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On the (Non)-Integrability of the Perturbed KdV Hierarchy with Generic Self-consistent Sources

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1. Introduction Nonholonomic deformations of integrable equations attracted the attention of the scientific community in the last few years. In [6], based on the Painlevé test, applied to a class of sixth-order nonlinear wave equations, a list of four equations that pass the test was obtained. Among the three known ones, there was a new equation in the list (later known as sixth-order KdV equation, or just KdV6):

$$\left(-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}\right)(v_t + v_{xxx} - 3v_x^2) = 0. \tag{1}$$

One can convert (1) into a "potential" form:

$$u_t + u_{xxx} - 6uu_x - w_x = 0, \tag{2}$$

$$-\frac{1}{4}w_{xxx} + uw_x + \frac{1}{2}wu_x = 0, \tag{3}$$

or equivalently

$$\left(-\frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial^{-1}\right)(u_t + u_{xxx} - 6uu_x) = 0. \tag{4}$$

Here ∂_x^{-1} is a notation for the left-inverse of ∂_x and

$$\Lambda = -\frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial^{-1} \tag{5}$$

is the recursion operator for the KdV hierarchy. In [7] B. Kupershmidt described (2) and (3) as a nonholonomic deformation of the KdV equation, written in a bi-Hamiltonian form. Later on, it was shown in [9] that the KdV6 equation is equivalent to a Rosochatius deformation of the KdV equation with self-consistent sources.
Here we are dealing with the potential form of the KdV6 equation (2), we study the class of inhomogeneous equations of KdV type

$$u_t + u_{xxx} - 6uu_x = W_x[u](x), \tag{6}$$

with an inhomogeneity/perturbation that presumably belongs to the same class of functions as the field $u(x)$ (i.e. decreasing fast enough, when $|x| \to \infty$).

Generally speaking, the perturbation, as a rule destroys the integrability of the considered nonlinear evolution equation (NLEE). The idea of perturbation through nonholonomic deformation, however, is to perturb an integrable NLEE with a driving force (deforming function), such that under suitable differential constraints on the perturbing function(s) the integrability of the entire system is preserved. In the case of local NLEE's, having a constraint given through differential relations (not by evolutionary equations) is equivalent to a nonholonomic constraint.
To the best of our knowledge, the most natural and efficient way for studying inhomogeneities/perturbations of NLEE integrable by the inverse scattering method is by using the expansions over the so-called "squared solutions" (squared eigenfunctions) or the so-called symplectic basis. The squared eigenfunctions of the spectral problem associated to an integrable equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform (IST) for the corresponding hierarchy as a Generalised Fourier transform (GFT). The Fourier modes for the GFT are the Scattering data. The expansion coefficients of the potential over the symplectic basis are the corresponding action-angle variables.

2. Generalised Fourier Transform for KdV Hierarchy The spectral problem for the KdV hierarchy is given by the Sturm-Liouville equation [8, 5]

$$-\Psi_{xx} + u(x)\Psi = k^2\Psi, \tag{7}$$

where $u(x)$ is a real-valued (Schwartz-class) potential on the whole axis and $k \in \mathbb{C}$ is spectral parameter. The continuous spectrum under these conditions corresponds to real k. We assume that the discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \ldots, N$ where κ_n is real.
The direct scattering problem for (7) is based on the so-called "Jost solutions" $f^+(x,k)$ and $\bar{f}^+(x,\bar{k})$,given by their asymptotics: $x \to \infty$ for all real $k \neq 0$ [8]:

$$\lim_{x\to\pm\infty} e^{-ikx} f^{\pm}(x,k) = 1, \qquad k \in \mathbb{R}\backslash\{0\}. \tag{8}$$

From the reality condition for $u(x)$ it follows that $\bar{f}^{\pm}(x,\bar{k}) = f^{\pm}(x,-k)$.
A key role in the interpretation of the inverse scattering method as a generalized Fourier transform plays the so-called 'generating' (recursion) operator: for the KdV hierarchy it has the form [1]:

$$L_{\pm} = -\frac{1}{4}\partial^2 + u(x) - \frac{1}{2}\int_{\pm\infty}^{x} \mathrm{d}\tilde{x}\, u'(\tilde{x}) \cdot . \tag{9}$$

