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Two component integrable systems modelling shallow water waves

ROSSEN I. IVANOV

The aim of this talk is to describe the derivation of shallow water model equations for the constant vorticity case and to demonstrate how these equations can be related to two integrable systems: a two component integrable generalization of the Camassa-Holm equation and the Kaup - Boussinesq system.

The motion of inviscid fluid is described by Euler’s equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0,$$

where $\rho$ is a constant density, $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at the point $(x, y, z)$ at the time $t$, $P$ is the pressure in the fluid, $\mathbf{g} = (0, 0, -g)$ is the constant Earth’s gravity acceleration.

We consider a motion of a shallow water over a flat bottom, which is located at $z = 0$. We assume that the motion is in the $x$-direction, and that the physical variables do not depend on $y$. Let $h$ be the mean level of the water and let $\eta(x, t)$ describes the shape of the water surface, i.e. the deviation from the average level. The pressure is $P = P_A + \rho g (h - z) + p(x, z, t)$, where $P_A$ is the constant atmospheric pressure, and $p$ is a pressure variable, measuring the deviation from the hydrostatic pressure distribution.

On the surface $z = h + \eta$, $P = P_A$ and therefore $p = \eta \rho g$. Taking $\mathbf{v} \equiv (u, 0, w)$ we can write the kinematic condition on the surface as (e.g. following [1]) $w = \eta_t + u \eta_x$ on $z = h + \eta$. Finally, there is no horizontal velocity at the bottom, thus $w = 0$ on $z = 0$.

Let us introduce now dimensionless parameters $\varepsilon = a/h$ and $\delta = h/\lambda$, where $a$ is the typical amplitude of the wave and $\lambda$ is the typical wavelength of the wave. Now we can introduce dimensionless quantities, according to the magnitude of the physical quantities, see [1, 2] for details: $x \rightarrow \lambda x$, $z \rightarrow zh$, $t \rightarrow \frac{\lambda}{\sqrt{g h}} t$, $\eta \rightarrow a \eta$, $u \rightarrow \varepsilon \sqrt{g h} u$, $w \rightarrow \varepsilon \delta \sqrt{g h} w$, $p \rightarrow \varepsilon \rho g h$.

Now let us notice that there is an exact solution of the governing equations of the form $u = \bar{U}(z)$, $0 \leq z \leq h$, $w \equiv 0$, $p \equiv 0$, $\eta \equiv 0$. This solution represents an arbitrary underlying 'shear' flow. In the presence of a shear flow the horizontal velocity of the fluid will be $\bar{U}(z) + u$. The scaling for such solution is clearly $u \rightarrow \varepsilon \sqrt{g h} (\bar{U}(z) + \varepsilon u)$, and the scaling for the other variables is as before. The system of equations is (the prime denotes derivative with respect to $z$):

$$u_t + \bar{U} u_x + w \bar{U}' + \varepsilon (u w_x + w u_x) = -p_x,$$
$$\delta^2 (w_t + \bar{U} w_x + \varepsilon (u w_x + w u_x)) = -p_z,$$
$$u_x + w_z = 0,$$
$$w = \eta_t + (\bar{U} + \varepsilon u) \eta_x,$$
$$p = \eta, \quad \text{on} \quad z = 1 + \varepsilon \eta,$$
$$w = 0, \quad \text{on} \quad z = 0.$$
The simplest nontrivial case is a linear shear, \( \hat{U}(z) = Az \), where \( A \) is a constant. We choose \( A > 0 \), so that the underlying flow is propagating in the positive direction of the \( x \)-coordinate.

The vorticity is \( \omega = (U + u)_z - w_x \) or in terms of the rescaled variables, \( \omega = A + \varepsilon (u_z - \delta^2 w_x) \). We are looking for a solution with constant vorticity \( \omega = A \), and therefore we require that \( u_z - \delta^2 w_x = 0 \). Together with the equation \( u_x + w_z = 0 \) it gives

\[
u = u_0 - \frac{\delta^2 z^2}{2} u_{0xx} + O(\varepsilon^2, \varepsilon^4, \varepsilon^6), \quad w = -zu_{0x} + \delta^2 z^3 \frac{3}{6} u_{0xxx} + O(\varepsilon^2, \varepsilon^4, \varepsilon^6),
\]

where \( u_0(x,t) \) is the leading order approximation for \( u \).

With these expressions we obtain the following from the condition on the surface, ignoring terms of order \( O(\varepsilon, \varepsilon^4, \varepsilon^6) \):

\[
\eta_t + A \eta_x + \left[ (1 + \varepsilon \eta) u_0 + \varepsilon \frac{A}{2} \eta^2 \right]_x = \delta^2 \frac{1}{6} u_{0xxx} = 0
\]

From the second of the Euler’s equations and the condition on the surface we have \( p = \eta - \delta^2 \left[ \frac{1}{2} u_{0xx} + \frac{1}{3} \frac{A}{2} A u_{0xx} \right] \), then the first of the Euler’s equations gives (Note that there is no \( z \)-dependence!)

\[
\left( u_0 - \frac{\delta^2}{2} u_{0xx} \right)_t + \varepsilon u_0 u_x + \eta_x - \delta^2 \frac{A}{3} u_{0xxx} = 0.
\]

The linearised equations are

\[
u_{0t} + \eta_x = 0, \quad \eta_t + A \eta_x + u_{0x} = 0,
\]

giving \( \eta_t + A \eta_x - \eta_{xx} = 0 \). This linear equation has a travelling wave solution \( \eta = \eta(x - ct) \) with a velocity \( c \) satisfying \( c^2 - Ac - 1 = 0 \), or

\[
c = \frac{1}{2} \left( A \pm \sqrt{4 + A^2} \right).
\]

If there is no shear (\( A = 0 \)), then \( c = \pm 1 \). In general, there is one positive and one negative solution, representing left and right running waves. Suppose that we have only one of these waves, then \( \eta = c u_0 + O(\varepsilon, \varepsilon^2) \) - e.g. from (3).

