2012

Singular Solutions of Cross-coupled EPDiff Equations: Waltzing Peakons and Compacton Pairs

Colin Cotter  
*Imperial College London*

Darryl Holm  
*Imperial College London*

Rossen Ivanov  
*Technological University Dublin*, rossen.ivanov@tudublin.ie

James Percival  
*Imperial College London*

Follow this and additional works at: [https://arrow.tudublin.ie/scschmatcon](https://arrow.tudublin.ie/scschmatcon)

Part of the Dynamic Systems Commons, Non-linear Dynamics Commons, Numerical Analysis and Computation Commons, Ordinary Differential Equations and Applied Dynamics Commons, and the Partial Differential Equations Commons

**Recommended Citation**

1. EPDiff equations. Let us define an one-parametric group of diffeomorphisms of \( \mathbb{R}^n \) with elements that satisfy

\[
\frac{\partial X(x, t)}{\partial t} = u(X(x, t), t), \quad X(x, 0) = x,
\]

or \( \dot{X} = u \circ X \) with \( x \in \mathbb{R}^n, \ t \in \mathbb{R}, \ X \in \text{Diff}(\mathbb{R}^n) \). Let us consider motion in \( \mathbb{R}^n \) with a velocity field \( u = \dot{X} \circ X^{-1} \); \( u(x, t): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and define a momentum variable \( m = Qu \) for some (inertia) operator \( Q \) (for example the Helmholtz operator \( Q = 1 - \partial_t \partial_t = 1 - \Delta \), where \( \partial_t = \frac{\partial}{\partial x^t} \)). Let us further define a Lagrangian

\[
L[u] = \frac{1}{2} \int m \cdot u \, d^n x.
\]

Since the velocity \( u = u^i \partial_i \in \text{Vect}(\mathbb{R}^n) \) is a vector field, \( m = m_i dx^i \otimes d^n x \) is a \( n + 1 \)-form density, we have a natural right-invariant bilinear form

\[
\langle m, u \rangle = \int m \cdot u \, d^n x.
\]

The Euler-Poincaré equation for the geodesic motion in this case is \([5, 6]\)

\[
\frac{d}{dt} \frac{\delta L}{\delta u} + \text{ad}_u^* \frac{\delta L}{\delta u} = 0, \quad u = G \ast m,
\]

where \( G \) is the Green function for the operator \( Q \). The corresponding Hamiltonian is

\[
H[m] = \langle m, u \rangle - L[u] = \frac{1}{2} \int m \cdot G \ast m \, d^n x,
\]

and the equation in Hamiltonian form \( (u = \frac{\delta H}{\delta m}) \) is

\[
\frac{\partial m}{\partial t} = -\text{ad}_m^* \frac{\delta H}{\delta m} m.
\]
The left Lie algebra of vector fields is \([u, v] = -(u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p\). For an arbitrary vector field \(v\) one can write [6]
\[
\langle ad_u m, v \rangle = \langle m, ad_u v \rangle = \langle m, [u, v] \rangle = -\int m(x) (u^k(\partial_k v^p) - v^k(\partial_k u^p))d^n x.
\]
and therefore (6) has the form, known as EPDiff equation:

\[
\frac{\partial m}{\partial t} + (u \cdot \nabla)m + m \cdot \partial_p u + m_p \text{div} u = 0. 
\] (7)

Due to the invariance of the Hamiltonian under the right action of the group Diff(\(R^n\)) there is a momentum conservation law according to the Noether’s Theorem (which can be verified directly with (1)):

\[
m_i(X(x, t), t)\partial_j X_i(x, t) \text{det} \left( \frac{\partial X}{\partial x} \right) = m_j(x, 0),
\] (8)

where \(\frac{\partial X}{\partial x}\) is the Jacobian matrix.

The Lie-Poisson bracket is

\[
\{A, B\}(m) = \langle m, \left[ \frac{\delta A}{\delta m_k}, \frac{\delta B}{\delta m_i} \right] \rangle = -\int m(x) \left( \frac{\delta A}{\delta m_k} \frac{\partial \delta B}{\partial m_i} - \frac{\delta B}{\delta m_k} \frac{\partial \delta A}{\partial m_i} \right) d^n x.
\] (9)

When \(n = 1\) the algebra (9), associated with the bracket is the algebra of vector fields on the circle. This algebra admits a generalization with a central extension, which is the famous Virasoro algebra [6, 8]. In two dimensions, \(n = 2\), the algebra, associated with the bracket is the algebra of vector fields on a torus [1].

