Essentially Indecomposable Modules Which Are Almost Free

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Essentially Indecomposable Modules Over a Complete Discrete Valuation Ring.

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1. Introduction.

Torsion-free modules over a complete discrete valuation ring R are markedly different from abelian groups or modules over an incomplete discrete valuation ring in that the only indecomposable modules which exist have rank 1 and so are isomorphic to R itself or the field, Q, of fractions of R ([7], p. 45). In this paper we investigate how close a reduced torsion-free R-module of infinite rank can come to being indecomposable. In particular we establish in § 4 the existence of essentially indecomposable modules with basic submodules of countable rank. The results here bear a strong resemblance to results on p-groups ([2], [5], [9] and [10]). Notation follows the standard works of Fuchs [3], [4] while set-theoretic concepts, which are kept to a minimum, may be found in Jech [6].

2. Maximal pure submodules.

Let R denote a complete discrete valuation ring of cardinality ν with unique prime p. For an infinite cardinal λ we let S_λ (or just S if no ambiguity is possible) denote a free R-module of rank λ. Clearly S_λ is not complete in the p-adic topology and we denote its completion by $\hat{S}_\lambda$ (or just $\hat{S}$).

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DEFINITION. A R-module X is said to be a maximal pure submodule of the complete R-module $\hat{S}$ if X is a pure submodule of $\hat{S}$ containing S and $\hat{S}/X \cong Q$, the field of fractions of R. We remark that if X is a maximal pure submodule of $\hat{S}$ then for any $x \in \hat{S} \setminus X$, we have $\hat{S} = \langle X, x \rangle_*$.

LEMMA 2.1. If G and K are pure submodules of $\hat{S}$ containing S then $G \cong K$ if and only if there is an automorphism θ of $\hat{S}$ with $G\theta = K$.

PROOF. The sufficiency is clear, we establish necessity. Let φ be an isomorphism from G onto K. Then φ extends uniquely to an endomorphism $\hat{\varphi}$ of $\hat{S}$. Similarly if ψ is the inverse of φ, it extends to an endomorphism $\hat{\psi}$ of $\hat{S}$. However since G and K are dense subsets of the Hausdorff space $\hat{S}$, it follows easily that $\hat{\varphi}\hat{\psi}$ and $\hat{\psi}\hat{\varphi}$ act as the identity on $\hat{S}$. Thus $\theta = \hat{\varphi}$ is the required automorphism.

Before examining the endomorphism rings of maximal pure submodules of $\hat{S}$, we introduce the concept of an inessential endomorphism (cf. [5]). Let X be a pure submodule of $\hat{S}$ containing S, then, as we noted in the proof of Lemma 2.1, any endomorphism φ of X has a unique extension $\hat{\varphi}$ to an endomorphism of $\hat{S}$. We define an endomorphism of X to be inessential if its unique extension to $\hat{S}$ maps $\hat{S}$ into X. It is easily seen that the difference of two inessential endomorphisms is inessential while the composition of two endomorphisms is inessential when either factor is inessential. Thus the inessential endomorphisms of X form a two-sided ideal $I(X)$ in the endomorphism ring $E(X)$ of X.

THEOREM 2.2. For any infinite cardinal λ, there exists an R-module G, with basic submodule of rank λ, such that $E(G)$ is the ring split extension of R by $I(G)$,

$$E(G) = R \oplus I(G) .$$

PROOF. Let S be a free R-module of rank λ and let G be any maximal pure submodule of $\hat{S}$. Clearly S is basic in G. We may identify $E(G)$ as a subring of $E(\hat{S})$ by identifying each endomorphism φ in $E(G)$ with its unique extension $\hat{\varphi}$ in $E(\hat{S})$. With this identification $I(G)$ is a left ideal of $E(\hat{S})$.

