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## Euler-Poincaré equations for G-Strands

Darryl Holm  
*Imperial College London*

Rossen Ivanov  
*Technological University Dublin, rossen.ivanov@tudublin.ie*

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## Euler-Poincaré equations for $G$ -Strands

Darryl D Holm<sup>a</sup>, Rossen I Ivanov<sup>b</sup>

<sup>a</sup> Department of Mathematics, Imperial College, London SW7 2AZ, UK

<sup>b</sup>Department of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland

E-mail: d.holm@imperial.ac.uk, rossen.ivanov@dit.ie

### Abstract.

The  $G$ -strand equations for a map  $\mathbb{R} \times \mathbb{R}$  into a Lie group  $G$  are associated to a  $G$ -invariant Lagrangian. The Lie group manifold is also the configuration space for the Lagrangian. The  $G$ -strand itself is the map  $g(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow G$ , where  $t$  and  $s$  are the independent variables of the  $G$ -strand equations. The Euler-Poincaré reduction of the variational principle leads to a formulation where the dependent variables of the  $G$ -strand equations take values in the corresponding Lie algebra  $\mathfrak{g}$  and its co-algebra,  $\mathfrak{g}^*$  with respect to the pairing provided by the variational derivatives of the Lagrangian.

We review examples of different  $G$ -strand constructions, including matrix Lie groups and diffeomorphism group. In some cases the  $G$ -strand equations are completely integrable 1+1 Hamiltonian systems that admit soliton solutions.

### 1. Introduction

We give a brief account of the  $G$ -strand construction, which gives rise to equations for a map  $\mathbb{R} \times \mathbb{R}$  into a Lie group  $G$  associated to a  $G$ -invariant Lagrangian. Our presentation reviews our previous works [7, 5, 6, 3, 8] and is aimed to illustrate the  $G$ -strand construction with several simple but instructive examples. The following examples are reviewed here:

- (i)  $SO(3)$ -strand equations for the so-called continuous spin chain. The equations reduce to the integrable chiral model in their simplest (bi-invariant) case.
- (ii)  $SO(3)$  - anisotropic chiral model, which is also integrable,
- (iii)  $\text{Diff}(\mathbb{R})$ -strand equations. These equations are in general non-integrable; however they admit solutions in 2 + 1 space-time with singular support (e.g., peakons). Peakon-antipeakon collisions governed by the  $\text{Diff}(\mathbb{R})$ -strand equations can be solved *analytically*, and potentially they can be applied in the theory of image registration.

### 2. Ingredients of Euler–Poincaré theory for Left $G$ -Invariant Lagrangians

Let  $G$  be a Lie group. A map  $g(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow G$  has two types of tangent vectors,  $\dot{g} := g_t \in TG$  and  $g' := g_s \in TG$ . Assume that the Lagrangian density function  $L(g, \dot{g}, g')$  is left  $G$ -invariant. The left  $G$ -invariance of  $L$  permits us to define  $l : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  by

$$L(g, \dot{g}, g') = L(g^{-1}g, g^{-1}\dot{g}, g^{-1}g') \equiv l(g^{-1}\dot{g}, g^{-1}g').$$

Conversely, this relation defines for any reduced lagrangian  $l = l(u, v) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  a left  $G$ -invariant function  $L : TG \times TG \rightarrow \mathbb{R}$  and a map  $g(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow G$  such that

$$u(t, s) := g^{-1}g_t(t, s) = g^{-1}\dot{g}(t, s) \quad \text{and} \quad v(t, s) := g^{-1}g_s(t, s) = g^{-1}g'(t, s).$$

**Lemma 1.** *The left-invariant tangent vectors  $u(t, s)$  and  $v(t, s)$  at the identity of  $G$  satisfy*

$$v_t - u_s = -\text{ad}_u v. \tag{1}$$

*Proof.* The proof is standard and follows from equality of cross derivatives  $g_{ts} = g_{st}$ . Equation (1) is usually called a **zero-curvature relation**. □

**Theorem 2** (Euler-Poincaré theorem for left-invariant Lagrangians).

With the preceding notation, the following two statements are equivalent:

- i Variational principle on  $TG \times TG$   $\delta \int_{t_1}^{t_2} L(g(t, s), \dot{g}(t, s), g'(t, s)) ds dt = 0$  holds, for variations  $\delta g(t, s)$  of  $g(t, s)$  vanishing at the endpoints in  $t$  and  $s$ . The function  $g(t, s)$  satisfies Euler-Lagrange equations for  $L$  on  $G$ , given by

$$\frac{\partial L}{\partial g} - \frac{\partial}{\partial t} \frac{\partial L}{\partial g_t} - \frac{\partial}{\partial s} \frac{\partial L}{\partial g_s} = 0.$$