2. Singular solutions. The Camassa-Holm (CH) equation [2] can be considered as a member of the family of EPDiff equations in \(n = 1\) dimension [5]:

\[
m_t + 2u_x m + um_x = 0, \quad m = u - u_{xx}.
\] (10)

The CH equation possesses the so-called \(N\)-peakon solution in the form

\[
u(x, t) = \frac{1}{2} \sum_{i=1}^{N} p_i(t) \exp(-|x - x_i(t)|),
\] (11)

provided \(p_i\) and \(x_i\) evolve according to the following system of ordinary differential equations:

\[
\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}.
\] (12)
where the Hamiltonian is $H = \frac{1}{4} \sum_{i,j=1}^{N} p_i p_j \exp(-|x_i - x_j|)$. The momentum is singular,

$$m(x, t) = \sum_{i=1}^{N} p_i(t) \delta(x - x_i(t)), \quad (13)$$

it defines the so-called singular momentum map, [5]. CH is an integrable equation and is very well studied - see e.g. the review article [4].

The singular momentum map (13) suggests the following measure-valued singular momentum solution Ansatz for the $n-$dimensional solutions of the EPDiff equation:

$$m(x, t) = \sum_{a=1}^{N} \int \mathbf{P}^a(s, t) \delta(x - \mathbf{Q}^a(s, t)) \, ds.$$ 

These singular momentum solutions, called “diffeons,” [7] are vector density functions supported in $\mathbb{R}^n$ on a set of $N$ surfaces (or curves) of codimension $(n - k)$ for $s \in \mathbb{R}^k$ with $k < n$. They may, for example, be supported on sets of points (vector peakons, $k = 0$), one-dimensional filaments (strings, $k = 1$), or two-dimensional surfaces (sheets, $k = 2$) in three dimensions.

3. Cross coupled CH equations and waltzing peakons. The Lagrangian for cross coupled CH (CCCH) is [3]

$$l(u, v) = \int_{\mathbb{R}} (uv + u_x v_x) \, dx.$$ 

The corresponding two-component EP equations in 1D on $\mathbb{R}$ are

$$\partial_t m = -a d_{h/\delta h m} m = -(vm)_x - mv_x \quad \text{with} \quad v := \frac{\delta h}{\delta m} = K \ast n,$$

$$\partial_t n = -a d_{h/\delta h n} n = -(un)_x - nu_x \quad \text{with} \quad u := \frac{\delta h}{\delta n} = K \ast m,$$

with $K(x, y) = \frac{1}{2} e^{-|x-y|}$ being the Green function of the Helmholtz operator. The CCCH Hamiltonian is

$$h(n, m) = \int_{\mathbb{R}} n K \ast m \, dx = \int_{\mathbb{R}} m K \ast n \, dx.$$ 

This Hamiltonian system has two-component singular momentum maps

$$m(x, t) = \sum_{a=1}^{M} m_a(t) \delta(x - q_a(t)), \quad n(x, t) = \sum_{b=1}^{N} n_b(t) \delta(x - r_b(t)).$$ 

The total momentum of CCCH is conserved, namely

$$\partial_t (u + v) + \partial_x (uv + K \ast (2uv + u_x v_x)) = 0.$$ 

3
4. Peakon solutions of the cross-flow equations. The CCCH equations are deformations of CH that support two different types of peakons, with velocities
\[ u(x, t) = \frac{1}{2} \sum_{a=1}^{M} m_a(t) e^{-|x-q_a(t)|}, \quad v(x, t) = \frac{1}{2} \sum_{b=1}^{N} n_b(t) e^{-|x-r_b(t)|}, \] (14)
and momenta,
\[ m(x, t) = \sum_{a=1}^{M} m_a(t) \delta(x - q_a(t)), \quad n(x, t) = \sum_{b=1}^{N} n_b(t) \delta(x - r_b(t)). \] (15)
The $2M+2N$ variables $(q_a, m_a)$, $a = 1, \ldots, M$, and $(r_b, n_b)$, $b = 1, \ldots, N$, are governed by the Hamilton’s canonical equations for the Hamiltonian function,
\[ H = \frac{1}{2} \sum_{a,b=1}^{M,N} m_a(t)n_b(t)e^{-|q_a(t)-r_b(t)|}, \] (16)
for the positions of the peakons, and
\[ \dot{m}_a(t) = -\frac{\partial H}{\partial q_a} = \frac{1}{2} m_a \sum_{b=1}^{N} n_b(t)e^{-|q_a(t) - r_b(t)|} = -m_a \frac{\partial v}{\partial x} \bigg|_{x=q_a}, \] (19)
\[ \dot{n}_b(t) = -\frac{\partial H}{\partial r_b} = -\frac{1}{2} n_b \sum_{a=1}^{M} m_a(t)e^{-|q_a(t) - r_b(t)|} = -n_b \frac{\partial u}{\partial x} \bigg|_{x=r_b} \] (20)
for their canonical momenta. Conserved quantities include the energy $H$ and the total momentum $\sum_a (m_a + n_a)$.

