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1. *Introduction*

In a remarkable paper [1] some twenty years ago, Corner showed that every countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free abelian group. This has been the starting point for many investigations of the so-called realization problem, which may be stated as follows.

Given an algebra A over a commutative ring R, when will A be the endomorphism algebra of an R-module G which belongs to some suitably restricted class $\mathscr{C}$. Complete characterizations of such algebras A have been obtained in the case where R is a complete discrete valuation ring and $\mathscr{C}$ is the class of torsion-free reduced R-modules [11] and also in the case where $R = \mathbb{Z}$ and $\mathscr{C}$ is the class of separable p-groups [10; 9, Section 109]. Such characterizations are, inevitably, much too complicated to lend themselves readily to applications. Consequently Corner [2] tackled the realization problem for primary abelian groups from a different angle. He showed that a suitably large class of rings A could not be realized as full endomorphism rings, but rather that the full endomorphism algebra would be the split extension of the given ring A by some ideal whose presence was unavoidable; in the case of primary groups this ideal is precisely the ideal of small endomorphisms [2]. This idea was subsequently extended to large primary groups in [6] and a similar result was produced in [8] for torsion-free modules over a complete discrete valuation ring.

The results in [2, 6, 8] are all capable of translation into results on endomorphism algebras in a suitable quotient category. Thus, for example, if $\mathscr{S}$ is the category having primary abelian groups as objects, and morphisms

$$\mathrm{Hom}_{\mathscr{S}}(G, H) = \mathrm{Hom}(G, H)/\mathrm{Hom}_s(G, H),$$

where $\mathrm{Hom}_s(G, H)$ consists of the small homomorphisms of G into H, then Corner's result is that if A is a ring whose additive group is the completion of a free p-adic module of at-most countable rank, then there exists a primary group G with $E_{\mathscr{S}}(G) = A$.

When dealing with mixed abelian groups (or more generally mixed R-modules), there is a natural category in which to work, viz. the category Walk (${}_R$Walk). The objects of ${}_R$Walk are R-modules and its morphisms are given by

$$\mathrm{Hom}_W(G, H) = \mathrm{Hom}(G, H)/\mathrm{Hom}_t(G, H),$$

where $\mathrm{Hom}_t(G, H)$ consists of the R-homomorphisms of G into H with torsion image (see [12]). Recently Dugas [4] has shown that each torsion-free reduced ring A is the Walk-endomorphism ring of a mixed abelian group G. The groups G so realized are all of large infinite rank even when the ring A is of comparatively small cardinality.

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The cardinalities of these groups have been significantly reduced in [3], which unifies the torsion, torsion-free and mixed cases of the realization problem.

Our approach will be to construct a (non-trivial) full embedding of the category of torsion-free reduced R-modules into the category ${}_R$Walk, where R is a principal ideal domain. As a consequence of this full embedding we may immediately lift established results from the category of reduced torsion-free R-modules to the category ${}_R$Walk. A typical, but by no means exhaustive, list of such results is contained in Corollaries 2.4–2.6. We note, in particular, that many of the results in [7] can now be established immediately. It is, by now, standard to use such realization results to exhibit a wide range of pathologies and so we desist from such repetition.

We conclude this introduction by noting that all unexplained terms may be found in the standard works of Fuchs [9]; our notation is in accord with [9] except that maps are written on the right.

2. *The embedding theorem*

Throughout let R be a principal ideal domain. We begin with an arbitrary reduced, separable torsion R-module T, and T' any pure extension of T by Q/R such that T' is also separable and reduced. Thus we have a pure-exact sequence of R-modules

$$0 \longrightarrow T \longrightarrow T' \longrightarrow Q/R \longrightarrow 0; \qquad (*)$$

these will be fixed for the rest of the section. Note that provided T has no torsion-complete p-component T_p, such a sequence exists (see [9, Corollary 68.5]).