Pick $x \in \hat{S} \setminus G$. Then for arbitrary φ in $E(G)$ we must have $q(x\hat{\varphi}) = tx + g$, some $q, t \in R$, $g \in G$. Since every element of R is a product of a power of p and a unit, there is no loss in generality in supposing

$q = p^r$, $t = p^s$. We consider two cases:

(i) $$r \leqslant s\,.$$

In this case $p^r(x\hat{\varphi} - p^{s-r}x) = g$. The purity of G in $\hat{S}$ implies that $x(\hat{\varphi} - p^{s-r}1)$ belongs to G. Since G is invariant under $\hat{\varphi} - p^{s-r}1$ and $\langle G, x\rangle_* = \hat{S}$, it is clear that $\hat{S}(\hat{\varphi} - p^{s-r}1)$ is contained in G. Thus $\hat{\varphi} - p^{s-r}1 \in I(G)$ and so $E(G) = R + I(G)$.

(ii) $$r > s\,.$$

We show that this case cannot arise. As before we can show that $x(p^{r-s}\hat{\varphi} - 1) \in G$ and deduce that $p^{r-s}\hat{\varphi} - 1 \in I(G)$. Suppose $p^{r-s}\hat{\varphi} - 1 = \theta$, where $\theta \in I(G)$. Since $r > s$, $p^{r-s}\hat{\varphi}$ belongs to the Jacobson radical of $E(\hat{S})$ (see [8]) and this forces θ to be a unit in $E(\hat{S})$. However since G is certainly not a homomorphic image of $\hat{S}$, $I(G)$ is a proper left ideal of $E(\hat{S})$ which contains a unit-contradiction. So case (ii) does not arise.

Since G is pure in $\hat{S}$, $R \cap I(G) = 0$ and *quâ* modules, $E(G) = R \oplus I(G)$. However this is clearly a ring split extension also and we have established the result.

3. Essentially-rigid systems of R-modules.

As we noted in the introduction indecomposable R-modules have rank 1 whereas indecomposable abelian groups of arbitrary large rank exist [11]. One useful tool in the investigation of indecomposable abelian groups was the concept of a rigid system of groups (see [4], § 88). In this section we define and explore an analagous concept for R-modules.

We extend the concept of inessential to homomorphisms between different reduced torsion-free R-modules X_i, X_j by defining $I_i(X_j) = \{\varphi \in \operatorname{Hom}(X_i, X_j) \mid \hat{X}_i\hat{\varphi} \leqslant X_j\}$ where $\hat{\varphi}$ denotes the unique extension of φ to a map $\hat{X}_i \to \hat{X}_j$.

Definition. A family $\{X_j\}$ $(j \in J)$ of reduced torsion-free R-modules is said to be essentially rigid if

$$\operatorname{Hom}(X_i, X_j) = \begin{cases} R \oplus I(X_j) & \text{if } i = j \\ I_i(X_j) & \text{if } i \neq j\,, \end{cases}$$

for all $i, j \in J$.

Suppose throughout this section that R is a complete discrete valuation ring of cardinality ν and λ is an infinite cardinal satisfying $\mu = \lambda^{\aleph_0} = 2^\lambda$ and $\nu \leqslant \mu$. For an infinite cardinal σ, let σ^+ denote the successor of σ. We can now state the main result of this section.

THEOREM 3.1. If λ is an infinite cardinal with the property that $\mu = \lambda^{\aleph_0} = 2^\lambda$ and R is a complete discrete valuation ring of cardinality $\nu \leqslant \mu$, then there exists an essentially-rigid family of R-modules having μ^+ members.

REMARK. (i) By assuming G.C.H. we may, of course replace μ^+ by 2^μ.

(ii) Cardinals of the form λ do exist for values of λ other than $\lambda = \aleph_0$ e.g. assuming G.C.H., any cardinal of cofinality $\aleph_0$ will do.

LEMMA 3.2. Let V be a vector space of dimension α, an infinite cardinal, over a field. Let $\{W_i\}$ $(i < \beta)$ be a family of subspaces of V indexed by the cardinal $\beta \leqslant \alpha$, such that $\dim W_i = \alpha$ for all $i < \beta$. Then there exist α^+ subspaces $\{U\}$ of V such that each subspace U is of codimension 1 in V and no subspace W_i is contained in any subspace U.

PROOF. By a result of Beaumont and Pierce ([1]), Lemma 5.2) there exists at least one such subspace, U_0 say. Suppose that the subspaces $\{U_i\}$ $(i < \zeta)$ have been constructed and $\zeta < \alpha^+$. Then the set of subspaces consisting of the given W_i together with the constructed subspaces U_i constitutes a family of at most α subspaces each of dimension α. Applying Beaumont and Pierce's result to this family yields another subspace of codimension 1. Call this subspace U_ζ. The result follows easily by transfinite induction.