- ii The constrained variational principle<sup>1</sup>

$$\delta \int_{t_1}^{t_2} l(\mathbf{u}(t, s), \mathbf{v}(t, s)) ds dt = 0$$

holds on  $\mathfrak{g} \times \mathfrak{g}$ , using variations of  $\mathbf{u} := g^{-1}g_t(t, s)$  and  $\mathbf{v} := g^{-1}g_s(t, s)$  of the forms

$$\delta \mathbf{u} = \dot{\mathbf{w}} + \text{ad}_{\mathbf{u}} \mathbf{w} \quad \text{and} \quad \delta \mathbf{v} = \mathbf{w}' + \text{ad}_{\mathbf{v}} \mathbf{w},$$

where  $\mathbf{w}(t, s) := g^{-1}\delta g \in \mathfrak{g}$  vanishes at the endpoints. The **Euler-Poincaré equations** hold on  $\mathfrak{g}^* \times \mathfrak{g}^*$  (**G-strand equations**)

$$\frac{d}{dt} \frac{\delta l}{\delta \mathbf{u}} - \text{ad}_{\mathbf{u}}^* \frac{\delta l}{\delta \mathbf{u}} + \frac{d}{ds} \frac{\delta l}{\delta \mathbf{v}} - \text{ad}_{\mathbf{v}}^* \frac{\delta l}{\delta \mathbf{v}} = 0 \quad \& \quad \partial_s \mathbf{u} - \partial_t \mathbf{v} = [\mathbf{u}, \mathbf{v}] = \text{ad}_{\mathbf{u}} \mathbf{v}$$

where  $(\text{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*)$  is defined via  $(\text{ad} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g})$  in the dual pairing  $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$  by,

$$\left\langle \text{ad}_{\mathbf{u}}^* \frac{\delta l}{\delta \mathbf{u}}, \mathbf{v} \right\rangle_{\mathfrak{g}^*} = \left\langle \frac{\delta l}{\delta \mathbf{u}}, \text{ad}_{\mathbf{u}} \mathbf{v} \right\rangle_{\mathfrak{g}}.$$

In 1901 Poincaré in his famous work proves that, when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the well known Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra. These equations are called now in his honor Euler-Poincaré equations. In modern language the contents of the Poincaré's article [12] is presented for example in [4, 11]. English translation of the article [12] can be found as Appendix D in [4].

### 3. G-strand equations on matrix Lie algebras

Denoting  $\mathbf{m} := \delta l / \delta \mathbf{u}$  and  $\mathbf{n} := \delta l / \delta \mathbf{v}$  in  $\mathfrak{g}^*$ , the G-strand equations become

$$\mathbf{m}_t + \mathbf{n}_s - \text{ad}_{\mathbf{u}}^* \mathbf{m} - \text{ad}_{\mathbf{v}}^* \mathbf{n} = 0 \quad \text{and} \quad \partial_t \mathbf{v} - \partial_s \mathbf{u} + \text{ad}_{\mathbf{u}} \mathbf{v} = 0.$$

For  $G$  a semisimple matrix Lie group and  $\mathfrak{g}$  its matrix Lie algebra these equations become

$$\begin{aligned} \mathbf{m}_t^T + \mathbf{n}_s^T + \text{ad}_{\mathbf{u}} \mathbf{m}^T + \text{ad}_{\mathbf{v}} \mathbf{n}^T &= 0, \\ \partial_t \mathbf{v} - \partial_s \mathbf{u} + \text{ad}_{\mathbf{u}} \mathbf{v} &= 0 \end{aligned} \tag{2}$$

where the ad-invariant pairing for semisimple matrix Lie algebras is given by

$$\langle \mathbf{m}, \mathbf{n} \rangle = \frac{1}{2} \text{tr}(\mathbf{m}^T \mathbf{n}),$$

the transpose gives the map between the algebra and its dual  $(\cdot)^T : \mathfrak{g} \rightarrow \mathfrak{g}^*$ . For semisimple matrix Lie groups, the adjoint operator is the matrix commutator. Examples are studied in [7, 6, 3].

<sup>1</sup> As with the basic Euler-Poincaré equations, this is not strictly a variational principle in the same sense as the standard Hamilton's principle. It is more like the Lagrange d'Alembert principle, because we impose the stated constraints on the variations allowed.

