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1. EPDiff equations. Let us define an one-parametric group of diffeomorphisms of \mathbb{R}^n with elements that satisfy

$$\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{x}, t), t), \quad \mathbf{X}(\mathbf{x}, 0) = \mathbf{x}, \quad (1)$$

or $\dot{X} = u \circ X$ with $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$, $X \in \text{Diff}(\mathbb{R}^n)$. Let us consider motion in \mathbb{R}^n with a velocity field $u = \dot{X} \circ X^{-1}$; $\mathbf{u}(\mathbf{x}, t): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and define a momentum variable $\mathbf{m} = Q\mathbf{u}$ for some (inertia) operator Q (for example the Helmholtz operator $Q = 1 - \partial_i \partial_i = 1 - \Delta$, where $\partial_i = \frac{\partial}{\partial x^i}$). Let us further define a Lagrangian

$$L[\mathbf{u}] = \frac{1}{2} \int \mathbf{m} \cdot \mathbf{u} \, d^n \mathbf{x}. \quad (2)$$

Since the velocity $\mathbf{u} = u^i \partial_i \in \text{Vect}(\mathbb{R}^n)$ is a vector field, $\mathbf{m} = m_i dx^i \otimes d^n \mathbf{x}$ is a $n + 1$ -form density, we have a natural right-invariant bilinear form

$$\langle \mathbf{m}, \mathbf{u} \rangle = \int \mathbf{m} \cdot \mathbf{u} \, d^n \mathbf{x}. \quad (3)$$

The Euler-Poincaré equation for the geodesic motion in this case is [5, 6]

$$\frac{d}{dt} \frac{\delta L}{\delta \mathbf{u}} + \text{ad}_{\mathbf{u}}^* \frac{\delta L}{\delta \mathbf{u}} = 0, \quad \mathbf{u} = G * \mathbf{m}, \quad (4)$$

where G is the Green function for the operator Q . The corresponding Hamiltonian is

$$H[\mathbf{m}] = \langle \mathbf{m}, \mathbf{u} \rangle - L[\mathbf{u}] = \frac{1}{2} \int \mathbf{m} \cdot G * \mathbf{m} \, d^n \mathbf{x}, \quad (5)$$

and the equation in Hamiltonian form ($\mathbf{u} = \frac{\delta H}{\delta \mathbf{m}}$) is

$$\frac{\partial \mathbf{m}}{\partial t} = -\text{ad}_{\frac{\delta H}{\delta \mathbf{m}}}^* \mathbf{m}. \quad (6)$$

The left Lie algebra of vector fields is $[\mathbf{u}, \mathbf{v}] = -(u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p$. For an arbitrary vector field \mathbf{v} one can write [6]

$$\begin{aligned} \langle \text{ad}_{\mathbf{u}}^* \mathbf{m}, \mathbf{v} \rangle &= \langle \mathbf{m}, \text{ad}_{\mathbf{u}} \mathbf{v} \rangle = \langle \mathbf{m}, [\mathbf{u}, \mathbf{v}] \rangle \\ &= -\langle m_l dx^l \otimes d^n \mathbf{x}, (u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p \rangle \\ &= -\int m_p (u^k(\partial_k v^p) - v^k(\partial_k u^p)) d^n \mathbf{x} \\ &= \int v^p (u^k(\partial_k m_p) + m_p(\partial_k u^k) + m_k(\partial_p u^k)) d^n \mathbf{x} \\ &= \langle ((\mathbf{u} \cdot \nabla) m_p + \mathbf{m} \cdot \partial_p \mathbf{u} + m_p \text{div} \mathbf{u}) dx^p \otimes d^n \mathbf{x}, \mathbf{v} \rangle, \end{aligned}$$

and therefore (6) has the form, known as *EPDiff equation*:

$$\frac{\partial m_p}{\partial t} + (\mathbf{u} \cdot \nabla) m_p + \mathbf{m} \cdot \partial_p \mathbf{u} + m_p \text{div} \mathbf{u} = 0. \quad (7)$$

Due to the invariance of the Hamiltonian under the right action of the group $\text{Diff}(\mathbb{R}^n)$ there is a momentum conservation law according to the Noether's Theorem (which can be verified directly with (1)) :

$$m_i(\mathbf{X}(\mathbf{x}, t), t) \partial_j X^i(\mathbf{x}, t) \det \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right) = m_j(x, 0), \quad (8)$$

where $\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right)_{ij} = \frac{\partial X^i}{\partial x^j}$ is the Jacobian matrix.