By introduction of a new variable \( \rho = 1 + \varepsilon \alpha \eta + \varepsilon^2 \beta \eta^2 + \varepsilon^3 \gamma u_{0xx} \), where

\[
\alpha = \frac{1}{3(1 + c^2)} + \frac{2c^2}{3(1 + c^2)} \left( 1 + \frac{Ac}{2} \right), \quad \beta = \frac{1 - (3 + c^2)(1 + \frac{Ac}{2})}{3(1 + c^2)}, \quad \gamma = \frac{\alpha}{6(c - A)},
\]

and a change of variables (rescaling) \( u_0 \rightarrow \frac{1}{\sqrt{c}} u_0, \quad x \rightarrow \frac{4}{\sqrt{c}} x, \quad t \rightarrow \frac{4}{\sqrt{c}} t \) where \( B = \frac{1}{2} + \frac{1}{6(c - A)} \left( A - \frac{1}{c} \right) \) the equations (1), (2) transform into the system

\[
\begin{align*}
(4) & \quad m_t + Am_x - u_{0xx} + 2mu_{0x} + u_0 m_x + \rho \rho_x = 0, & m = u_0 - u_{0xx} \\
(5) & \quad \rho t + A \rho_x + (\rho u_0)_x = 0,
\end{align*}
\]

Before the rescaling we had \( \eta \varepsilon \eta = \rho - 1 - \varepsilon^2 \beta \varepsilon u_0^2 - \varepsilon^2 \gamma u_{0xx} \). Since in the leading order \( \eta = c u_0 \) the rescaling of \( \eta \) is \( \eta \rightarrow \frac{1}{\alpha} \eta \). Thus in terms of the rescaled variables \( \eta = \rho - 1 - \frac{\beta c^2}{\alpha} u_0^2 - B^2 \frac{2}{\alpha} u_{0xx} \).
The system (4), (5) is an integrable 2-component Camassa-Holm system that appears in [3], generalizing the famous Camassa-Holm equation [4]. The Lax representation for this system is (ζ is a spectral parameter)

\[ \Psi_{xx} = \left( -\zeta^2 \rho^2 + \zeta (m - \frac{A}{2}) + \frac{1}{4} \right) \Psi, \]

\[ \Psi_t = \left( \frac{1}{2} \zeta - u_0 - A \right) \Psi_x + \frac{1}{2} u_{0x} \Psi. \]

An alternative derivation for the case of zero vorticity, based on the Green-Naghdi equations is reported in [5].

Another integrable system matching the water waves asymptotic equations to the first order of the small parameters \( \varepsilon, \delta \) is the Kaup - Boussinesq system. We describe briefly its derivation. Introducing \( V = u - \delta^2 (\frac{1}{2} - \frac{A}{2}) u_{xx} \) the equation (2) can be written as \( V_t + \varepsilon V V_x + \eta_x = 0 \). Equation (1) in the first order in \( \varepsilon, \delta \) is

\[ \eta_t + \left[ A \eta + (1 + \varepsilon \eta) u_0 + \varepsilon \frac{A}{2} \eta^2 \right]_x - \delta^2 \frac{1}{6} u_{0xxx} = 0 \]

and with a shift \( \eta \rightarrow \eta - \frac{1}{\varepsilon} \) it becomes

\[ \eta_t + \varepsilon (1 + \frac{Ac}{2}) (\eta u_0)_x - \delta^2 \frac{1}{6} u_{0xxx} = 0 \quad \text{or} \quad \eta_t + \varepsilon \left( 1 + \frac{c^2}{2} \right) \eta V_x - \delta^2 \frac{1}{6} V_{xxx} = 0. \]

Further rescaling leads to the Kaup - Boussinesq system

\[ V_t + V V_x + \eta_x = 0, \quad \eta_t - \frac{1}{4} V_{xxx} + \frac{1 + \varepsilon c^2}{2} (\eta V)_x = 0, \]

which is integrable iff \( A = 0 \) \( (c^2 = 1) \) with a Lax pair [6]

\[ \Psi_{xx} = -\left( (\zeta - \frac{1}{2} V^2 - \eta) + \frac{1}{2} V \right) \Psi, \quad \Psi_t = -(\zeta + \frac{1}{2} V) \Psi_x + \frac{1}{2} V_x \Psi. \]

It is interesting to investigate further which specific properties of the original governing equations are preserved in the 'integrable' approximate models. For example the 2-component Camassa-Holm system for certain initial data admits breaking waves solutions [5].

References