**4. Lie-Poisson Hamiltonian formulation**

Legendre transformation of the Lagrangian  $\ell(\mathbf{u}, \mathbf{v}) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  yields the Hamiltonian  $h(\mathbf{m}, \mathbf{v}) : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$

$$h(\mathbf{m}, \mathbf{v}) = \langle \mathbf{m}, \mathbf{u} \rangle - \ell(\mathbf{u}, \mathbf{v}). \tag{3}$$

Its partial derivatives imply

$$\frac{\delta \ell}{\delta \mathbf{u}} = \mathbf{m}, \quad \frac{\delta h}{\delta \mathbf{m}} = \mathbf{u} \quad \text{and} \quad \frac{\delta h}{\delta \mathbf{v}} = -\frac{\delta \ell}{\delta \mathbf{v}} = \mathbf{v}.$$

These derivatives allow one to rewrite the Euler-Poincaré equation solely in terms of momentum  $\mathbf{m}$  as

$$\begin{aligned} \partial_t \mathbf{m} &= \text{ad}_{\delta h / \delta \mathbf{m}}^* \mathbf{m} + \partial_s \frac{\delta h}{\delta \mathbf{v}} - \text{ad}_{\mathbf{v}}^* \frac{\delta h}{\delta \mathbf{v}}, \\ \partial_t \mathbf{v} &= \partial_s \frac{\delta h}{\delta \mathbf{m}} - \text{ad}_{\delta h / \delta \mathbf{m}} \mathbf{v}. \end{aligned} \tag{4}$$

Assembling these equations into Lie-Poisson Hamiltonian form gives,

$$\frac{\partial}{\partial t} \begin{bmatrix} \mathbf{m} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \text{ad}^*(\cdot) \mathbf{m} & \partial_s - \text{ad}_{\mathbf{v}}^* \\ \partial_s + \text{ad}_{\mathbf{v}} & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \mathbf{m} \\ \delta h / \delta \mathbf{v} \end{bmatrix} \tag{5}$$

The Hamiltonian matrix in equation (5) also appears in the Lie-Poisson brackets for Yang-Mills plasmas, for spin glasses and for perfect complex fluids, such as liquid crystals.

**5. Example: The Euler-Poincaré PDEs for the  $SO(3)$ -strand and the chiral model. The 2-time spatial and body angular velocities on  $\mathfrak{so}(3)$**

Let us make the following explicit identification:

$$\mathbf{u} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \in \mathfrak{g} \quad \leftrightarrow \quad \mathbf{u} \equiv \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{R}^3 \tag{6}$$

and similarly for  $\mathbf{v}$ . In terms of the corresponding group element  $g(s, t)$ , describing rotation,  $\mathbf{u}(t, s) = g^{-1} \partial_t g(t, s)$  and  $\mathbf{v}(t, s) = g^{-1} \partial_s g(t, s)$  resemble 2 body angular velocities. For  $G = SO(3)$  and Lagrangian  $\ell(\mathbf{u}, \mathbf{v}) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ , in 1 + 1 space-time the Euler-Poincaré equation becomes

$$\frac{\partial}{\partial t} \frac{\delta \ell}{\delta \mathbf{u}} + \mathbf{u} \times \frac{\delta \ell}{\delta \mathbf{u}} = - \left( \frac{\partial}{\partial s} \frac{\delta \ell}{\delta \mathbf{v}} + \mathbf{v} \times \frac{\delta \ell}{\delta \mathbf{v}} \right), \tag{7}$$

and its auxiliary equation becomes

$$\frac{\partial}{\partial t} \mathbf{v} = \frac{\partial}{\partial s} \mathbf{u} + \mathbf{v} \times \mathbf{u}. \tag{8}$$

The Hamiltonian form of these equations on  $\mathfrak{so}(3)^*$  are obtained from the Legendre transform relations

$$\frac{\delta \ell}{\delta \mathbf{u}} = \mathbf{m}, \quad \frac{\delta h}{\delta \mathbf{m}} = \mathbf{u} \quad \text{and} \quad \frac{\delta h}{\delta \mathbf{v}} = -\frac{\delta \ell}{\delta \mathbf{v}}.$$

Hence, the Euler-Poincaré equation implies the Lie-Poisson Hamiltonian structure in vector form

$$\partial_t \begin{bmatrix} \mathbf{m} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{m} \times & \partial_s + \mathbf{v} \times \\ \partial_s + \mathbf{v} \times & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \mathbf{m} \\ \delta h / \delta \mathbf{v} \end{bmatrix}.$$