LEMMA 3.3. If S is free of rank λ then there exist μ^+ maximal pure submodules of $\hat{S}$ with the property that none of them contains a submodule isomorphic to $\hat{S}$.

PROOF. Since S is free of rank λ, $|\hat{S}| = \max(\lambda^{\aleph_0}, \nu^{\aleph_0})$. But $\nu \leqslant \mu$ implies that $\nu^{\aleph_0} \leqslant \mu^{\aleph_0} = (\lambda^{\aleph_0})^{\aleph_0} = \lambda^{\aleph_0} = \mu$. So $\hat{S}/S$ is a Q-vector space of dimension $\mu = \lambda^{\aleph_0}$. Now let $\{W_k\}$ $(k \in K)$ be the collection of submodules of $\hat{S}$ which are isomorphic to $\hat{S}$. Each of these submodules is determined by an endomorphism of $\hat{S}$. However each endomorphism of $\hat{S}$ is completely determined by its action on S which has rank λ, so $|E(\hat{S})| = \mu^{\lambda} = (2^{\lambda})^{\lambda} = 2^{\lambda} = \mu$. Hence $\{W_k\}$ $(k \in K)$ is a family of at most μ submodules of $\hat{S}$.

Let $\overline{W}_k = \langle W_k + S\rangle_* / S$. Then $\{\overline{W}_k\}$ $(k \in K)$ is a family of at most μ subspaces of the Q-vector space $\hat{S}/S$ which has dimension μ. Since $\overline{W}_k \cong (Q \otimes_R (W_k + S))/Q \otimes_R S$, it follows that each $\overline{W}_k$ has dimension μ. By Lemma 3.2 we can find μ^+ subspaces U such that no $\overline{W}_k$ is contained in any U and, moreover, each U has codimension 1 in $\hat{S}/S$. If G is a submodule of $\hat{S}$ with $G/S = U$, then G is a maximal pure submodule of $\hat{S}$ and clearly no W_k is contained in any G. Thus we have constructed the required family of μ^+ maximal pure submodules.

Let $\mathcal{G}_\lambda$ denote the collection of all maximal pure submodules of $\hat{S}_\lambda$ which do not contain an isomorphic copy of $\hat{S}_\lambda$.

LEMMA 3.4. If $\{G_\alpha\}$ $(\alpha < \beta)$ is a subset of $\mathcal{G}_\lambda$ and $|\beta| \leqslant \mu$, then there exist μ^+ submodules G in $\mathcal{G}_\lambda$ such that $\mathrm{Hom}(G_\alpha, G) = I_\alpha(G)$ for all $\alpha < \beta$.

PROOF. The proof is similar to that of Lemma 3.3. Suppose $\{W_{\alpha i}\}$ denotes the set of endomorphic images of G_α which have rank μ. Then for each α, the set $\{W_{\alpha i}\}$ is of cardinality at most μ. Since $|\beta| \leqslant \mu$, the union of all such collections is a set of at most μ submodules of $\hat{S}$. Call this set $\mathfrak{W}$. Now let $\mathfrak{V}$ denote the set of endomorphic images of $\hat{S}$ which are isomorphic to $\hat{S}$. Then $\mathfrak{W} \cup \mathfrak{V}$ is a collection of at most μ submodules of $\hat{S}$, say $\mathfrak{W} \cup \mathfrak{V} = \{X_i\}$ $(i < \mu)$. Note that each X_i has rank μ. Set $\overline{X}_i = \langle X_i + S\rangle_*/S$; then $\{\overline{X}_i\}$ $(i < \mu)$ is a collec tion of μ subspaces of the Q-vector space $\hat{S}/S$ and each $\overline{X}_i$ has dimension μ. By Lemma 3.2 there exist μ^+ maximal subspaces U such that no $\overline{X}_i$ is contained in a U. Choose a maximal pure submodule G such that $G/S = U$. Clearly $G \in \mathcal{G}_\lambda$.