5. The coupled peakon pair. The simplest possible case is $M = N = 1$. Introducing the new variables $X = \frac{q + r}{2}$, $Y = q - r$, respectively the mean position of the peaks and their separation distance. The evolution equations in terms of the new variables are
\[ \dot{X} = \frac{(m + n)}{4} e^{-|Y|}, \quad \dot{Y} = \frac{n - m}{2} e^{-|Y|}. \]
Thus we can define the behavior of the exponential function of the absolute separation of the peaks,
\[ \frac{d}{dt} e^{\frac{|Y|}{2}} = \text{sgn}(Y) \frac{n - m}{2}, \] (21)
From (19) - (20)
\[ \dot{m} = -\dot{n} = \text{sgn}(Y) \frac{mn}{2} e^{-|Y|} = \text{sgn}(Y)E, \]
where \( E = H|_{t=0} \) is the (constant) value of the Hamiltonian, that is to say the total energy of the coupled pair. Differentiating (21) again with respect to time gives
\[ \frac{d^2}{dt^2} (e^{Y|}) = -\text{sgn}^2(Y)E + 2\delta(Y)(n - m)^2. \]
On integrating for a particular signature of \( Y|_{t=0} = Y_0 \neq 0 \),
\[ e^{Y|} = -\frac{1}{2} m_0 n_0 e^{-|Y_0|} t^2 + \frac{1}{2} \text{sgn}(Y_0)(n_0 - m_0) t + e^{Y_0}, \]
where
\[ m_0 = m|_{t=0}, \quad n_0 = n|_{t=0} \quad Y_0 = Y|_{t=0}. \]
If \( m_0 \) and \( n_0 \) have the same signature then eventually we will have \( |Y| = 0 \), regardless of the value of \( |Y_0| \). Thus, when \( m_0 \) and \( n_0 \) share the same signature the half period of their waltzing motion can be found by setting \( Y_0 = 0 \) and looking for when \( e^{Y|} \) attains unity, namely \( t = 2\frac{n_0 - m_0}{m_0 n_0} \).

It will be noted that at this time
\[ m|_{t=2\frac{n_0 - m_0}{m_0 n_0}} = m_0 + m_0 n_0 \left( \frac{m_0 - n_0}{m_0 n_0} \right) = n_0, \]
and similarly
\[ n|_{t=2\frac{n_0 - m_0}{m_0 n_0}} = m_0, \]
so that the two types of peakons do indeed exchange momentum amplitudes over a half cycle, see Fig. 1. The explicit solutions as well as other examples with waltzing peakons and compactons are given in [3].

6. Cross-coupled EPDiff in higher dimensions. The straightforward generalization to higher dimensions is
\[ l = \int (u \cdot v + (\nabla u) \cdot (\nabla v)) \, d^n x, \]
\[ \frac{dm}{dt} = -ad_{\delta t / \delta m} m = -v \cdot \nabla m - (\nabla v)^T \cdot m - m \text{ div } v, \quad m = u - \Delta u, \]
\[ \frac{dn}{dt} = -ad_{\delta t / \delta n} n = -u \cdot \nabla n - (\nabla u)^T \cdot n - n \text{ div } u, \quad n = v - \Delta v. \]
Numerical studies in \( n = 2 \) dimensions show that the waltzing pairs also appear in higher dimensions. In the case with rotational symmetry the two concentric waves \( u \) and \( v \) have 'waltzing' fronts and also rotate with respect to each other.

Acknowledgements The work of R.I. is supported by the Science Foundation of Ireland (SFI), under Grant No. 09/RFP/MTH2144.
Figure 1: Plot showing velocity fields of a peakon-peakon pair with $m_0 = 5$, $n_0 = 0.5$, $l_0 = 0$ (solid lines). The dotted path indicates the subsequent path of the two peaks in the frame travelling at the particles mean velocity $\dot{X} = \frac{\partial z}{\partial t}$. For these initial conditions the total period for one orbit of the cycle is $T = 3.6$. Also shown is the form of the two peakons at subsequent times $t = 0.45 + 1.8n$, $n \in \mathbb{Z}$.

References