This Poisson structure appears in various other theories, such as complex fluids and filament dynamics. When

$$\ell = \frac{1}{2} \int (\mathbf{u} \cdot A \mathbf{u} + \mathbf{v} \cdot B \mathbf{v}) ds \tag{9}$$

this is the  $SO(3)$  *spin-chain model*, which is in general non-integrable- eq. (7) and (8) give:

$$\frac{\partial}{\partial t} A\mathbf{u} + \mathbf{u} \times A\mathbf{u} + \frac{\partial}{\partial s} B\mathbf{v} + \mathbf{v} \times B\mathbf{v} = 0, \tag{10}$$

$$\frac{\partial}{\partial t} \mathbf{v} = \frac{\partial}{\partial s} \mathbf{u} + \mathbf{v} \times \mathbf{u}. \tag{11}$$

When  $A = -B = 1$ , this is the  $SO(3)$  *chiral model*, which is an integrable Hamiltonian system.

$$\mathbf{u}_t - \mathbf{v}_s = 0, \tag{12}$$

$$\mathbf{v}_t - \mathbf{u}_s + \mathbf{u} \times \mathbf{v} = 0. \tag{13}$$

### 6. Integrability

Some of the  $G$ -strands models are well known integrable models. They have a *zero-curvature representation* for two operators  $L$  and  $M$  of the form

$$L_t - M_s + [L, M] = 0, \tag{14}$$

which is the compatibility condition for a pair of linear equations

$$\psi_s = L\psi, \quad \text{and} \quad \psi_t = M\psi.$$

For the  $SO(3)$  chiral model for example these operators are

$$\begin{aligned} L &= \frac{1}{4} \left[ (1 + \lambda)(\mathbf{u} - \mathbf{v}) - \left(1 + \frac{1}{\lambda}\right) (\mathbf{u} + \mathbf{v}) \right], \\ M &= -\frac{1}{4} \left[ (1 + \lambda)(\mathbf{u} - \mathbf{v}) + \left(1 + \frac{1}{\lambda}\right) (\mathbf{u} + \mathbf{v}) \right]. \end{aligned} \tag{15}$$

Another integrable matrix example:  $SO(3)$  anisotropic Chiral model [2]

$$\begin{aligned} \partial_t \mathbf{v}(t, s) - \partial_s \mathbf{u}(t, s) + \mathbf{u} \times P\mathbf{v} - \mathbf{v} \times P\mathbf{u} &= 0, \\ \partial_s \mathbf{v}(t, s) - \partial_t \mathbf{u}(t, s) - \mathbf{v} \times P\mathbf{v} + \mathbf{u} \times P\mathbf{u} &= 0. \end{aligned} \tag{16}$$

$P = \text{diag}(P_1, P_2, P_3)$  is a constant diagonal matrix. Under the linear change of variables

$$\mathbf{X} = \mathbf{u} - \mathbf{v} \quad \text{and} \quad \mathbf{Y} = -\mathbf{u} - \mathbf{v} \tag{17}$$

equations (16) acquire the form of the following  $SO(3)$  anisotropic chiral model,

$$\begin{aligned} \partial_t \mathbf{X}(t, s) + \partial_s \mathbf{X}(t, s) + \mathbf{X} \times P\mathbf{Y} &= 0, \\ \partial_t \mathbf{Y}(t, s) - \partial_s \mathbf{Y}(t, s) + \mathbf{Y} \times P\mathbf{X} &= 0. \end{aligned} \tag{18}$$

The system (18) represents two *cross-coupled* equations for  $\mathbf{X}$  and  $\mathbf{Y}$ . These equations preserve the magnitudes  $|\mathbf{X}|^2$  and  $|\mathbf{Y}|^2$ , so they allow the further assumption that the vector fields  $(\mathbf{X}, \mathbf{Y})$  take values on the product of unit spheres  $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ . The anisotropic chiral model is an integrable system and its Lax pair in terms of  $(\mathbf{u}, \mathbf{v})$  utilizes the following isomorphism between  $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$  and  $\mathfrak{so}(4)$ :

$$A(\mathbf{u}, \mathbf{v}) = \begin{pmatrix} 0 & u_3 & -u_2 & v_1 \\ -u_3 & 0 & u_1 & v_2 \\ u_2 & -u_1 & 0 & v_3 \\ -v_1 & -v_2 & -v_3 & 0 \end{pmatrix}. \tag{19}$$

The system (16) can be recovered as a compatibility condition of the operators

$$L = \partial_s - A(\mathbf{v}, \mathbf{u})(\lambda \text{Id} + J), \quad (20)$$

$$M = \partial_t - A(\mathbf{u}, \mathbf{v})(\lambda \text{Id} + J), \quad (21)$$

where the diagonal matrix  $J$  is defined by

$$J = -\frac{1}{2} \text{diag}(P_1, P_2, P_3, P_1 + P_2 + P_3). \quad (22)$$

This Lax pair is due to Bordag and Yanovski [1]. The  $O(3)$  anisotropic chiral model can be derived as an Euler-Poincaré equation from a Lagrangian with quadratic kinetic and potential energy. The details are presented in [7].

