx)\,dx. \quad (C.2.3)$$

In order to define a smoothing kernel, the function k given in $(C.2.1)$ must be an even function. For, if $k(x)$ is even, then the corresponding bilateral Laplace transform $[E(z)]^{-1}$ is also even. This fact follows readily from

$$[E(z)]^{-1}$$

$$= \int_{-\infty}^{\infty} k(x)\exp(-zx)\,dx = \int_{-\infty}^{\infty} k(-x)\exp(-zx)\,dx$$

$$= \int_{-\infty}^{\infty} k(x)\exp(zx)\,dx = [E(-z)]^{-1}$$

Conversely, if $[E(z)]^{-1}$ is even, then its inverse bilateral transform is even since a component of convergence of $(C.2.3)$ contains the imaginary axis. This follows from the fact that the component of convergence of each one of the functions which compose $E(z)$ contains completely the imaginary axis. Further, it follows that

$$[E(iu)]^{-1} = K(u), \quad (C.2.4)$$

where $K(u)$ is the FT of k. From the evenness of $[E(z)]^{(-1)}$ it follows that $K(u)$ is real, hence k is even.

But $E(z)$ is even iff $b = 0$ and $a(2\lambda - 1) = -a(2\lambda)$, $\lambda = 1, 2, \ldots$. Therefore $E(z)$ is taken to be

$$E(z) = \exp(-cz^2) \prod_{\lambda=1}^{\infty} \Big(1 - \frac{z^2}{a^2(\lambda)}\Big), \qquad (C.2.5)$$

with $c, a(\lambda) \in \mathbf{R}$, $c \geq 0$, and

$$\sum_{\lambda=1}^{\infty} a^{-2}(\lambda) < \infty$$

Equation $(C.2.4)$ establishes the relationship between the bilateral Laplace transform and the Fourier transform of k. Thus, any analysis associated with use of the bilateral Laplace transform can be undertaken in terms of the Fourier transform.

Using equation $(C.2.4)$ the Fourier transform of $(C.2.1)$ is given by

$$k(x) \leftrightarrow K(u) = [E(iu)]^{-1} = \exp(-cu^2) \prod_{\lambda=1}^{\infty} \Big(\frac{a^2(\lambda)}{a^2(\lambda) + u^2}\Big), \qquad (C.2.6)$$

where $\leftrightarrow$ denotes transformation from real to Fourier space, $c, a(\lambda) \in \mathbf{R}$, $c \geq 0$, and $\sum_{\lambda=1}^{\infty} a^{-2}(\lambda) < \infty$.

Because equation $(C.2.6)$ is a variation diminishing function by construction and $|K(0)| \leq 1$, then the following result holds.

Theorem C.2.1 (VDSKs)

k defined as in equation $(C.2.6)$

$\Longrightarrow$

1. *k is a smoothing kernel belonging to SK_1,*
2. *k is variation diminishing,*
3. *$k(x) \geq 0$, $x \in \mathbf{R}$.*

In order to make a complete study of the VDSKs, such kernels will be divided in three classes: *The Finite VDSKs*, *The Non-Finite VDSKs*, and *The Gaussian VDSK*.

C.3 The Finite VDSKs

The finite and the non-finite VDSKs are kernels which can be synthesized from the following basic function:

$$\mathrm{e}(x) = \frac{1}{2} \exp(-|x|), \ x \in \mathbf{R}. \qquad (C.3.1)$$

The finite VDSKs are made up by a finite number of convolutions of functions $a(\lambda)\,\mathrm{e}[a(\lambda)x]$, $\lambda = 1, 2, \ldots$. Clearly $\mathrm{e}(x)$ is a VDSK with mean $\nu = 0$ and variance $\sigma^2 = 2$ and its Fourier transform is given by

$$\mathrm{e}(x) \leftrightarrow \frac{1}{1 + u^2}. \qquad (C.3.2)$$

Note that if $a > 0$, then $a\,\mathrm{e}(ax)$ is again a VDSK. Using the similarity property of the Fourier transform and equation $(C.3.2)$, its Fourier transform is given by

$$a\,\mathrm{e}(ax) \leftrightarrow \frac{a^2}{a^2 + u^2}. \qquad (C.3.3)$$

Its mean ν again vanishes and its variance takes the value $\sigma^2 = 2/a^2$.

