1989-01-01

Maximal order abelian subgroups of symmetric groups

J. M. Burns

Brendan Goldsmith
Technological University Dublin, brendan.goldsmith@tudublin.ie

Recommended Citation

Burns, J. M., & Goldsmith, B. (1989). Maximal order abelian subgroups of symmetric groups. *Journal of the London Mathematical Society*, vol. 21, pg. 70-72. doi:10.1112/blms/21.1.70

MAXIMAL ORDER ABELIAN SUBGROUPS OF SYMMETRIC GROUPS

J. M. BURNS AND B. GOLDSMITH

1. *Introduction*

Colleagues working in the area of statistical mechanics recently posed the following question: 'Are all Abelian subgroups of maximal order in the symmetric group S_n isomorphic?' Their problem arose from considerations of reducible representations constructed from tensor products of unitary representations arising in the study of the statistical mechanics of systems of n quantum spins. In particular they wanted to know what the situation was as $n \to \infty$. A complete classification of the maximal order Abelian subgroups of S_n can be derived from recent general results of Kovacs and Praeger [2]; our objective here, however, is to give a straightforward proof of this classification using the concept of the trace of an Abelian group introduced recently by Hoffman [1]. Our notation will be standard; in particular a cyclic group of order n will be denoted by Z_n and the product of k copies of such a group will be denoted by Z_n^k. Our principal result is:

THEOREM 1. *Let G be an Abelian subgroup of maximal order of the symmetric group S_N. Then*

(i) $G \cong Z_3^k$ *if* $N = 3k$,
(ii) $G \cong Z_2 \times Z_3^k$ *if* $N = 3k+2$,
(iii) *either* $G \cong Z_4 \times Z_3^{k-1}$ *or* $G \cong Z_2 \times Z_2 \times Z_3^{k-1}$ *if* $N = 3k+1$.

2. *The trace of a finite abelian group*

As is well known, any finite Abelian group G can be expressed (in multiplicative notation) as a direct product of cyclic groups of prime power order,

$$G = Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_k}$$

and the unordered sequence $(m_1, m_2, \ldots, m_k)$ completely determines (up to isomorphism) G. Then, as observed in Hoffman [1], any symmetric function of the m_i is an invariant of G. In particular the function $T(G) = \sum_{i=1}^{k} m_i$ is an invariant, the *trace* of G. (Note that if G is trivial then we set $T(G) = 0$.) We shall use the following result from [1].

PROPOSITION 2. *If an Abelian group G is imbedded in the symmetric group S_n then $T(G) \leqslant n$.*

Received 5 January 1988.

1980 *Mathematics Subject Classification* 20B35.

Bull. London Math. Soc. 21 (1989) 70–72

Notice that the converse is trivially true. Thus it follows that the problem of determining the maximal order Abelian subgroups of a symmetric group S_N reduces to solving:

Maximize the product $\prod m_i$ of the prime powers m_i subject to the constraint $\sum m_i \leqslant N$. (*)

It is in this form that we tackle the problem.

3. *The classification*

Before presenting the combinatorial solution we note the following heuristical argument. It is clear (see Lemma 3 below) that the constraint $\leqslant N$ in (*) is rather spurious; maximality will require equality. A standard application of Lagrange multiplier techniques would indicate that all (or as many as possible) of the integers m_i should be chosen equal (to m say). Thus we want to maximize an expression of the form m^k subject to $km = N$. But this amounts to maximizing $m^{N/m}$ and, regarding this as a real function, an elementary calculation gives this function as having a maximum at e and it is decreasing from e onwards. Moreover since $2^{N/2} < 3^{N/3}$ we would expect the solution of the integer-valued problem to be of the form 3^k. We now give a rigorous demonstration of Theorem 1.

LEMMA 3. *The solution to* (*) *occurs when* $\sum m_i = N$.

Proof. If $\sum_{i=1}^k m_i < N$ let $N - \sum_{i=1}^k m_i = t$ where $t \geqslant 1$. Then

$$m_1 + t = p_1^{r_1} \dots p_s^{r_s} = p_1^{r_1} u$$

where the p_i are primes. Then the prime powers $m_2, \dots, m_k, q_1, \dots, q_u$ (where each $q_i = p_1^{r_1}$) have sum N but their product exceeds $\prod_{i=1}^k m_i$ since

$$(p_1^{r_1})^u \geqslant u(p_1^{r_1}) = m_1 + t > m_1.$$

Suppose the prime powers $2^{t_1}, 2^{t_2}, \dots, 2^{t_l}, \lambda_1, \lambda_2, \dots, \lambda_k$ are a solution of:

$$\text{Maximize } \prod m_i \text{ subject to } \sum m_i = N. \qquad (**)$$

Then we have the following.

LEMMA 4. *The terms* $\lambda_i (1 \leqslant i \leqslant k)$ *are all powers of* 3.