The Lie-Poisson bracket is

$$\begin{aligned} \{A, B\}(\mathbf{m}) &= \left\langle \mathbf{m}, \left[\frac{\delta A}{\delta \mathbf{m}}, \frac{\delta B}{\delta \mathbf{m}} \right] \right\rangle \\ &= -\int m_i \left(\frac{\delta A}{\delta m_k} \partial_k \frac{\delta B}{\delta m_i} - \frac{\delta B}{\delta m_k} \partial_k \frac{\delta A}{\delta m_i} \right) d^n \mathbf{x}. \end{aligned} \quad (9)$$

When $n = 1$ the algebra (9), associated with the bracket is the algebra of vector fields on the circle. This algebra admits a generalization with a central extension, which is the famous Virasoro algebra [6, 8]. In two dimensions, $n = 2$, the algebra, associated with the bracket is the algebra of vector fields on a torus [1].

2. Singular solutions. The Camassa-Holm (CH) equation [2] can be considered as a member of the family of EPDiff equations in $n = 1$ dimension [5]:

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx}. \quad (10)$$

The CH equation possesses the so-called N -peakon solution in the form

$$u(x, t) = \frac{1}{2} \sum_{i=1}^N p_i(t) \exp(-|x - x_i(t)|), \quad (11)$$

provided p_i and x_i evolve according to the following system of ordinary differential equations:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad (12)$$

where the Hamiltonian is $H = \frac{1}{4} \sum_{i,j=1}^N p_i p_j \exp(-|x_i - x_j|)$. The momentum is singular,

$$m(x, t) = \sum_{i=1}^N p_i(t) \delta(x - x_i(t)), \quad (13)$$

it defines the so-called singular momentum map, [5]. CH is an integrable equation and is very well studied - see e.g. the review article [4].

The singular momentum map (13) suggests the following measure-valued singular momentum solution Ansatz for the n -dimensional solutions of the EPDiff equation:

$$\mathbf{m}(\mathbf{x}, t) = \sum_{a=1}^N \int \mathbf{P}^a(s, t) \delta(\mathbf{x} - \mathbf{Q}^a(s, t)) ds.$$

These singular momentum solutions, called “diffeons,” [7] are vector density functions supported in \mathbb{R}^n on a set of N surfaces (or curves) of codimension $(n - k)$ for $s \in \mathbb{R}^k$ with $k < n$. They may, for example, be supported on sets of points (vector peakons, $k = 0$), one-dimensional filaments (strings, $k = 1$), or two-dimensional surfaces (sheets, $k = 2$) in three dimensions.

3. Cross coupled CH equations and waltzing peakons. The Lagrangian for cross coupled CH (CCCH) is [3]

$$l(u, v) = \int_{\mathbb{R}} (uv + u_x v_x) dx.$$

The corresponding two-component EP equations in 1D on \mathbb{R} are

$$\partial_t m = -\text{ad}_{\delta h / \delta m}^* m = -(vm)_x - mv_x \quad \text{with} \quad v := \frac{\delta h}{\delta m} = K * n,$$

$$\partial_t n = -\text{ad}_{\delta h / \delta n}^* n = -(un)_x - nu_x \quad \text{with} \quad u := \frac{\delta h}{\delta n} = K * m.$$

with $K(x, y) = \frac{1}{2} e^{-|x-y|}$ being the Green function of the Helmholtz operator. The CCCH Hamiltonian is

$$h(n, m) = \int_{\mathbb{R}} n K * m dx = \int_{\mathbb{R}} m K * n dx.$$

This Hamiltonian system has **two**-component singular momentum maps

$$m(x, t) = \sum_{a=1}^M m_a(t) \delta(x - q_a(t)), \quad n(x, t) = \sum_{b=1}^N n_b(t) \delta(x - r_b(t)).$$