Now, if X is an arbitrary R-module, then $(*)$ yields another pure-exact sequence (see [9, Theorem 60.4]):

$$0 \longrightarrow T \otimes X \longrightarrow T' \otimes X \longrightarrow Q/R \otimes X \longrightarrow 0. \qquad (*_X)$$

Since $Q/R \otimes X$ is canonically an epimorphic image of $Q \otimes X$ we can form the pullback $H(X)$ of $(*_X)$ with respect to this canonical epimorphism η_X. This yields the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \otimes X & \longrightarrow & H(X) & \xrightarrow{\pi_X} & Q \otimes X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \eta_X} & & \\ 0 & \longrightarrow & T \otimes X & \longrightarrow & T' \otimes X & \longrightarrow & Q/R \otimes X & \longrightarrow & 0 \end{array}$$

in which σ_X is epic since η_X is epic. Note that, by the construction of a pullback, $\operatorname{Ker} \sigma_X$ is mapped isomorphically onto $\operatorname{Ker} \eta_X$ by π_X. Also $\operatorname{Ker} \eta_X$ is canonically isomorphic to $X/t(X)$. The R-module $H(X)$ has the same torsion-free rank as X and its torsion submodule is isomorphic to $T \otimes X$. Note that, if X is torsion-free reduced, then $H(X)$ is reduced and hence non-split. If $U(M) = \bigcap_{0 \neq r \in R} rM$ denotes the first Ulm submodule of an R-module, then the purity of $(*)$ implies the following result.

LEMMA 2.1. $\operatorname{Ker} \sigma_X = U(H(X))$.

Proof. Note first that it follows from [9, Theorem 61.1] that $T'_p \otimes X \cong T'_p \otimes B_p$, where B_p is a p-basic submodule of X. Thus $T'_p \otimes X = \bigoplus_p T'_p \otimes B_p$ and since T' is separable it follows readily that $U(T' \otimes X) = 0$. But $U(H(X))\sigma_X \subseteq U(T' \otimes X) = 0$ and hence $U(H(X)) \subseteq \operatorname{Ker} \sigma_X$.

Conversely, let m be an arbitrary element of $\operatorname{Ker} \sigma_X$ and let r be an arbitrary non-zero element of R. Then there is an element $y \in H(X)$ with $m - ry = z \in t(H(X))$. But then

$$z = z\sigma_X = m\sigma_X - ry\sigma_X = -ry\sigma_X \in r(T' \otimes X) \cap T \otimes X = r(T \otimes X)$$

by the purity of the sequence $(*_X)$. Hence $m \in rH(X)$. Since r is arbitrary non-zero, we have $m \in U(H(X))$ and so $\operatorname{Ker} \sigma_X \subseteq U(H(X))$.

We remark that the construction of $H(X)$ is functorial: every $f \in \operatorname{Hom}(X, Y)$ yields homomorphisms $Q \otimes X \to Q \otimes Y$ and $T' \otimes X \to T' \otimes Y$ which in turn give rise to a unique homomorphism $H(f): H(X) \to H(Y)$ by the universal property of the pullback. We denote this functor by H. In order to place our construction in a functorial setting let U be the subfunctor of the identity defined by $U(X) = \bigcap_{0 \neq r \in R} rX$ and $U(f) = f|U(X)$, the restriction of f to $U(X)$; let F be the functor defined by $F(X) = X/t(X)$ and $F(f) = \bar{f}$, where $\bar{f}$ is the mapping induced by f on the quotient.

Proposition 2.2. *The functors UH and F are naturally equivalent.*

Proof. By Lemma 2.1 $UH(X) = \operatorname{Ker} \sigma_X$ and since π_X maps $\operatorname{Ker} \sigma_X$ isomorphically onto the kernel of η_X the assertion follows from the observation that $\operatorname{Ker} \eta_X \cong X/t(X)$.

In the following let ${}_R\mathscr{C}$ denote the category of torsion-free reduced R-modules.

