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AN ESSENTIALLY SEMI-RIGID CLASS OF MODULES

B. GOLDSMITH

Introduction

A class \mathcal{C} of abelian groups is said to be rigid if $\text{Hom}(A, B) = 0$ for all pairs of different groups $A, B \in \mathcal{C}$. The class \mathcal{C} is said to be semi-rigid if $\text{Hom}(A, B) \neq 0$ implies $\text{Hom}(B, A) = 0$ for all pairs of different groups $A, B \in \mathcal{C}$. (This is a slight modification of the concept used by Fuchs in [4] where in addition it is required that for each $A \in \mathcal{C}$, $\text{Hom}(A, A)$ should be a subgroup of the group of rationals \mathbb{Q} .) It is known from model theory that the existence of a proper (that is, not a set) rigid class of groups is not provable in ZFC (see for example [9, §17, 196–203]. In a recent paper [5], Göbel and Shelah have established the existence of a semi-rigid class of cotorsion-free abelian groups. In this note we wish to investigate the analogous problem for reduced torsion-free modules over a complete discrete valuation ring R . Since all such modules are necessarily separable, there is no possibility of finding a rigid or semi-rigid class. However, exploiting the concept of inessential homomorphism used previously in [1, 2, 6, 7], it is natural to raise the question of the existence of essentially rigid and semi-rigid classes. We show in this note that the existence of an essentially semi-rigid class is easily established.

2. Notation and some preliminaries

Throughout we shall suppose that R is a complete discrete valuation ring of cardinality v having unique prime ideal p . Our notation will follow the standard works of Fuchs [3, 4]; set-theoretic concepts may be found in Jech [8].

For an infinite cardinal λ , let S_λ denote a free R -module of rank λ ; note that $|S_\lambda| = \lambda v$. Let \hat{S}_λ denote the completion of S_λ in the p -adic topology. (All topological references will be to this topology.) It follows then that $|\hat{S}_\lambda| = |S_\lambda|^{\aleph_0} = \lambda^{\aleph_0} v^{\aleph_0}$. Define the class Γ by

$$\Gamma = \{\lambda \mid \lambda \text{ is an infinite cardinal with } \lambda^{\aleph_0} = 2^\lambda \geq v\}.$$

LEMMA 1. (i) Γ is a proper class.

(ii) If $\lambda \in \Gamma$ then $|\hat{S}_\lambda| = 2^\lambda$.

Proof. (ii) is trivial. To show that (i) holds note that if α is any cardinal $\geq v$, then $\kappa = \sup\{\alpha, 2^\alpha, 2^{2^\alpha}, \dots\}$ belongs to Γ . (In fact κ is a strong limit of cofinality ω .) The following definition was introduced in [7].

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DEFINITION. An R -module G is said to be a *maximal pure submodule* of the complete R -module \hat{S} if G is a pure submodule of \hat{S} containing S and $\hat{S}/G \cong Q$, the field of fractions of R .

For each $\lambda \in \Gamma$ let $\{G_{\lambda i}\}$ ($i \in I_\lambda$) denote the family of maximal pure submodules of \hat{S}_λ .

LEMMA 2. For each $\lambda \in \Gamma$, $|I_\lambda| = 2^{2^\lambda}$.

Proof. For $\lambda \in \Gamma$, $\hat{S}_\lambda/S_\lambda$ is a Q -vector space of dimension 2^λ . By a well-known result on vector spaces there are precisely $|Q|^{2^\lambda}$ subspaces of $\hat{S}_\lambda/S_\lambda$ of codimension one. Since a maximal pure submodule is just the preimage of a subspace of codimension one, there are 2^{2^λ} maximal pure submodules of \hat{S}_λ , that is, $|I_\lambda| = 2^{2^\lambda}$.

LEMMA 3. If $\alpha, \beta \in \Gamma$ and $\alpha \geq \beta$, then $|\text{Hom}(\hat{S}_\alpha, \hat{S}_\beta)| \leq 2^\alpha$.

Proof. Every homomorphism from \hat{S}_α to \hat{S}_β is determined by its effect on the basic submodule S_α , of rank α . Hence there are at most $|\hat{S}_\beta|^\alpha$ such homomorphisms. If $\alpha, \beta \in \Gamma$ and $\alpha \geq \beta$ then this upper bound becomes $2^{\beta\alpha} = 2^\alpha$.