The total momentum of CCCH is conserved, namely

$$\partial_t (u + v) + \partial_x (uv + K * (2uv + u_x v_x)) = 0.$$

4. Peakon solutions of the cross-flow equations. The CCCH equations are deformations of CH that support two different types of peakons, with velocities

$$u(x, t) = \frac{1}{2} \sum_{a=1}^M m_a(t) e^{-|x-q_a(t)|}, \quad v(x, t) = \frac{1}{2} \sum_{b=1}^N n_b(t) e^{-|x-r_b(t)|}, \quad (14)$$

and momenta,

$$m(x, t) = \sum_{a=1}^M m_a(t) \delta(x - q_a(t)), \quad n(x, t) = \sum_{b=1}^N n_b(t) \delta(x - r_b(t)). \quad (15)$$

The $2M+2N$ variables (q_a, m_a) , $a = 1, \dots, M$, and (r_b, n_b) , $b = 1, \dots, N$, are governed by the Hamilton's canonical equations for the Hamiltonian function,

$$H = \frac{1}{2} \sum_{a,b=1}^{M,N} m_a(t) n_b(t) e^{-|q_a(t)-r_b(t)|}, \quad (16)$$

namely,

$$\dot{q}_a(t) = \frac{\partial H}{\partial m_a} = \frac{1}{2} \sum_{b=1}^N n_b(t) e^{-|q_a(t)-r_b(t)|} = v(q_a(t), t), \quad (17)$$

$$\dot{r}_b(t) = \frac{\partial H}{\partial n_b} = \frac{1}{2} \sum_{a=1}^M m_a(t) e^{-|q_a(t)-r_b(t)|} = u(r_b(t), t), \quad (18)$$

for the positions of the peakons, and

$$\dot{m}_a(t) = -\frac{\partial H}{\partial q_a} = \frac{1}{2} m_a \sum_{b=1}^N n_b \operatorname{sgn}(q_a - r_b) e^{-|q_a(t)-r_b(t)|} = -m_a \frac{\partial v}{\partial x} \Big|_{x=q_a}, \quad (19)$$

$$\dot{n}_b(t) = -\frac{\partial H}{\partial r_b} = -\frac{1}{2} n_b \sum_{a=1}^M m_a \operatorname{sgn}(q_a - r_b) e^{-|q_a(t)-r_b(t)|} = -n_b \frac{\partial u}{\partial x} \Big|_{x=r_b} \quad (20)$$

for their canonical momenta. Conserved quantities include the energy H and the total momentum $\sum_a (m_a + n_a)$.

5. The coupled peakon pair. The simplest possible case is $M = N = 1$. Introducing the new variables $X = \frac{q+r}{2}$, $Y = q - r$, respectively the mean position of the peaks and their separation distance. The evolution equations in terms of the new variables are

$$\dot{X} = \frac{(m+n)}{4} e^{-|Y|}, \quad \dot{Y} = \frac{n-m}{2} e^{-|Y|}.$$

Thus we can define the behavior of the exponential function of the absolute separation of the peaks,

$$\frac{d}{dt} e^{|Y|} = \operatorname{sgn}(Y) \frac{n-m}{2}, \quad (21)$$

From (19) - (20)

$$\dot{m} = -\dot{n} = \text{sgn}(Y) \frac{mn}{2} e^{-|Y|} = \text{sgn}(Y) E,$$

where $E = H|_{t=0}$ is the (constant) value of the Hamiltonian, that is to say the total energy of the coupled pair. Differentiating (21) again with respect to time gives

$$\frac{d^2}{dt^2} (e^{|Y|}) = -\text{sgn}^2(Y) E + 2\delta(Y) (n - m)^2.$$

On integrating for a particular signature of $Y|_{t=0} = Y_0 \neq 0$,

$$e^{|Y|} = -\frac{1}{2} m_0 n_0 e^{-|Y_0|} t^2 + \frac{1}{2} \text{sgn}(Y_0) (n_0 - m_0) t + e^{|Y_0|},$$