Now consider $\mathrm{Hom}(G_\alpha, G)$ for any α. If $\varphi\colon G_\alpha \to G$ is not inessential then $\mathrm{Ker}\,\hat{\varphi}$ is contained in G_α which forces $\mathrm{Ker}\,\hat{\varphi}$ to have rank less than μ. But then $\mathrm{Im}\,\varphi \cong G_\alpha/\mathrm{Ker}\,\varphi$ is an endomorphic image of G_α of rank μ and is contained in G-contradiction. So we conclude that $\mathrm{Hom}(G_\alpha, G) = I_\alpha(G)$ for each α.

LEMMA 3.5. Given any maximal pure submodule G of $\hat{S}_\lambda$, there are at most μ maximal pure submodules G_i of $\hat{S}_\lambda$ for which $\mathrm{Hom}(G_i, G) \neq I_i(G)$.

PROOF. Suppose there exists a family $\{G_i\}$ $(i \in J)$ of more than μ submodules. For each $i \in J$, pick a homomorphism $\varphi_i: G_i \to G$. Then $\{\hat{\varphi}_i\}$ $(i \in J)$ is a family of more than μ endomorphisms of $\hat{S}_\lambda$. Since $|E(\hat{S}_\lambda)| = \mu$, we must have $\hat{\varphi}_i = \hat{\varphi}_j$ for some $i \neq j \in J$. But then

$$\hat{S}\hat{\varphi}_i = (G_i + G_j)\hat{\varphi}_i \leqslant G_i\hat{\varphi}_i + G_j\hat{\varphi}_j = G .$$

Thus φ_i is inessential-contradiction. This establishes the lemma.

PROOF OF THEOREM 3.1. Choose G_0 to be any member of $\mathcal{G}_\lambda$. Suppose the essentially-rigid family $\{G_\alpha\}$ $(\alpha < \beta)$ has been constructed for $\beta < \mu^+$. By Lemma 3.4 there exist μ^+ maximal pure submodules G such that $\mathrm{Hom}\,(G_\alpha, G) = I_\alpha(G)$. However for each $\alpha < \beta$, there are, by Lemma 3.5, at most μ of these submodules G for which $\mathrm{Hom}\,(G, G_\alpha) \neq I(G_\alpha)$. Then deleting all such submodules G deletes at most μ submodules from the original collection since $\beta \leqslant \mu$. So there exists $G \in \mathcal{G}_\lambda$ with $\mathrm{Hom}\,(G, G_\alpha) = I(G_\alpha)$ and $\mathrm{Hom}\,(G_\alpha, G) = I_\alpha(G)$ for all $\alpha < \beta$. Set $G_\beta = G$. Then the family $\{G_\alpha\}$ $(\alpha \leqslant \beta)$ is essentially-rigid. The proof is completed by transfinite induction.

4. Essentially-indecomposable modules.

In this section we show that a slightly stronger result than Theorem 3.1 can be deduced and apply this new result to the construction of essentially indecomposable modules.

We shall use the term basic rank of a homomorphism to denote the rank of a basic submodule of the image of the homomorphism.

DEFINITION. If S is a free R-module of infinite rank λ and X is a pure submodule of $\hat{S}$ containing S, then we define

$$I_\lambda(X) = \{\varphi \in E(X) \,|\, \hat{S}\hat{\varphi} \leqslant X \text{ and } \hat{\varphi} \text{ has basic rank} < \lambda\}$$

Clearly $I_\lambda(X)$ is an ideal in $E(X)$.

THEOREM 4.1. If R is a complete discrete valuation ring of cardinality ν and λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$ and $\nu \leqslant \mu$, then there exists a family of μ^+ R-modules $\{G_j\}$ $(j \in J)$ such that

(i) for each $j \in J$, $E(G_j) = R \oplus I_\lambda(G_j)$;

(ii) for distinct $j, k \in J$, every homomorphism $G_j \to G_k$ is inessential and has basic rank less than λ

Proof. This stronge result comes by observing in the proof of Theorem 3.1 that all of the modules constructed actually belong to $\mathcal{G}_\lambda$. Since the image of an inessential homomorphism is complete, it must be the completion of a free module of rank less than λ. This gives the desired result.