Let $a(1), a(2), \ldots, a(n) > 0$ be constants, some or all of which may be coincident. The following VDSKs are introduced

$$k_\lambda(x) = a(\lambda)\,\mathrm{e}[a(\lambda)x], \ \lambda = 1, 2, \ldots, n. \qquad (C.3.4)$$

The combination of these functions by convolution gives a new VDSKs with properties quantified in the following theorem.

Theorem C.3.1 (Properties of The Finite VDSKs)

1. *$a(\lambda) > 0$, $\lambda = 1, 2, \ldots$,*
2. *$k_\lambda(x) = a(\lambda)\,\mathrm{e}[a(\lambda)x]$,*
3. *$k = k_1 \otimes k_2 \otimes \cdots \otimes k_n$,*
4. *$K(u) = \prod_{\lambda=1}^{n} \big(a^2(\lambda)/(a^2(\lambda) + u^2)\big)$*

$\Longrightarrow$

A. *k is a VDSK,*

B. *$k(x) \leftrightarrow K(u)$,*

C. *k has mean $\nu = 0$,*

D. *k has variance $\sigma^2 = \sum_{\lambda=1}^{n} \big(2/a^2(\lambda)\big) < \infty$.*

Proof. A. The assertion follows from mathematical induction.

B. It follows from Convolution Theorem and mathematical induction.

C. Let $k_\lambda(x) \leftrightarrow K_\lambda(u)$. Then because each k_λ is a VDSK, it follows that the respective mean, ν_λ, is given by

$$\nu_\lambda = iK'_\lambda(0) = 0, \ \lambda = 1, 2, \ldots, n$$

Moreover, if $n = 2$, then the mean ν of k is given by

$$\nu = iK'(0) = i(K_1K_2)'(0) = i(K_1{K_2}' + {K_1}'K_2)(0) = i(0) = 0$$

The assertion follows from this result and mathematical induction.

D. Let $k_\lambda(x) \leftrightarrow K_\lambda(u)$. Then because k_λ is a VDSK, it follows that the respective variance, σ^2_λ, is given by

$$\sigma^2_\lambda = -K''(0) = \frac{2}{a^2\lambda}, \ \lambda = 1, 2, \ldots, n$$

Furthermore, from the result given in C above, if $n = 2$, then the mean σ^2 of k is given by

$$\sigma^2 = -K''(0) = -(K_1K_2)''(0)$$

$$= (-K_1K_2'' - 2K_1'K_2' - K_1''K_2)(0) = \frac{2}{a^2(1)} + \frac{2}{a^2(2)}$$

The assertion follows from this result and mathematical induction. From the explicit expression of $K(u)$ given in Theorem C.3.1. it follows that

$$K(u) = \prod_{\lambda=1}^{n} \left(\frac{a^2(\lambda)}{a^2(\lambda) + u^2} \right)$$

$$= \prod_{\lambda=1}^{n} \left(\frac{a(\lambda)}{a(\lambda) - iu} \right) \left(\frac{-a(\lambda)}{-a(\lambda) - iu} \right)$$

$$= \prod_{\lambda=1}^{n} \left(\frac{a(\lambda)}{a(\lambda) - iu} \right) \prod_{\lambda=1}^{n} \left(\frac{-a(\lambda)}{-a(\lambda) - iu} \right)$$

$$= \prod_{\lambda=1}^{2n} \left(\frac{d(\lambda)}{d(\lambda) - iu} \right)$$

where $d(\lambda) = a(\lambda)$ for $\lambda = 1, 2, \ldots, n$ and $d(\lambda) = -a(\lambda)$ for $\lambda = n+1, n+2, \ldots, 2n$. Thus k is of degree $2n$ and the following theorem holds.

Theorem C.3.2 (Degree of Differentiability of The Finite VDSKs)

k a finite VDSK,

$\Longrightarrow$

1. *$k \in C^{2n-2}(\mathbf{R}, \mathbf{R})$,*
2. *$k \in C^{2n-1}(\mathbf{R}, \mathbf{R})$ except at $x = 0$, where*

$$k^{2n-1}(0^+),\ k^{2n-1}(0^-)$$

both exist.