**Remark 3.** If  $P = \text{Id}$ , equations (16) recover the  $SO(3)$  chiral model.

### 7. The $\text{Diff}(\mathbb{R})$ -strand system

The constructions described briefly in the previous sections can be easily generalized in cases where the Lie group is the group of the Diffeomorphisms. Consider Hamiltonian which is a right-invariant bilinear form given by the  $H^1$  Sobolev inner product

$$H(u, v) \equiv \frac{1}{2} \int_{\mathcal{M}} (uv + u_x v_x) dx. \quad (23)$$

The manifold  $\mathcal{M}$  is  $\mathbb{S}^1$  or in the case when the class of smooth functions vanishing rapidly at  $\pm\infty$  is considered, we will allow  $\mathcal{M} \equiv \mathbb{R}$ .

Let us introduce the notation  $u(g(x)) \equiv u \circ g$ . If  $g(x) \in G$ , where  $G \equiv \text{Diff}(\mathcal{M})$ , then

$$H(u, v) = H(u \circ g, v \circ g)$$

is a right-invariant  $H^1$  metric.

Let us further consider an one-parametric family of diffeomorphisms,  $g(x, t)$  by defining the  $t$ -evolution as

$$\dot{g} = u(g(x, t), t), \quad g(x, 0) = x, \quad \text{i.e.} \quad \dot{g} = u \circ g \in T_g G; \quad (24)$$

$u = \dot{g} \circ g^{-1} \in \mathfrak{g}$ , where  $\mathfrak{g}$ , the corresponding Lie-algebra is the algebra of vector fields,  $\text{Vect}(\mathcal{M})$ . Now we recall the following result:

**Theorem 4.** (A. Kirillov, 1980, [9, 10]) *The dual space of  $\mathfrak{g}$  is a space of distributions but the subspace of local functionals, called the regular dual  $\mathfrak{g}^*$  is naturally identified with the space of quadratic differentials  $m(x)dx^2$  on  $\mathcal{M}$ . The pairing is given for any vector field  $u\partial_x \in \text{Vect}(\mathcal{M})$  by*

$$\langle m dx^2, u \partial_x \rangle = \int_{\mathcal{M}} m(x) u(x) dx$$

The coadjoint action coincides with the action of a diffeomorphism on the quadratic differential:

$$\text{Ad}_g^* : m dx^2 \mapsto m(g) g_x^2 dx^2$$

and

$$\text{ad}_u^* = 2u_x + u\partial_x$$

Indeed, a simple computation shows that

$$\begin{aligned} \langle \text{ad}_{u\partial_x}^* m dx^2, v \partial_x \rangle &= \langle m dx^2, [u\partial_x, v\partial_x] \rangle = \int_{\mathcal{M}} m(u_x v - v_x u) dx = \\ \int_{\mathcal{M}} v(2mu_x + um_x) dx &= \langle (2mu_x + um_x) dx^2, v \partial_x \rangle, \end{aligned}$$

i.e.  $\text{ad}_u^* m = 2u_x m + um_x$ .

The  $\text{Diff}(\mathbb{R})$ -strand system arises when we choose  $G = \text{Diff}(\mathbb{R})$ . For a two-parametric group we have two tangent vectors

$$\partial_t g = u \circ g \quad \text{and} \quad \partial_s g = v \circ g,$$

where the symbol  $\circ$  denotes composition of functions.

In this right-invariant case, the  $G$ -strand PDE system with reduced Lagrangian  $\ell(u, v)$  takes the form,

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + \frac{\partial}{\partial s} \frac{\delta \ell}{\delta v} &= -\text{ad}_u^* \frac{\delta \ell}{\delta u} - \text{ad}_v^* \frac{\delta \ell}{\delta v}, \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial s} &= \text{ad}_u v. \end{aligned} \tag{25}$$

Of course, the distinction between the maps  $(u, v) : \mathbb{R} \times \mathbb{R} \rightarrow \mathfrak{g} \times \mathfrak{g}$  and their pointwise values  $(u(t, s), v(t, s)) \in \mathfrak{g} \times \mathfrak{g}$  is clear. Likewise, for the variational derivatives  $\delta \ell / \delta u$  and  $\delta \ell / \delta v$ .