3. An essentially semi-rigid class

The concept of an inessential homomorphism is a modification of a concept introduced by Corner at the Montpellier Symposium 1967; it has been used previously for working with modules over a complete discrete valuation ring by Goldsmith [7] and Dugas, Göbel and Goldsmith [1]; in a wider context it has also been exploited by Dugas and Göbel [2].

DEFINITION. A homomorphism $\phi: G \rightarrow H$, where G, H are reduced torsion-free R -modules, is said to be *inessential* if the unique extension $\hat{\phi}$ of ϕ from $\hat{G} \rightarrow \hat{H}$ is such that $\hat{G}\hat{\phi} \subseteq H$. The collection of inessential homomorphisms from G to H is denoted by $\text{Ines}(G, H)$.

DEFINITION. A class \mathcal{C} of R -modules is said to be *essentially semi-rigid* if $\text{Hom}(A, B) \neq \text{Ines}(A, B)$ implies $\text{Hom}(B, A) = \text{Ines}(B, A)$ for all pairs of different modules $A, B \in \mathcal{C}$. An *essentially rigid* class is defined similarly.

Note. If A is complete then every homomorphism $A \rightarrow B$ is automatically inessential and so when we raise the problem of the existence of essentially rigid and semi-rigid classes we are, of course, looking for non-trivial examples, that is, examples in which the modules in the class are not complete.

THEOREM. If R is a complete discrete valuation ring then there exists a non-trivial essentially semi-rigid class of R -modules.

Proof. The essentially semi-rigid class will be constructed by choosing inductively one module from each of the sets $\{G_{\lambda i}\}$ ($i \in I_\lambda, \lambda \in \Gamma$). Thus suppose that the collection $\{G_\lambda\}$ ($\lambda < \zeta$), where $\lambda, \zeta \in \Gamma$, has been constructed with the property that

$$\text{Hom}(G_\lambda, G_\alpha) = \text{Ines}(G_\lambda, G_\alpha) \quad \text{for } \alpha < \lambda < \zeta.$$

Fix any $\alpha < \zeta$ and consider the collection $\{\text{Hom}(G_{\zeta_i}, G_\alpha)\} (i \in I_\zeta)$. Define the subset J_α of I_ζ by

$$J_\alpha = \{i \in I_\zeta \mid \text{Hom}(G_{\zeta_i}, G_\alpha) \neq \text{Ines}(G_{\zeta_i}, G_\alpha)\}.$$

We claim that $|J_\alpha| \leq 2^\zeta$. For suppose that $|J_\alpha| > 2^\zeta$ and pick a set of homomorphisms $\{\phi_i: G_{\zeta_i} \rightarrow G_\alpha\} (i \in J_\alpha)$ such that each ϕ_i is not inessential. Since each homomorphism ϕ_i extends uniquely to a homomorphism $\hat{\phi}_i: \hat{S}_\zeta \rightarrow \hat{S}_\alpha$, we deduce from Lemma 3 that $\hat{\phi}_i = \hat{\phi}_j$ for some $i \neq j \in J_\alpha$. However, it follows easily from the property of being maximal pure that $\hat{S}_\zeta = G_{\zeta_i} + G_{\zeta_j}$ and hence $\hat{S}_\zeta \hat{\phi}_i \subseteq G_{\zeta_i} \hat{\phi}_i + G_{\zeta_j} \hat{\phi}_j \subseteq G_\alpha$, contrary to the choice of ϕ_i . Thus $|J_\alpha| \leq 2^\zeta$. Since $|I_\zeta| = 2^{2^\zeta}$ we may pick $i_0 \in I_\zeta \setminus \bigcup_{\alpha < \zeta} J_\alpha$ and

clearly $\text{Hom}(G_{\zeta_{i_0}}, G_\alpha) = \text{Ines}(G_{\zeta_{i_0}}, G_\alpha)$ for all $\alpha < \zeta$. Set $G_\zeta = G_{\zeta_{i_0}}$. The construction is completed by transfinite induction. Since the class of R -modules $\{G_\lambda\} (\lambda \in \Gamma)$ has the property $\text{Hom}(G_\lambda, G_\alpha) = \text{Ines}(G_\lambda, G_\alpha)$ for $\alpha < \lambda$, and each G_λ is a maximal pure submodule of \hat{S}_λ (and hence is not complete), it follows that $\{G_\lambda\} (\lambda \in \Gamma)$ is the desired non-trivial essentially semi-rigid class.

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