where

$$m_0 = m|_{t=0}, \quad n_0 = n|_{t=0} \quad Y_0 = Y|_{t=0}.$$

If m_0 and n_0 have the same signature then eventually we will have $|Y| = 0$, regardless of the value of $|Y_0|$. Thus, when m_0 and n_0 share the same signature the half period of their waltzing motion can be found by setting $Y_0 = 0$ and looking for when $e^{|Y|}$ attains unity, namely $t = 2 \frac{n_0 - m_0}{n_0 m_0}$. It will be noted that at this time

$$m|_{t=2 \frac{n_0 - m_0}{m_0 n_0}} = m_0 + m_0 n_0 \left(\frac{m_0 - n_0}{m_0 n_0} \right) = n_0,$$

and similarly

$$n|_{t=2 \frac{m_0 - n_0}{m_0 n_0}} = m_0,$$

so that the two types of peakons do indeed exchange momentum amplitudes over a half cycle, see Fig. 1. The explicit solutions as well as other examples with waltzing peakons and compactons are given in [3].

6. Cross-coupled EPDiff in higher dimensions. The straightforward generalization to higher dimensions is

$$l = \int (\mathbf{u} \cdot \mathbf{v} + (\nabla \mathbf{u}) \cdot (\nabla \mathbf{v})) d^n x,$$

$$\frac{d\mathbf{m}}{dt} = -\text{ad}_{\delta l / \delta \mathbf{m}}^* \mathbf{m} = -\mathbf{v} \cdot \nabla \mathbf{m} - (\nabla \mathbf{v})^T \cdot \mathbf{m} - \mathbf{m} \text{div } \mathbf{v}, \quad \mathbf{m} = \mathbf{u} - \Delta \mathbf{u},$$

$$\frac{d\mathbf{n}}{dt} = -\text{ad}_{\delta l / \delta \mathbf{n}}^* \mathbf{n} = -\mathbf{u} \cdot \nabla \mathbf{n} - (\nabla \mathbf{u})^T \cdot \mathbf{n} - \mathbf{n} \text{div } \mathbf{u}, \quad \mathbf{n} = \mathbf{v} - \Delta \mathbf{v}.$$

Numerical studies in $n = 2$ dimensions show that the waltzing pairs also appear in higher dimensions. In the case with rotational symmetry the two concentric waves (\mathbf{u} and \mathbf{v}) have 'waltzing' fronts and also rotate with respect to each other.

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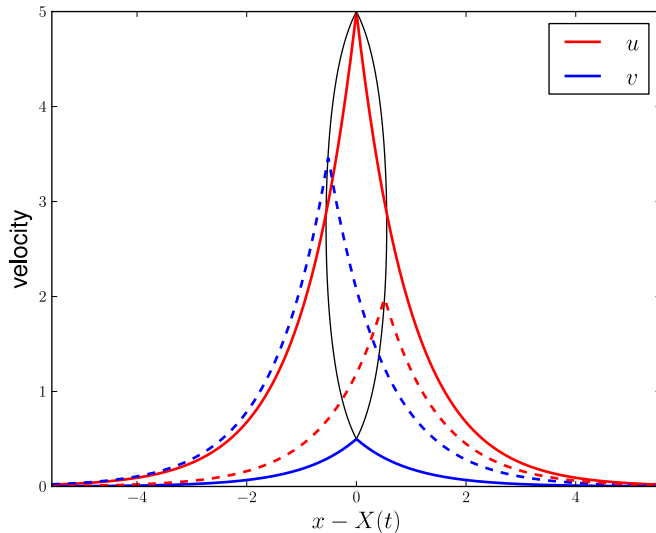


Figure 1: Plot showing velocity fields of a peakon-peakon pair with $m_0 = 5$, $n_0 = 0.5$, $l_0 = 0$ (solid lines). The dotted path indicates the subsequent path of the two peaks in the frame travelling at the particles mean velocity $\dot{X} = \frac{q+r}{2}$. For these initial conditions the total period for one orbit of the cycle is $T = 3.6$. Also shown is the form of the two peakons at subsequent times $t = 0.45 + 1.8n$, $n \in \mathbb{Z}$

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