### 8. The $\text{Diff}(\mathbb{R})$ -strand Hamiltonian structure

Upon setting  $m = \delta \ell / \delta u$  and  $n = \delta \ell / \delta v$ , the right-invariant  $\text{Diff}(\mathbb{R})$ -strand equations in (25) for maps  $\mathbb{R} \times \mathbb{R} \rightarrow G = \text{Diff}(\mathbb{R})$  in one spatial dimension may be expressed as a system of two 1+2 PDEs in  $(t, s, x)$ ,

$$\begin{aligned} m_t + n_s &= -\text{ad}_u^* m - \text{ad}_v^* n = -(um)_x - mu_x - (vn)_x - nv_x, \\ v_t - u_s &= -\text{ad}_v u = -uv_x + vu_x. \end{aligned} \tag{26}$$

The Hamiltonian structure for these  $\text{Diff}(\mathbb{R})$ -strand equations is obtained by Legendre transforming to

$$h(m, v) = \langle m, u \rangle - \ell(u, v).$$

One may then write the equations (26) in Lie-Poisson Hamiltonian form as

$$\frac{d}{dt} \begin{bmatrix} m \\ v \end{bmatrix} = \begin{bmatrix} -\text{ad}^*(\cdot)m & \partial_s + \text{ad}_v^* \\ \partial_s - \text{ad}_v & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta m = u \\ \delta h / \delta v = -n \end{bmatrix}. \tag{27}$$

### 9. Peakon solutions of the $\text{Diff}(\mathbb{R})$ -strand equations

With the following choice of Lagrangian,

$$\ell(u, v) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{2} \|v\|_{H^1}^2, \tag{28}$$

the corresponding Hamiltonian is positive-definite and the  $\text{Diff}(\mathbb{R})$ -strand equations (26) admit peakon solutions in *both* momenta

$$m = u - u_{xx} \quad \text{and} \quad n = -(v - v_{xx}),$$

with continuous velocities  $u$  and  $v$ . This is a two-component generalization of the CH equation.

**Theorem 5.** *The  $\text{Diff}(\mathbb{R})$ -strand equations (26) admit singular solutions expressible as linear superpositions summed over  $a \in \mathbb{Z}$*

$$\begin{aligned} m(s, t, x) &= \sum_a M_a(s, t) \delta(x - Q^a(s, t)), \\ n(s, t, x) &= \sum_a N_a(s, t) \delta(x - Q^a(s, t)), \\ u(s, t, x) &= K * m = \sum_a M_a(s, t) K(x, Q^a), \\ v(s, t, x) &= -K * n = -\sum_a N_a(s, t) K(x, Q^a), \end{aligned} \tag{29}$$

that are peakons in the case that  $K(x, y) = \frac{1}{2} e^{-|x-y|}$  is the Green function the inverse Helmholtz operator  $(1 - \partial_x^2)$ :

$$(1 - \partial_x^2)K(x, 0) = \delta(x)$$



The solution parameters  $\{Q^a(s, t), M_a(s, t), N_a(s, t)\}$  with  $a \in \mathbb{Z}$  that specify the singular solutions (29) are determined by the following set of evolutionary PDEs in  $s$  and  $t$ , in which we denote  $K^{ab} := K(Q^a, Q^b)$  with integer summation indices  $a, b, c, e \in \mathbb{Z}$ :

$$\begin{aligned} \partial_t Q^a(s, t) &= u(Q^a, s, t) = \sum_b M_b(s, t) K^{ab}, \\ \partial_s Q^a(s, t) &= v(Q^a, s, t) = -\sum_b N_b(s, t) K^{ab}, \\ \partial_t M_a(s, t) &= -\partial_s N_a - \sum_c (M_a M_c - N_a N_c) \frac{\partial K^{ac}}{\partial Q^a} \quad (\text{no sum on } a), \\ \partial_t N_a(s, t) &= -\partial_s M_a + \sum_{b,c,e} (N_b M_c - M_b N_c) \frac{\partial K^{ec}}{\partial Q^e} (K^{eb} - K^{cb})(K^{-1})_{ae}. \end{aligned} \tag{30}$$

The last pair of equations in (30) may be solved as a system for the momenta, i.e., Lagrange multipliers  $(M_a, N_a)$ , then used in the previous pair to update the support set of positions  $Q^a(t, s)$ .