Theorem 2.3. *Let R be a principal ideal domain, let T be a separable reduced torsion R-module and let T' be a pure extension of T by Q/R such that T' is separable and reduced. Then there is a full embedding $\bar{H}: {}_R\mathscr{C} \to {}_R\mathrm{Walk}$ such that*

(i) *$\bar{H}(X)$ is reduced, non-split and of the same torsion-free rank as X,*
(ii) *$t(\bar{H}(X)) \cong T \otimes X$,*
(iii) *$\bar{H}(X)/t(\bar{H}(X))$ is divisible,*
(iv) *$U\bar{H}(X) = X$ and $\bar{H}(X)/U\bar{H}(X) = T' \otimes X$.*

Proof. Let $\bar{H}(X) = H(X)$ for $X \in {}_R\mathscr{C}$ and $\bar{H}(f) = H(f) + \operatorname{Hom}_t(H(X), H(Y))$ for $f: X \to Y$. The only assertion still to be verified is that $\bar{H}$ is a full embedding. By Proposition 2.2 UH is naturally equivalent to F which is the identity functor on ${}_R\mathscr{C}$. Therefore we may identify X and $UH(X)$. Consider the homomorphisms $h: \operatorname{Hom}(X, Y) \to \operatorname{Hom}(H(X), H(Y))$ and $u: \operatorname{Hom}(H(X), H(Y)) \to \operatorname{Hom}(X, Y)$ induced by H and U respectively. Then hu is the identity on $\operatorname{Hom}(X, Y)$, thus h is monic and u is epic. Furthermore $\operatorname{Ker} u = \operatorname{Hom}_t(H(X), H(Y))$ since $g|UH(X) = 0$ implies that $\operatorname{Im} g$ is torsion as an epimorphic image of the torsion module $H(X)/UH(X) \cong T' \otimes X$. On the other hand, if $\operatorname{Im} g$ is torsion, then $g(UH(X)) = 0$ because $UH(Y) \cap t(H(Y)) = 0$. Thus we conclude that the map $f \mapsto \bar{H}(f)$ is an isomorphism and $\bar{H}$ is a full embedding.

Remarks. (a) An alternative way to construct the functor H is the following. Let $M = H(R)$, a mixed module of torsion-free rank one. Then it is readily seen that the functors H and $M\otimes -$ are naturally equivalent.

(b) As indicated in the above proof, $E(H(X))$ is the split extension of $E(X)$ by $\mathrm{Hom}_t(H(X), H(X))$, that is, there are ring homomorphisms

$$E(X) \xrightarrow{h} E(H(X)) \xrightarrow{u} E(X)$$

such that $hu = \mathrm{id}_{E(X)}$ and $\ker u = Hom_t(H(X), H(X))$.

COROLLARY 2.4. *Let R be a principal ideal domain. If A is a countable reduced torsion-free R-algebra then there are $2^{\aleph_0}$ countable mixed R-modules M_i with $M_i/t(M_i)$ divisible, $E_W(M_i) \cong A$ and $\mathrm{Hom}_W(M_i, M_j) = 0$ for $i \neq j$.*

Proof. By an unpublished extension of a well-known theorem of Corner [1] there exist countable reduced torsion-free modules with $E(X_i) \cong A$ and $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$. Now Theorem 2.3 yields the assertion by choosing an appropriate torsion module T, for example an unbounded countable direct sum of cyclics.

In the finite rank case Corner's result gives the following.

COROLLARY 2.5. *Let R be a principal ideal domain and let A be a countable reduced torsion-free algebra of finite rank n. Then there exists a reduced mixed module M of torsion-free rank $2n$ such that $M/t(M)$ is divisible and $E_W(M) = A$.*

COROLLARY 2.6. *If R is a principal ideal domain and not a complete discrete valuation ring and A is any cotorsion-free R-algebra, then there exists a reduced mixed R-module M with $M/t(M)$ divisible and $E_W(M) = A$.*

Proof. This is a consequence of [5, Corollary 5.4], which ensures the existence of a cotorsion-free R-module X with $E(X) = A$.

REMARK. It was shown in [3] that X can be chosen to be of cardinality $|A|^{\aleph_0}$. Thus M can be made to be of cardinality $|A|^{|R|}$.

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