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ON ENDOMORPHISM ALGEBRAS OF MIXED MODULES

B. FRANZEN AND B. GOLDSMITH

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1. Introduction

In a remarkable paper [1] some twenty years ago, Corner showed that every countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free abelian group. This has been the starting point for many investigations of the so-called realization problem, which may be stated as follows.

Given an algebra A over a commutative ring R , when will A be the endomorphism algebra of an R -module G which belongs to some suitably restricted class \mathcal{C} . Complete characterizations of such algebras A have been obtained in the case where R is a complete discrete valuation ring and \mathcal{C} is the class of torsion-free reduced R -modules [11] and also in the case where $R = \mathbb{Z}$ and \mathcal{C} is the class of separable p -groups [10; 9, Section 109]. Such characterizations are, inevitably, much too complicated to lend themselves readily to applications. Consequently Corner [2] tackled the realization problem for primary abelian groups from a different angle. He showed that a suitably large class of rings A could not be realized as full endomorphism rings, but rather that the full endomorphism algebra would be the split extension of the given ring A by some ideal whose presence was unavoidable; in the case of primary groups this ideal is precisely the ideal of small endomorphisms [2]. This idea was subsequently extended to large primary groups in [6] and a similar result was produced in [8] for torsion-free modules over a complete discrete valuation ring.

The results in [2, 6, 8] are all capable of translation into results on endomorphism algebras in a suitable quotient category. Thus, for example, if \mathcal{S} is the category having primary abelian groups as objects, and morphisms

$$\text{Hom}_{\mathcal{S}}(G, H) = \text{Hom}(G, H)/\text{Hom}_s(G, H),$$

where $\text{Hom}_s(G, H)$ consists of the small homomorphisms of G into H , then Corner's result is that if A is a ring whose additive group is the completion of a free p -adic module of at-most countable rank, then there exists a primary group G with $E_{\mathcal{S}}(G) = A$.

When dealing with mixed abelian groups (or more generally mixed R -modules), there is a natural category in which to work, viz. the category Walk (${}_R\text{Walk}$). The objects of ${}_R\text{Walk}$ are R -modules and its morphisms are given by

$$\text{Hom}_W(G, H) = \text{Hom}(G, H)/\text{Hom}_t(G, H),$$

where $\text{Hom}_t(G, H)$ consists of the R -homomorphisms of G into H with torsion image (see [12]). Recently Dugas [4] has shown that each torsion-free reduced ring A is the Walk -endomorphism ring of a mixed abelian group G . The groups G so realized are all of large infinite rank even when the ring A is of comparatively small cardinality.

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The cardinalities of these groups have been significantly reduced in [3], which unifies the torsion, torsion-free and mixed cases of the realization problem.

Our approach will be to construct a (non-trivial) full embedding of the category of torsion-free reduced R -modules into the category ${}_R\text{Walk}$, where R is a principal ideal domain. As a consequence of this full embedding we may immediately lift established results from the category of reduced torsion-free R -modules to the category ${}_R\text{Walk}$. A typical, but by no means exhaustive, list of such results is contained in Corollaries 2.4–2.6. We note, in particular, that many of the results in [7] can now be established immediately. It is, by now, standard to use such realization results to exhibit a wide range of pathologies and so we desist from such repetition.

We conclude this introduction by noting that all unexplained terms may be found in the standard works of Fuchs [9]; our notation is in accord with [9] except that maps are written on the right.

2. The embedding theorem

Throughout let R be a principal ideal domain. We begin with an arbitrary reduced, separable torsion R -module T , and T' any pure extension of T by Q/R such that T' is also separable and reduced. Thus we have a pure-exact sequence of R -modules

$$0 \longrightarrow T \longrightarrow T' \longrightarrow Q/R \longrightarrow 0; \tag{*}$$

these will be fixed for the rest of the section. Note that provided T has no torsion-complete p -component T_p , such a sequence exists (see [9, Corollary 68.5]).