The eigenfunctions of the recursion operator are the squared eigenfunctions of the spectral problem (7):

$$F^{\pm}(x,k) \equiv (f^{\pm}(x,k))^2, \qquad F_n^{\pm}(x) \equiv F(x, i\kappa_n), \tag{10}$$

For our purposes, it is more convenient to adopt a special set of "squared solutions", called symplectic basis [2, 5]. It has the property that the expansion coefficients of

the potential $u(x)$ over the symplectic basis are the so-called action-angle variables for the corresponding NLEE.
For the KdV hierarchy, the symplectic basis is given by:

$$\mathcal{P}(x,k) = \mp\left(\mathcal{R}^{\pm}(k)F^{\pm}(x,k) - \mathcal{R}^{\pm}(-k)F^{\pm}(x,-k)\right), \tag{11}$$
$$\mathcal{Q}(x,k) = \mathcal{R}^{-}(k)F^{-}(x,k) + \mathcal{R}^{+}(k)F^{+}(x,k), \tag{12}$$
$$P_n(x) = -R_n^{\pm}F_n^{\pm}(x), \qquad Q_n(x) = -\frac{1}{2k_n}\Big(R_n^{+}\dot{F}_n^{-}(x) - R_n^{-}\dot{F}_n^{+}(x)\Big). \tag{13}$$

Its elements satisfy the following canonical relations:

$$\begin{aligned} [\![\mathcal{P}(k_1),\mathcal{Q}(k_2)]\!] &= \delta(k_1-k_2), & [\![\mathcal{P}(k_1),\mathcal{P}(k_2)]\!] &= [\![\mathcal{Q}(k_1),\mathcal{Q}(k_2)]\!] = 0, \\ [\![P_m,Q_n]\!] &= \delta_{mn}, & [\![P_m,P_n]\!] &= [\![Q_m,Q_n]\!] = 0, \end{aligned} \tag{14}$$

$(k_1>0, k_2>0)$ with respect to the skew-symmetric product

$$[\![f,g]\!] \equiv \frac{1}{2}\int_{-\infty}^{\infty}(f(x)g_x(x) - g(x)f_x(x))\mathrm{d}x = \int_{-\infty}^{\infty} f(x)g_x(x)\mathrm{d}x, \tag{15}$$

The symplectic basis satisfies the completeness relation [5]:

$$\begin{aligned} \frac{\theta(x-y)-\theta(y-x)}{2} = \frac{1}{2\pi}\int_0^{\infty}\Big(\mathcal{P}(x,k)\mathcal{Q}(y,k) - \mathcal{Q}(x,k),\mathcal{P}(y,k)\Big)\frac{\mathrm{d}k}{\beta(k)} \\ -\sum_{n=1}^{N}\Big(P_n(x)Q_n(y) - Q_n(x)P_n(y)\Big), \end{aligned} \tag{16}$$

where $\beta(k) = 2ikb(k)b(-k)$. Notice that the integration over k is from 0 to ∞. It follows that every function $X(x)$ from the same class as the potential $u(x)$ (i.e. smooth and vanishing fast enough when $x \to \pm\infty$) can be expanded over the symplectic basis:

$$\begin{aligned} X(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{\mathrm{d}k}{\beta(k)}\left(\mathcal{P}(x,k)\phi_X(k) - \mathcal{Q}(x,k)\rho_X(k)\right) \\ -\sum_{n=1}^{N}\left(P_n(x)\phi_{n,X} - Q_n(x)\rho_{n,X}\right). \end{aligned} \tag{17}$$

The expansion coefficients can be recovered from the so-called inversion formulas:

$$\begin{aligned} \phi_X(k) &= [\![\mathcal{Q}(y,k),X(y)]\!], & \rho_X(k) &= [\![\mathcal{P}(y,k),X(y)]\!], \\ \phi_{n,X} &= [\![Q_n(y),X(y)]\!], & \rho_{n,X} &= [\![P_n(y),X(y)]\!]. \end{aligned} \tag{18}$$