The asymptotic behaviour of k and its Fourier transform, K, will be now studied.

Theorem C.3.3 (Asymptotic Behaviour of The Fourier transform of The Finite VDSKs)

1. *k a finite VDSK,*
2. *$k(x) \leftrightarrow K(u)$*

$\Longrightarrow$

$$|K(u)| = O(|u|^{-2n}),\ |u| \to \infty$$

Proof. k is made up of a finite convolution operations of functions $k_\lambda(x) = a(\lambda)\, \mathrm{e}[a(\lambda)x]$, where $a(\lambda) > 0$, $\lambda = 1, 2, \ldots, n$; and whose FT, $K_\lambda(u)$, satisfy the inequality

$$|K_\lambda(u)| = \left| \frac{a^2(\lambda)}{a^2(\lambda) + u^2} \right| \leq \frac{a^2(\lambda)}{|u|^2},\ \lambda = 1, 2, \ldots, n$$

Thus

$$|K(u)| = \left| \prod_{\lambda=1}^{n} K_\lambda(u) \right| \leq \prod_{\lambda=1}^{n} \left(\frac{a^2(\lambda)}{|u|^2} \right) = |u|^{-2n} \prod_{\lambda=1}^{n} a^2(\lambda). \tag{C.3.5}$$

From the above theorem we construct the following corollarys.

Corollary C.3.4 (Absolute and Quadratic Integrability of The Fourier transform of The Finite VDSKs)

1. *k a finite VDSK,*
2. *$k(x) \leftrightarrow K(u)$*

$\Longrightarrow$

$K(u) \in L(\mathbf{R}, \mathbf{R}) \cap L^2(\mathbf{R}, \mathbf{R})$.

Corollary C.3.5 (Absolute and Quadratic Integrability of The Finite VDSKs)

k a finite VDSK,

$\Longrightarrow$

$k(x) \in L(\mathbf{R}, \mathbf{R}) \cap L^2(\mathbf{R}, \mathbf{R})$.

The Fourier transform $K(u)$ of the Fourier transform of k is given by

$$K(u) \leftrightarrow 2\pi k(-x)$$

Since k is a even function then

$$K(u) \leftrightarrow 2\pi k(x)$$

This result in conjunction with Corollary C.3.4. and Riemann-Lebesgue Lemma proves the following theorem.

Theorem C.3.6 (Asymptotic Behaviour of The Finite VDSKs)

k a finite VDSK

$\Longrightarrow$

$k(x) \to 0$ as $|x| \to \infty$.

C.4 The Non-Finite VDSKs

We now study kernels k holding the property

$$k(x) \leftrightarrow K(u) = \prod_{\lambda=1}^{\infty} \left(\frac{a^2(\lambda)}{a^2(\lambda) + u^2} \right) \tag{C.4.1}$$

which are non-finite kernels. In particular, the infinite product in equation $(C.4.1)$ may have only a finite number of factors, so that the finite VDSKs of the last section are included. Kernels holding equation $(C.4.1)$ can be synthesized from the basic kernel

$$\mathrm{e}(x) = \frac{1}{2}\exp(-|x|),\ x \in \mathbf{R}$$

The non-finite VDSKs are composed of a non-finite number of functions $a(\lambda)\,\mathrm{e}[a(\lambda)x]$, $\lambda = 1, 2, \ldots$. The properties of such kernels are given in the following theorem.

Theorem C.4.1 (Properties of The Non-Finite VDSKs)

1. *$a(\lambda) > 0$, $\lambda = 1, 2, \ldots,$*
2. *$k_\lambda(x) = a(\lambda)\,\mathrm{e}[a(\lambda)x]$,*
3. *$k = k_1 \otimes k_2 \otimes \cdots \otimes k_n \ldots,$*
4. *$K(u) = \prod_{\lambda=1}^{\infty} \left(a^2(\lambda)/(a^2(\lambda) + u^2)\right)$*

$\Longrightarrow$

A. *k is a VDSK,*

B. *$k(x) \leftrightarrow K(u)$,*

C. *k has mean $\nu = 0$,*

D. *k has variance $\sigma^2 = \sum_{\lambda=1}^{\infty} \left(2/a^2(\lambda)\right) < \infty$.*

Since k (Theorem C.4.1) is made up by a non-finite number of convolution operationw, then it is of degree infinity, which leads to the following.