Now, if X is an arbitrary R -module, then (*) yields another pure-exact sequence (see [9, Theorem 60.4]):

$$0 \longrightarrow T \otimes X \longrightarrow T' \otimes X \longrightarrow Q/R \otimes X \longrightarrow 0. \tag{*_X}$$

Since $Q/R \otimes X$ is canonically an epimorphic image of $Q \otimes X$ we can form the pullback $H(X)$ of $*_X$ with respect to this canonical epimorphism η_X . This yields the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T \otimes X & \longrightarrow & H(X) & \xrightarrow{\pi_X} & Q \otimes X \longrightarrow 0 \\ & & \parallel & & \downarrow \sigma_X & & \downarrow \eta_X \\ 0 & \longrightarrow & T \otimes X & \longrightarrow & T' \otimes X & \longrightarrow & Q/R \otimes X \longrightarrow 0 \end{array}$$

in which σ_X is epic since η_X is epic. Note that, by the construction of a pullback, $\text{Ker } \sigma_X$ is mapped isomorphically onto $\text{Ker } \eta_X$ by π_X . Also $\text{Ker } \eta_X$ is canonically isomorphic to $X/t(X)$. The R -module $H(X)$ has the same torsion-free rank as X and its torsion submodule is isomorphic to $T \otimes X$. Note that, if X is torsion-free reduced, then $H(X)$ is reduced and hence non-split. If $U(M) = \bigcap_{0 \neq r \in R} rM$ denotes the first Ulm submodule of an R -module, then the purity of (*) implies the following result.

LEMMA 2.1. $\text{Ker } \sigma_X = U(H(X))$.

Proof. Note first that it follows from [9, Theorem 61.1] that $T'_p \otimes X \cong T'_p \otimes B_p$, where B_p is a p -basic submodule of X . Thus $T'_p \otimes X = \bigoplus_p T'_p \otimes B_p$ and since T' is separable it follows readily that $U(T' \otimes X) = 0$. But $U(H(X))\sigma_X \subseteq U(T' \otimes X) = 0$ and hence $U(H(X)) \subseteq \text{Ker } \sigma_X$.

Conversely, let m be an arbitrary element of $\text{Ker } \sigma_X$ and let r be an arbitrary non-zero element of R . Then there is an element $y \in H(X)$ with $m - ry = z \in t(H(X))$. But then

$$z = z\sigma_X = m\sigma_X - ry\sigma_X = -ry\sigma_X \in r(T' \otimes X) \cap T \otimes X = r(T \otimes X)$$

by the purity of the sequence $(*_X)$. Hence $m \in rH(X)$. Since r is arbitrary non-zero, we have $m \in U(H(X))$ and so $\text{Ker } \sigma_X \subseteq U(H(X))$.

We remark that the construction of $H(X)$ is functorial: every $f \in \text{Hom}(X, Y)$ yields homomorphisms $Q \otimes X \rightarrow Q \otimes Y$ and $T' \otimes X \rightarrow T' \otimes Y$ which in turn give rise to a unique homomorphism $H(f): H(X) \rightarrow H(Y)$ by the universal property of the pullback. We denote this functor by H . In order to place our construction in a functorial setting let U be the subfunctor of the identity defined by $U(X) = \bigcap_{0 \neq r \in R} rX$ and $U(f) = f|U(X)$, the restriction of f to $U(X)$; let F be the functor defined by $F(X) = X/t(X)$ and $F(f) = \bar{f}$, where \bar{f} is the mapping induced by f on the quotient.

PROPOSITION 2.2. *The functors UH and F are naturally equivalent.*

Proof. By Lemma 2.1 $UH(X) = \text{Ker } \sigma_X$ and since π_X maps $\text{Ker } \sigma_X$ isomorphically onto the kernel of η_X the assertion follows from the observation that $\text{Ker } \eta_X \cong X/t(X)$.

In the following let ${}_R\mathcal{C}$ denote the category of torsion-free reduced R -modules.