In particular, if $X(x) = u(x)$ is a solution of the spectral problem, one can compute [5]:

$$[\![\mathcal{P}(y,k),u(y)]\!] = 0, \qquad [\![\mathcal{Q}(y,k),u(y)]\!] = -4ik\beta(k), \tag{19}$$
$$[\![P_n(y),u(y)]\!] = 0, \qquad [\![Q_n(y),u(y)]\!] = 4ik_n. \tag{20}$$

Thus, from (19) one gets:

$$u(x) = \frac{2}{\pi i}\int_0^\infty \mathcal{P}(x,k)\mathrm{d}k - \sum_{n=1}^{N} 4ik_n P_n(x). \tag{21}$$

The expression for the variation of the potential is

$$\begin{aligned}\delta u(x) &= \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{\beta(k)}\left(\mathcal{P}_x(x,k)\delta\phi(k) - \mathcal{Q}_x(x,k)\delta\rho(k)\right)\\ &- \sum_{n=1}^{N}\left(P_{n,x}\delta\phi_n - Q_{n,x}\delta\rho_n\right)\end{aligned} \tag{22}$$

with expansion coefficients

$$\rho(k) \equiv -2ik\ln|a(k)| = -2ik\ln(1-\mathcal{R}^-(k)\mathcal{R}^-(-k)), \quad k>0, \tag{23}$$

$$\phi(k) \equiv 2i\beta(k)\arg b(k) = \beta(k)\ln\frac{\mathcal{R}^-(k)}{\mathcal{R}^+(k)}, \tag{24}$$

$$\rho_n = -\lambda_n = -k_n^2, \qquad \phi_n = 2\ln b_n = \ln\frac{R_n^-}{R_n^+}. \tag{25}$$

These are known as action ($\rho(k)$) - angle ($\phi(k)$) variables for KdV equation [10, 8]. Due to the time-evolution of u, $\delta u(x,t) = u_t\delta t + Q((\delta t)^2)$, etc. the equations of the KdV hierarchy

$$u_t + \partial_x\Omega(\Lambda)u(x,t) = 0, \tag{26}$$

with (21) and (22) are equivalent to a system of trivial linear ordinary differential equations for the canonical variables (which can be considered as scattering data):

$$\begin{aligned}\phi_t &= 4i\beta(k)\Omega(k^2), \qquad & \rho_t(k) &= 0,\\ \phi_{n,t} &= 4ik_n\Omega(k_n^2), & \rho_{n,t} &= 0.\end{aligned} \tag{27}$$

3. Perturbations to the equations of the KdV hierarchy Let us consider a general perturbation $W_x[u]$ to an equation from the KdV hierarchy:

$$u_t + \partial_x\Omega(\Lambda)u(x,t) = W_x[u]. \tag{28}$$

The function $W_x[u]$ is assumed to belong to the class of admissible potentials for the associated spectral problem (7) (Schwartz class functions, in our case).
The expansion of the perturbation over the symplectic basis is:

$$\begin{aligned}W_x[u] = \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{\beta(k)}\left(\mathcal{P}_x(x,k)\phi_W(k) - \mathcal{Q}_x(x,k)\rho_W(k)\right)&\\ - \sum_{n=1}^{N}\left(P_{n,x}(x)\phi_{n,W} - Q_{n,x}(x)\rho_{n,W}\right).&\end{aligned} \tag{29}$$

The substitution of the above expansion (29) in (28) together with (21) and (22) leads to a modification of the time evolution (27) of the scattering data as follows:

$$\phi_t = 4i\beta(k)\Omega(k^2) + \phi_W(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), \tag{30}$$

$$\rho_t(k) = \rho_W(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), \tag{31}$$

$$\phi_{n,t} = 4ik_n\Omega(k_n^2) + \phi_{n,W}(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), \tag{32}$$

$$\rho_{n,t} = \rho_{n,W}(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)). \tag{33}$$

Since $W = W[u]$ and u depend on the scattering data, we observe that the expansion coefficients of the perturbation ($\phi_W(k) = [\![\mathcal{Q}(y,k), W(y)]\!]$ etc.) also depend on the scattering data. Thus for generic W the new dynamical system (30) – (33) for the scattering data can be extremely complicated and non-integrable in general. This reflects the obvious fact that the perturbed integrable equations are, in general, not integrable.