Theorem C.4.2 (Degree of Differentiability of The Non-Finite VDSKs)

k a non-finite VDSK

$\Longrightarrow$

$k \in C^\infty(\mathbf{R}, \mathbf{R})$.

The asymptotic behaviour of the Fourier transform of a non-finite kernel is established in the following theorem.

Theorem C.4.3 (Asymptotic Behaviour of The Fourier transform of The Non-Finite VDSKs)

1. *k a non-finite VDSK,*
2. *$k(x) \leftrightarrow K(u)$,*
3. *$R, p > 0$*

$\Longrightarrow$

$|K(u)| = O(|u|^{-2p})$, $|u| \to \infty$.

Proof. Choose $N > p$ and so large that $|a(\lambda)| \geq R$ when $\lambda > N$ which is possible since $|a(\lambda)| \to \infty$ as $\lambda \to \infty$. Set

$$K_N(u) = \prod_{\lambda=N+1}^{\infty} \left(\frac{a^2(\lambda)}{a^2(\lambda) + u^2}\right)$$

By equation $(C.3.5)$, it follows that

$$|K(u)| \leq \frac{|K_N(u)|}{|u|^{2N}} \prod_{\lambda=1}^{N} a^2(\lambda)$$

Because $|K_N(u)|$ never vanishes and is continuous for all $u \in \mathbf{R}$, then it has a positive lower bound. Hence, for a suitable constant M

$$|K(u)| \leq \frac{M}{|u|^{2N}}$$

In particular, if $p = 1$ in the above theorem and because k is a variation diminishing function, the following corollary results.

Corollary C.4.4 (Absolute Integrability of The Non-Finite Kernels and Their FT)

1. *k a non-finite VDSK,*
2. *$k(x) \leftrightarrow K(u)$*

$\Longrightarrow$

$k, K \in L(\mathbf{R}, \mathbf{R})$.

Application of the symmetry property of the Fourier transform, the Riemann-Lebesgue Lemma and the above corollary proves the following theorem.

Theorem C.4.5 (Asymptotic Behaviour of The Non-Finite VDSKs)

k a non-finite VDSK

$\Longrightarrow$

$k(x) \to 0$ as $|x| \to \infty$.

Some examples of non-finite VDSKs are:

$$\frac{\pi}{4}\,\mathrm{sech}^2(\frac{\pi x}{2}) \leftrightarrow u\,\mathrm{csch}\,u$$

$$= \prod_{\lambda=1}^{\infty} \left(\frac{\lambda^2\pi^2}{\lambda^2\pi^2 + u^2}\right), \qquad (C.4.2)$$

$$\frac{1}{2}\,\mathrm{sech}(\frac{\pi x}{2}) \leftrightarrow \mathrm{sech}\,u = \prod_{\lambda=1}^{\infty} \left(\frac{(2\lambda-1)^2\pi^2}{(2\lambda-1)^2\pi^2 + u^2}\right). \qquad (C.4.3)$$

Note that a non-finite VDSK does not necessarily belongs to $L^2(\mathbf{R}, \mathbf{R})$, e.g. the kernel given by equation $(C.4.3)$.

C.5 The Gaussian VDSK

The Gaussian VDSK, k, is defined by the relation

$$k(x) \leftrightarrow K(u) = \exp(-cu^2),\ c > 0. \qquad (C.5.1)$$

With $c \to 1/4c^2$, the Gaussian VDSK is now defined as

$$k(x) \leftrightarrow K(u) = \exp(-u^2/4c^2),\ c > 0. \qquad (C.5.2)$$

The basic properties of the above kernel follow directly and are collated together in the following theorem.