THEOREM 2.3. *Let R be a principal ideal domain, let T be a separable reduced torsion R -module and let T' be a pure extension of T by Q/R such that T' is separable and reduced. Then there is a full embedding $\bar{H}: {}_R\mathcal{C} \rightarrow {}_R\text{Walk}$ such that*

- (i) $\bar{H}(X)$ is reduced, non-split and of the same torsion-free rank as X ,
- (ii) $t(\bar{H}(X)) \cong T \otimes X$,
- (iii) $\bar{H}(X)/t(\bar{H}(X))$ is divisible,
- (iv) $U\bar{H}(X) = X$ and $\bar{H}(X)/U\bar{H}(X) = T' \otimes X$.

Proof. Let $\bar{H}(X) = H(X)$ for $X \in {}_R\mathcal{C}$ and $\bar{H}(f) = H(f) + \text{Hom}_t(H(X), H(Y))$ for $f: X \rightarrow Y$. The only assertion still to be verified is that \bar{H} is a full embedding. By Proposition 2.2 UH is naturally equivalent to F which is the identity functor on ${}_R\mathcal{C}$. Therefore we may identify X and $UH(X)$. Consider the homomorphisms $h: \text{Hom}(X, Y) \rightarrow \text{Hom}(H(X), H(Y))$ and $u: \text{Hom}(H(X), H(Y)) \rightarrow \text{Hom}(X, Y)$ induced by H and U respectively. Then hu is the identity on $\text{Hom}(X, Y)$, thus h is monic and u is epic. Furthermore $\text{Ker } u = \text{Hom}_t(H(X), H(Y))$ since $g|UH(X) = 0$ implies that $\text{Im } g$ is torsion as an epimorphic image of the torsion module $H(X)/UH(X) \cong T' \otimes X$. On the other hand, if $\text{Im } g$ is torsion, then $g(UH(X)) = 0$ because $UH(Y) \cap t(H(Y)) = 0$. Thus we conclude that the map $f \mapsto \bar{H}(f)$ is an isomorphism and \bar{H} is a full embedding.

REMARKS. (a) An alternative way to construct the functor H is the following. Let $M = H(R)$, a mixed module of torsion-free rank one. Then it is readily seen that the functors H and $M \otimes -$ are naturally equivalent.

(b) As indicated in the above proof, $E(H(X))$ is the split extension of $E(X)$ by $\text{Hom}_t(H(X), H(X))$, that is, there are ring homomorphisms

$$E(X) \xrightarrow{h} E(H(X)) \xrightarrow{u} E(X)$$

such that $hu = \text{id}_{E(X)}$ and $\ker u = \text{Hom}_t(H(X), H(X))$.

COROLLARY 2.4. *Let R be a principal ideal domain. If A is a countable reduced torsion-free R -algebra then there are 2^{\aleph_0} countable mixed R -modules M_i with $M_i/t(M_i)$ divisible, $E_W(M_i) \cong A$ and $\text{Hom}_W(M_i, M_j) = 0$ for $i \neq j$.*

Proof. By an unpublished extension of a well-known theorem of Corner [1] there exist countable reduced torsion-free modules with $E(X_i) \cong A$ and $\text{Hom}(X_i, X_j) = 0$ for $i \neq j$. Now Theorem 2.3 yields the assertion by choosing an appropriate torsion module T , for example an unbounded countable direct sum of cyclics.

In the finite rank case Corner's result gives the following.

COROLLARY 2.5. *Let R be a principal ideal domain and let A be a countable reduced torsion-free algebra of finite rank n . Then there exists a reduced mixed module M of torsion-free rank $2n$ such that $M/t(M)$ is divisible and $E_W(M) = A$.*

COROLLARY 2.6. *If R is a principal ideal domain and not a complete discrete valuation ring and A is any cotorsion-free R -algebra, then there exists a reduced mixed R -module M with $M/t(M)$ divisible and $E_W(M) = A$.*

Proof. This is a consequence of [5, Corollary 5.4], which ensures the existence of a cotorsion-free R -module X with $E(X) = A$.

REMARK. It was shown in [3] that X can be chosen to be of cardinality $|A|^{\aleph_0}$. Thus M can be made to be of cardinality $|A|^{|R|}$.

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