Recall that $E_0(G)$ denotes the ideal of $E(G)$ consisting of all endomorphisms of finite rank.

Corollary 4.2 (G.C.H.). If R is a complete discrete valuation ring of cardinality $2^{\aleph_0}$, then there exists a family of $2^{2^{\aleph_0}}$ R modules $\{G_j\}$ $(j \in J)$ each with basic submodules of rank $\aleph_0$ such that

(i) for each $j \in J$, $E(G_j) = R \oplus E_0(G_j)$;

(ii) for distinct $j, k \in J$, every homomorphism $G_j \to G_k$ has finite rank.

Proof. Since $2^{\aleph_0} \leqslant (\aleph_0)^{\aleph_0} \leqslant (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, we see that $\lambda = \aleph_0$ satisfies the cardinality requirements of Theorem 4.1. However if a homomorphism from G_j has finite basic rank then it clearly also has finite rank. The result now follows from Theorem 4.1 and G.C.H.

Definition. If λ is an infinite cardinal we say that a reduced torsion-free R-module G is λ-essentially indecomposable if in any decomposition $G = A \oplus B$, one of A, B is the completion of a free module of rank less then λ.

In the case $\lambda = \aleph_0$ we are requiring that in any direct decomposition one of the summands is complete of finite rank. A module with this property is said to be essentially indecomposable (cf. essentially indecomposable p-groups, [9], § 15).

The existence of λ-essentially indecomposable modules follows rather easily from Theorem 4.1 in the case $\lambda^{\aleph_0} = 2^\lambda$. For if G is one of the modules constructed in Theorem 4.1 and we have a decomposition $G = A \oplus B$ with associated projections π_1 and π_2, then one of π_1, π_2 belongs to $I_\lambda(G)$ since the quotient $E(G)/I_\lambda(G)$ is a domain. If $\pi_1 \in I_\lambda(G)$ then clearly A is the completion of a free R-module of rank less then λ. In particular if $\lambda = \aleph_0$ we have established the existence of $2^{2^{\aleph_0}}$ essentially indecomposable R-modules for any complete discrete valuation ring R of cardinality $2^{\aleph_0}$.

We conclude this section by constructing an essentially indecomposable module which is not a maximal pure submodule of a complete module.

PROPOSITION 4.3. If R is a complete discrete valuation ring of cardinality $2^{\aleph_0}$ and S is a free R-module of countably infinite rank, then there exists a pure submodule H of $\hat{S}$ containing S with $\hat{S}/H \cong Q \oplus Q$ and such that $E(H) = R \oplus E_0(H)$.

PROOF. Choose distinct maximal pure submodules G and G_1 belonging to the family constructed in Corollary 4.2. Set $H = G \cap G_1$. Clearly $S \leqslant H \leqslant \hat{S}$ and both inclusions are pure. Also $\hat{S}/H \cong Q \oplus Q$. Let $\hat{S} = \langle H, x, y \rangle_*$ where $G = \langle H, x \rangle_*$ and $G_1 = \langle H, y \rangle_*$. Let φ be any endomorphism of H. Then as in the proof of Theorem 2.2 we may write

$$q(x\hat{\varphi}) = h_0 + \alpha x + \beta y$$

$$q(y\hat{\varphi}) = h_1 + \gamma x + \delta y$$

where $q, \alpha, \beta, \gamma, \delta \in R$ and $h_0, h_1 \in H$.

Now $x(q\hat{\varphi} - \alpha 1) \in G_1$ and so $q\hat{\varphi} - \alpha 1$ maps G into G_1. From the properties of G and G_1 we conclude that $q\hat{\varphi} - \alpha 1$ is an inessential endomorphism. It follows as in the proof of Theorem 2.2, that $\hat{\varphi}|G$ belongs to $R \oplus I(G_1)$. Hence we can write $\varphi = r + \theta$ where $r \in R$, $\theta \in I(G_1)$. But then $\theta \in E(H) \cap I(G_1) = E(H) \cap E_0(G) = E_0(H)$. So $\varphi \in R \oplus E_0(H)$ and $E(H) \leqslant R \oplus E_0(H)$. The reverse inequality is clearly true so $E(H) = R \oplus E_0(H)$.

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