Theorem C.5.1 (Basic Properties of The Gaussian VDSK)

1. $k(x) = c\,\mathrm{gauss}(cx)$, $c > 0$,
2. $K(u) = \exp(-u^2/4c^2)$, $c > 0$,
3. $p > 0$

$\Longrightarrow$

A. k *is a VDSK,*

B. $k(x) \leftrightarrow K(u)$,

C. k *has mean* $\nu = 0$,

D. k *has variance* $\sigma^2 = 1/2c^2$,

E. $k, K \in L(\mathbf{R}, \mathbf{R}) \cap L^2(\mathbf{R}, \mathbf{R})$,

F. $k, K \in C^\infty(\mathbf{R}, \mathbf{R})$,

G. $|k(x)| = o(|x|^{-p})$,

H. $|K(u)| = o(|u|^{-p})$.

If in equation $(C.5.1)$, c is considered as a variable, say t, then after taking the inverse Fourier transform with respect to x we obtain a real valued function of two variables, i.e.

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-x^2/4t). \qquad (C.5.3)$$

This new function is the familiar *source solution* of the diffusion equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) k(x,t) = 0 \qquad (C.5.4)$$

C.6 Geometric Properties of The VDSKs

We consider the general geometric properties shared by the finite, non-finite and the Gaussian VDSKs where k denotes either a finite, non-finite or Gaussian VDSK throughout.

Theorem C.6.1 (Geometric Properties of The VDSKs)

1. k *a VDSK,*
2. $f\colon \mathbf{R} \to \mathbf{R}$ *bounded and convex (concave)*

$\Longrightarrow$

A. For $a, b \in \mathbf{R}$

$$V[k(x) \otimes f(x) - a - bx] \le V[f(x) - a - bx], \quad (C.6.1)$$

B. $(k \otimes f)(x)$ *is convex (concave).*

Proof. A. Inequality $(C.6.1)$ follows by a direct application of the variation diminishing property of k.

B. It is well known that f is convex iff

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) - f(x) \ge 0,$$

for all $x \in \mathbf{R}$, $h > 0$. Because k is a non-negative function, then

$$\Delta_h^2[(k \otimes f)(x)] = \Delta_h^2 \left[\int_{-\infty}^{\infty} k(y) f(x-y)\, dy\right]$$

$$= \int_{-\infty}^{\infty} k(y) \Delta_h^2 f(x-y)\, dy \ge 0$$

Thus the inequality follows. The case for which f is concave follows using a similar argument but $\Delta_h^2 f(x) \le 0$, for all $x \in \mathbf{R}$, $h > 0$.

The geometric significance of inequality (C.6.1) is that the number of intersections of the straight line $y = a+bx$, $a, b \in \mathbf{R}$, with $(k \otimes f)(x)$ does not exceed the number of intersections of $y = a + bx$ with $y = f(x)$. As a special instance of such an inequality, it follows that $(k \otimes f)(x)$ is non-negative if f is non-negative.

Corollary C.6.2 (Non-Negativity of $k \otimes f$)

1. k *a VDSK,*
2. $f\colon \mathbf{R} \to \mathbf{R}$, $f \ge 0$, *and bounded*

$\Longrightarrow$

$(k \otimes f)(x) \ge 0$, $x \in \mathbf{R}$.

From the above results, it is clear that if f is composed of a succession of alternating convex or concave arcs, then $k \otimes f$ is also made up of a similar succession of convex or concave arcs equal in number to those of f. Thus, a VDSK is *shape preserving.*

Appendix D: Relationships between the Hurst Exponent and the Topological, Fractal and Fourier Dimensions

Suppose we cut up some simple one-, two- and three-dimensional Euclidean objects (a line, a square surface and a cube, for example), make exact copies of them and then keep on repeating the copying process. Let N be the number of copies that we make at each stage and let r be the length of each of the copies, i.e. the scaling ratio. Then we have

$$Nr^{D_T} = 1, \quad D_T = 1, 2, 3, ...$$

where D_T is the topological dimension. The similarity or fractal dimension is that value of D_F which is usually (but not always) a non-integer dimension 'greater' that its topological dimension (i.e. 0,1,2,3,... where 0 is the dimension of a point on a line) and is given by

$$D_F = -\frac{\log(N)}{\log(r)}$$

The fractal dimension is that value that is strictly greater than the topological dimension. In each case, as the value of the fractal dimension increases, the fractal becomes increasingly 'space-filling' in terms of the topological dimension which the fractal dimension is approaching. A fractal exhibits structures that are self-similar. A self-similar deterministic fractal is one where a change in the scale of a function $f(x)$ (which may be a multi-dimensional function) by a scaling factor λ produces a smaller version, reduced in size by λ, i.e.