**10. Single-peakon solution of the of the Diff( $\mathbb{R}$ )-strand system**

The single-peakon solution of the Diff( $\mathbb{R}$ )-strand equations (26) is straightforward to obtain from (30). Combining the equations in (30) for a single peakon shows that  $Q^1(s, t)$  satisfies the Laplace equation,

$$(\partial_s^2 - \partial_t^2)Q^1(s, t) = 0.$$

Thus, any function  $h(s, t)$  that solves the wave equation provides a solution  $Q^1 = h(s, t)$ . From the first two equations in (30)

$$M_1(s, t) = \frac{1}{K_0} h_t(s, t) \quad N_1(s, t) = \frac{1}{K_0} h_s(s, t),$$

where  $K_0 = K(0, 0)$ .

The solutions for the single-peakon parameters  $Q^1, M_1$  and  $N_1$  depend only on one function  $h(s, t)$ , which in turn depends on the  $(s, t)$  boundary conditions. The shape of the Green's function comes into the corresponding solutions for the peakon profiles

$$u(s, t, x) = M_1(s, t)K(x, Q^1(s, t)), \quad v(s, t, x) = -N_1(s, t)K(x, Q^1(s, t)).$$

**11. Peakon-Antipeakon collisions on a Diff( $\mathbb{R}$ )-strand**

Denote the relative spacing  $X(s, t) = Q^1 - Q^2$  for the peakons at positions  $Q^1(t, s)$  and  $Q^2(t, s)$  on the real line and the Green's function  $K = K(X)$ . Then the first two equations in (30) imply

$$\begin{aligned} \partial_t X &= (M_1 - M_2)(K_0 - K(X)), \\ \partial_s X &= -(N_1 - N_2)(K_0 - K(X)), \end{aligned} \tag{31}$$

where  $K_0 = K(0)$ .

The second pair of equations in (30) may then be written as

$$\begin{aligned} \partial_t M_1 &= -\partial_s N_1 - (M_1 M_2 - N_1 N_2)K'(X), \\ \partial_t M_2 &= -\partial_s N_2 + (M_1 M_2 - N_1 N_2)K'(X), \\ \partial_t N_1 &= -\partial_s M_1 + (N_1 M_2 - M_1 N_2) \frac{K_0 - K}{K_0 + K} K'(X), \\ \partial_t N_2 &= -\partial_s M_2 + (N_1 M_2 - M_1 N_2) \frac{K_0 - K}{K_0 + K} K'(X). \end{aligned} \tag{32}$$

Asymptotically, when the peakons are far apart, the system (32) simplifies, since  $\frac{K_0 - K}{K_0 + K} \rightarrow 1$  and  $K'(X) \rightarrow 0$  as  $|X| \rightarrow \infty$ .

The system (32) has two immediate conservation laws obtained from their sums and differences,

$$\begin{aligned}\partial_t(M_1 + M_2) &= -\partial_s(N_1 + N_2), \\ \partial_t(N_1 - N_2) &= -\partial_s(M_1 - M_2).\end{aligned}\tag{33}$$

These may be resolved by setting

$$\begin{aligned}M_1 - M_2 &= \frac{\partial_t X}{K_0 - K}, & N_1 - N_2 &= -\frac{\partial_s X}{K_0 - K}, \\ M_1 + M_2 &= \partial_s \phi, & N_1 + N_2 &= -\partial_t \phi,\end{aligned}\tag{34}$$

and introducing two potential functions,  $X$  and  $\phi$ , for which equality of cross derivatives will now produce the system of equations (31) and (32).

### 12. A simplification.

A simplification arises if  $\phi = 0$ , in which case the collision is perfectly antisymmetric, as seen from equation (34). This is the peakon-antipeakon collision, for which the equation for  $X$  reduces to

$$(\partial_t^2 - \partial_s^2)X + \frac{K'}{2(K_0 - K)}(X_t^2 - X_s^2) = 0.\tag{35}$$

This equation can be easily rearranged to produce a linear equation:

$$(\partial_t^2 - \partial_s^2)F(X) = 0, \quad \text{where} \quad F(X) = \int_{X_0}^X (K_0 - K(Y))^{-1/2} dY.\tag{36}$$