Proof. Suppose, without loss of generality, that $\lambda_1 = p^t$ where p is a prime $\geqslant 5$. Let $n = p^{t-1} \geqslant 1$ and note that $n \geqslant t$. Then we may write $p = 3s + r$ where $s \geqslant 1$ and $r = 1, 2$. If $r = 2$ and we replace λ_1 by $q_1 = q_2 = \dots = q_{ns} = 3, m_1 = m_2 = \dots = m_n = 2$ then the set of prime powers $2^{t_1}, \dots, 2^{t_l}, \lambda_2, \dots, \lambda_k, m_1 = \dots = m_n, q_1 = \dots = q_{ns}$ has sum N but has product which exceeds $2^{\Sigma t_i} \prod \lambda_i$ since $3^{sn} 2^n = (3^s 2)^n > (3s+2)^n \geqslant p^t$ since $n \geqslant t$-contradiction. However if $r = 1$, (note then that $s \geqslant 2$) and we replace λ_1 by $q_1 = \dots = q_{ns} = 3$ then the terms $2^{t_1}, \dots, 2^{t_l}, \lambda_2, \dots, \lambda_k, q_1, \dots, q_{ns}$ have sum $\leqslant N$. Thus, although this set may not be an optimal choice, its product exceeds the supposed maximal product since $3^{ns} > (3s+1)^n$ (because $s \geqslant 2$) and $(3s+1)^n = p^n \geqslant p^t$—contradiction. This contradiction to the maximality property in (**) establishes the lemma.

LEMMA 5. (i) $t_i \leqslant 2$ *for each* i; (ii) $\sum_{i=1}^l t_i \leqslant 2$.

Proof. (i) Assume, without loss of generality, that $t_1 > 2$ and express $2^{t_1} = 3s+r$ where $r = 1, 2$ and $s \geqslant 2$. Then if $r = 2$ replace the term 2^{t_1} by $q_1 = \ldots = q_s = 3$ and $m_1 = 2$. These terms have sum 2^t but their product is $3^s \cdot 2$ which exceeds $3s+2$, giving a contradiction. Finally if $r = 1$ replace 2^{t_1} by $q_1 = \ldots = q_s = 3$. The resulting system, which is not optimal since it has sum $N-1$, has a product which exceeds the supposed maximal product since $3^s > 3s+1$ (because $s \geqslant 2$), again a contradiction. Thus we have established part (i) of the Lemma.

Assume in (ii) that $\sum_{i=1}^{l} t_i > 2$ and express this sum as $3s+r$. We consider separately the three possible cases $s = 0, 1, 2$. Observe firstly however that $\sum_{i=1}^{l} 2^{t_i} = 2\sum_{i=1}^{l} t_i$ if each $t_i \leqslant 2$.

(a) If $r = 0$ replace $2^{t_1}, \ldots, 2^{t_l}$ by $q_1 = \ldots = q_{2s} = 3$ and note that

$$\sum_{i=1}^{l} 2^{t_i} = 6s = q_1 + \ldots + q_{2s}.$$

However since $3^{2s} > 2^{3s}$ this would contradict the maximality property in (**).

(b) If $r = 1$ replace $2^{t_1}, \ldots, 2^{t_l}$ by $q_1 = \ldots = q_{2s} = 3, m_1 = 2$. Again this leaves the sum unchanged but the new product would exceed the supposed maximal product since $3^{2s} \cdot 2 > 2^{3s+1}$.

(c) This case is handled exactly as in (b) only using $m_1 = 2^2$.

LEMMA 6. *The maximum power of* 3 *occurring in any term* λ_i *is* 1.

Proof. Suppose, without loss of generality, that $\lambda_1 = 3^k$ where $k > 1$. Then replace the set $\{\lambda_1, \ldots, \lambda_k\}$ by $\{m_0, m_1, m_2, \lambda_2, \ldots, \lambda_k\}$ where $m_0 = m_1 = m_2 = 3^{k-1}$. The set $\{m_0, m_1, m_2, \lambda_2, \ldots, \lambda_k, 2^{t_1}, \ldots, 2^{t_l}\}$ satisfies the additive condition in (**) but $m_0 m_1 m_2 = 3^{3k-3} > 3^k$ if $k > 1$. So we must conclude $k = 1$.

The classification now follows immediately from an examination of the congruence of N modulo 3 using the fact that $3^{k-1} \cdot 4 > 3^k \cdot 1$.

References

1. M. HOFFMAN, 'An invariant of finite abelian groups', *Amer. Math. Monthly* 94 (1987) 664–666.
2. L. G. KOVACS and C. E. PRAEGER, 'Finite permutation groups with large abelian quotients', Research Report 13 (1987) Australian National University Mathematics Research Report Series.

Dublin Institute of Technology
Kevin Street
Dublin 8
Ireland

Dublin Institute of Technology
Kevin Street
Dublin 8
Ireland
and
Dublin Institute for Advanced Studies
Burlington Road
Dublin 4
Ireland