$$f(\lambda x) = \lambda f(x)$$

A self-affine deterministic fractal is one where a change in the scale of a function $f(x)$ by a factor λ produces a smaller version reduced in size by a factor λ^q, $q > 0$, i.e.

$$f(\lambda x) = \lambda^q f(x)$$

For stochastic fields, the expression

$$\Pr[f(\lambda x)] = \lambda^q \Pr[f(x)]$$

describes a statistically self-affine field - a random scaling fractal. As we zoom into the fractal, the shape changes, but the distribution of lengths remains the same.

There is no unique method for computing the fractal dimension. The methods available are broadly categorized into two families: (i) Size-measure relationships, based on recursive length or area measurements of a curve or surface using different measuring scales; (ii) application of relationships based on approximating or fitting a curve or surface to a known fractal function or statistical property, such as the variance.

Consider a simple Euclidean straight line λ of length $L(\lambda)$ over which we 'walk' a shorter 'ruler' of length δ. The number of steps taken to cover the line $N[L(\lambda), \delta]$ is then L/δ which is not always an integer for arbitrary L and δ. Since

$$\begin{aligned} N[L(\lambda), \delta] &= \frac{L(\lambda)}{\delta} = L(\lambda)\delta^{-1}, \\ \Rightarrow 1 &= \frac{\ln L(\lambda) - \ln N[L(\lambda), \delta]}{\ln \delta} \\ &= -\left(\frac{\ln N[L(\lambda), \delta] - \ln L(\lambda)}{\ln \delta}\right) \end{aligned}$$

which expresses the topological dimension $D_T = 1$ of the line. In this case, $L(\lambda)$ is the Lebesgue measure of the line and if we normalize by setting $L(\lambda) = 1$, the latter equation can then be written as

$$1 = -\lim_{\delta \to 0} \left[\frac{\ln N(\delta)}{\ln \delta}\right]$$

since there is less error in counting $N(\delta)$ as δ becomes smaller. We also then have $N(\delta) = \delta^{-1}$. For extension to a fractal curve f, the essential point is that the fractal dimension should satisfy an equation of the form

$$N[F(f), \delta] = F(f)\delta^{-D_F}$$

where $N[F(f), \delta]$ is 'read' as the number of rulers of size δ needed to cover a fractal set f whose measure is $F(f)$ which can be any valid suitable measure of the curve. Again we may normalize, which amounts to defining a new measure F' as some constant multiplied by the old measure to get

$$D_F = -\lim_{\delta \to 0} \left[\frac{\ln N(\delta)}{\ln \delta}\right]$$

where $N(\delta)$ is taken to be $N[F'(f), \delta]$ for notational convenience. Thus a piecewise continuous field has precise fractal properties over all scales. However, for the discrete (sampled) field

$$D = -\left\langle \frac{\ln N(\delta)}{\ln \delta} \right\rangle$$

where we choose values δ_1 and δ_2 (i.e. the upper and lower bounds) satisfying $\delta_1 < \delta < \delta_2$ over which we apply an averaging processes denoted by $\langle \ \rangle$. The most common approach is to utilise a bi-logarithmic plot of $\ln N(\delta)$ against $\ln \delta$, choose values δ_1 and δ_2 over which the plot is uniform and apply an appropriate data fitting algorithm (e.g. a least squares estimation method or, as used in this paper, Orthogonal Linear Regression) within these limits.

The relationship between the Fourier dimension q and the fractal dimension D_F can be determined by considering this method for analysing a statistically self-affine field. For a fractional Brownian process (with unit step length)

$$A(t) = t^H, \quad H \in (0, 1]$$

where H is the Hurst dimension. Consider a fractal curve covering a time period $\Delta t = 1$ which is divided up into $N = 1/\Delta t$ equal intervals. The amplitude increments ΔA are then given by