When  $K(Y) = \frac{1}{2}e^{-|Y|}$ , we have

$$F(X) = \sqrt{2} \int_{X_0}^X \frac{1}{\sqrt{1 - e^{-|Y|}}} dY.\tag{37}$$

We can take for simplicity  $X_0 = 0$ , this would change  $F(X)$  only by a constant. The computation gives

$$F(X) = 2\sqrt{2} \operatorname{sign}(X) \cosh^{-1} \left( e^{|X|/2} \right)$$

. Hence the solution  $X(t, s)$  can be expressed in terms of any solution  $h(t, s)$  of the linear wave equation  $(\partial_t^2 - \partial_s^2)h(t, s) = 0$  as

$$X(t, s) = \pm \ln \left( \cosh^2(h(t, s)) \right).\tag{38}$$

$h(t, s)$  is any solution of the wave equation.

$$M_1 = -M_2 = \frac{\partial_t X}{2(K_0 - K(X))}, \quad N_1 = -N_2 = -\frac{\partial_s X}{2(K_0 - K(X))}.$$

### Complex Diff( $\mathbb{R}$ )-strand equations

The Diff( $\mathbb{R}$ )-strands may also be *complexified*. Upon complexifying  $(s, t) \in \mathbb{R}^2 \rightarrow (z, \bar{z}) \in \mathbb{C}$  where  $\bar{z}$  denotes the complex conjugate of  $z$  and setting  $\partial_z g = u \circ g$  and  $\partial_{\bar{z}} g = \bar{u} \circ g$  the Euler-Poincaré  $G$ -strand equations in (26) become

$$\begin{aligned}\frac{\partial}{\partial z} \frac{\delta \ell}{\delta u} + \frac{\partial}{\partial \bar{z}} \frac{\delta \ell}{\delta \bar{u}} &= -\operatorname{ad}_u^* \frac{\delta \ell}{\delta u} - \operatorname{ad}_{\bar{u}}^* \frac{\delta \ell}{\delta \bar{u}}, \\ \frac{\partial \bar{u}}{\partial z} - \frac{\partial u}{\partial \bar{z}} &= \operatorname{ad}_u \bar{u}.\end{aligned}\tag{39}$$

Here the Lagrangian  $\ell$  is taken to be real:

$$\ell(u, \bar{u}) = \frac{1}{2} \|\nu\|_{H^1}^2 = \frac{1}{2} \int u (1 - \partial_x^2) \bar{u} dx.\tag{40}$$

Upon setting  $m = \delta\ell/\delta u$ ,  $\bar{m} = \delta\ell/\delta\bar{u}$ , for the real Lagrangian  $\ell$ , equations (39) may be rewritten as

$$\begin{aligned}m_z + \bar{m}_{\bar{z}} &= -\text{ad}_u^* m - \text{ad}_{\bar{u}}^* \bar{m} = -(um)_x - mu_x - (\bar{u}\bar{m})_x - \bar{m}\bar{u}_x, \\ \bar{u}_z - u_{\bar{z}} &= -\text{ad}_u u = -u\bar{u}_x + \bar{u}u_x,\end{aligned}\tag{41}$$

where the independent coordinate  $x \in \mathbb{R}$  is on the real line, although coordinates  $(z, \bar{z}) \in \mathbb{C}$  are complex, as are solutions  $u$ , and  $m = u - u_{xx}$ . This is a possible complexification of the Camassa-Holm equation. These equations are invariant under two involutions,  $P$  and  $C$ , where

$$P : (x, m) \rightarrow (-x, -m) \quad \text{and} \quad C : \text{Complex conjugation.}$$

They admit singular solutions just as before, modulo  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ . For real variables  $m = \bar{m}$ ,  $u = \bar{u}$  and real evolution parameter  $z = \bar{z} =: t$ , they reduce to the CH equation. Their travelling wave solutions and other possible CH complexifications are studied in [5].

## Conclusions

The  $G$ -strand equations comprise a system of PDEs obtained from the Euler-Poincaré (EP) variational equations for a  $G$ -invariant Lagrangian, coupled to an auxiliary *zero-curvature* equation. Once the  $G$ -invariant Lagrangian has been specified, the system of  $G$ -strand equations in (2) follows automatically in the EP framework. For matrix Lie groups, some of the the  $G$ -strand systems are integrable. The single-peakon and the peakon-antipeakon solution of the  $\text{Diff}(\mathbb{R})$ -strand equations (26) depends on a single function of  $s, t$ . The *complex*  $\text{Diff}(\mathbb{R})$ -strand equations and their peakon collision solutions have also been solved by elementary means. The stability of the single-peakon solution under perturbations into the full solution space of equations (26) would be an interesting problem for future work.

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