4. KdV Hirarchy with Self-consistent Sources (SCS). Let us investigate the integrability of the following equation:

$$\Lambda^*(u_t + \partial_x\Omega(\Lambda)u(x,t)) = 0, \tag{34}$$

where the star is a notation for a Hermitian conjugation. KdV6 in (4) is a particular case of this equation with $\Omega(\Lambda) = -4\Lambda$. In order to simplify our further analysis, instead of the equation (34) we study the following one:

$$(\Lambda^* - \lambda_1)(u_t + \partial_x\Omega(\Lambda)u(x,t)) = 0, \tag{35}$$

where λ_1 is a constant. The corresponding analogue for KdV6 is

$$v_{6x} + v_{txxx} - 2v_tv_{xx} - 4v_xv_{xt} - 10v_xv_{4x} - 20v_{xx}v_{xxx} + 30v_x^2v_{xx} + 4\lambda_1(v_{xt} + v_{xxxx} - 6v_xv_{xx}) = 0. \tag{36}$$

Since the operator ∂ does not have a kernel when u is Schwartz class, (34) is equivalent to

$$u_t + \partial_x\Omega(\Lambda)u(x,t) = \begin{cases} (c_1P_1(x,t) + c_2Q_1(x,t))_x & \text{for} \quad \lambda_1 = k_1^2 < 0, \\ (c_1\mathcal{P}(x,k_1,t) + c_2\mathcal{Q}(x,k_1,t))_x & \text{for} \quad \lambda_1 = k_1^2 > 0, \end{cases} \tag{37}$$

where $c_{1,2} = c_{1,2}(t,k_1;\rho(k_1,t),\phi(k_1,t),\rho_n(t),\phi_n(t))$ are x-independent functions, but the important observation is that the time-dependence could be implicit through the scattering data of the potential $u(x,t)$. Equation (37) is a perturbed equation from the KdV hierarchy. The perturbation in the right-hand side of (37) is in the eigenspace of the recursion operator corresponding to the eigenvalue λ_1, i.e. it is given by 'squared' eigenfunctions of the spectral problem (7) at λ_1. Such a special perturbation is often called 'self-consistent sources' perturbation. For simplicity we use the symplectic basis, see the precise definitions (11) – (13). Typically the SCS in the literature is taken with $c_2 = 0$; such perturbations do not violate integrability.

For example, if $\lambda_1 > 0$ is a continuous spectrum eigenvalue, the dynamical system (30) – (33) has the form

$$\phi_t = 4i\beta(k)\Omega(k^2) + 2\pi\beta(k)c_1(t,k_1;\rho(k_1,t),\phi(k_1,t),\rho_n(t),\phi_n(t))\delta(k-k_1), \quad (38)$$

$$\rho_t(k) = -2\pi\beta(k)c_2(t,k_1;\rho(k_1,t),\phi(k_1,t),\rho_n(t),\phi_n(t))\delta(k-k_1), \quad (39)$$

$$\phi_{n,t} = 4ik_n\Omega(k_n^2), \quad (40)$$

$$\rho_{n,t} = 0. \quad (41)$$

Similar equations can be written for the time evolution of the action-angle variables on the discrete spectrum of the Lax operator L.
It is clear, that dynamical systems like (38) – (41) can not be integrable for a general functional dependence of $c_{1,2}$ on the scattering data. Thus the equations (34), including KdV6, are not *completely* integrable. In other words, there are solutions, which can not be obtained via the Inverse Scattering Method, since the aforementioned dynamical systems for the scattering data are not always integrable.

5. Conclusions and Outlook Here we have used the expansion over the eigenfunctions of the recursion operator for the KdV hierarchy for studying nonholonomic deformations of the corresponding NLEE from the hierarchy. We have shown, that in the case of self-consistent sources, the corresponding perturbed NLEE is integrable, but not completely integrable.
The approach presented in this article can be applied also to the study of inhomogeneous versions of NLEE, related to other linear spectral problems, e.g. the Camassa-Holm equation, various difference and matrix generalizations of KdV-like and Zakharov - Shabat spectral problems, various non-Hamiltonian systems, etc.

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