$$\Delta A = \Delta t^H = \frac{1}{N^H} = N^{-H}$$

The number of lengths $\delta = N^{-1}$ required to cover each interval is

$$\Delta A \Delta t = \frac{N^{-H}}{N^{-1}} = N^{1-H}$$

so that

$$N(\delta) = N N^{1-H} = N^{2-H}$$

Now, since

$$N(\delta) = \frac{1}{\delta^{D_F}}, \quad \delta \to 0,$$

then, by inspection,

$$D_F = 2 - H$$

Thus, a Brownian process, where $H = 1/2$, has a fractal dimension of 1.5. For higher topological dimensions D_T

$$D_F = D_T + 1 - H$$

This algebraic equation provides the relationship between the fractal dimension D_F, the topological dimension D_T and the Hurst dimension H. We can now determine the relationship between the Fourier dimension q and the fractal dimension D_F.

Consider a fractal signal $f(x)$ over an infinite support with a finite sample $f_X(x)$, given by

$$f_X(x) = \begin{cases} f(x), & 0 < x < X; \\ 0, & \text{otherwise.} \end{cases}$$

A finite sample is essential as otherwise the power spectrum diverges. Moreover, if $f(x)$ is a random function then for any experiment or computer simulation we must necessarily take a finite sample. Let $F_X(k)$ be the Fourier transform of $f_X(x)$, $P_X(k)$ be the power spectrum and $P(k)$ be the power spectrum of $f(x)$. Then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_X(k) \exp(ikx) dk,$$

$$P_X(k) = \frac{1}{X} |F_X(k)|^2$$

and

$$P(k) = \lim_{X \to \infty} P_X(k)$$

The power spectrum gives an expression for the power of a signal for particular harmonics. $P(k)dk$ gives the power in the range k to $k + dk$. Consider a function $g(x)$, obtained from $f(x)$ by scaling the x-coordinate by some $a > 0$, the f-coordinate by $1/a^H$ and then taking a finite sample as before, i.e.

$$g_X(x) = \begin{cases} g(x) = \frac{1}{a^H} f(ax), & 0 < x < X; \\ 0, & \text{otherwise.} \end{cases}$$

Let $G_X(k)$ and $P'_X(k)$ be the Fourier transform and power spectrum of $g_X(x)$, respectively. We then obtain an expression for G_X in terms of F_X,

$$G_X(k) = \int_0^X g_X(x) \exp(-ikx) dx =$$

$$\frac{1}{a^{H+1}} \int_0^X f(s) \exp\left(-\frac{iks}{a}\right) ds$$

where $s = ax$. Hence

$$G_X(k) = \frac{1}{a^{H+1}} F_X\left(\frac{k}{a}\right)$$

and the power spectrum of $g_X(x)$ is

$$P'_X(k) = \frac{1}{a^{2H+1}} \frac{1}{aX} \left| F_X\left(\frac{k}{a}\right)\right|^2$$

and, as $X \to \infty$,

$$P'(k) = \frac{1}{a^{2H+1}} P\left(\frac{k}{a}\right)$$

Since $g(x)$ is a scaled version of $f(x)$, their power spectra are equal, and so

$$P(k) = P'(k) = \frac{1}{a^{2H+1}} P\left(\frac{k}{a}\right)$$

If we now set $k = 1$ and then replace $1/a$ by k we get

$$P(k) \propto \frac{1}{k^{2H+1}} = \frac{1}{k^q}$$

Now since $q = 2H + 1$ and $D_F = 2 - H$, we have

$$D_F = 2 - \frac{q-1}{2} = \frac{5-q}{2}$$

The fractal dimension of a fractal signal can be calculated directly from q using the above relationship. This method also generalizes to higher topological dimensions giving

$$q = 2H + D_T$$

Thus, since

$$D_F = D_T + 1 - H$$

then $q = 5 - 2D_F$ for a fractal signal and $q = 8 - 2D_F$ for a fractal surface so that, in general,

$$q = 2(D_T + 1 - D_F) + D_T = 3D_T + 2 - 2D_F$$

and

$$D_F = D_T + 1 - H = D_T + 1 - \frac{q - D_T}{2} = \frac{3D_T + 2 - q}{2}$$

Acknowledgments

The author is supported by the Science Foundation Ireland Stokes Professorship Programme. The case study reported in this paper was undertaken at the request of Augustus Asset Managers Limited http://www.augustus.co.uk/.

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