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POISSON STRUCTURES OF EQUATIONS ASSOCIATED WITH GROUPS OF DIFFEOMORPHISMS

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A class of equations describing the geodesic flow for a right-invariant metric on the group of diffeomorphisms of \mathbb{R}^n is reviewed from the viewpoint of their Lie-Poisson structures. A subclass of these equations is analogous to the Euler equations in hydrodynamics (for $n = 3$), preserving the volume element of the domain of fluid flow. An example in $n = 1$ dimension is the Camassa-Holm equation, which is a geodesic flow equation on the group of diffeomorphisms, preserving the H^1 metric.

Keywords: Lie group, Virasoro group, group of diffeomorphisms, Lie-Poisson bracket, vector fields

1. Camassa-Holm equation

The Camassa-Holm (CH) equation can be considered as a member of the family of EPDiff equations, that is, Euler-Poincaré equations, associated with the diffeomorphism group in n -dimensions.¹⁸ Let us consider first the CH equation in the form

$$q_t + 2u_x q + u q_x = 0, \quad q = u - u_{xx} + \omega, \quad (1)$$

with ω an arbitrary parameter. The traveling wave solutions of (1) are smooth solitons⁵ if $\omega > 0$ and peaked solitons (peakons) if $\omega = 0$.^{4,13,14,24,28}

CH is a bi-hamiltonian equation, i.e. it admits two compatible Hamiltonian structures^{4,15} $J_1 = -(q\partial + \partial q)$, $J_2 = -(\partial - \partial^3)$:

$$q_t = J_2 \frac{\delta H_2[q]}{\delta q} = J_1 \frac{\delta H_1[q]}{\delta q}, \quad (2)$$

$$H_1 = \frac{1}{2} \int q u dx, \quad (3)$$

$$H_2 = \frac{1}{2} \int (u^3 + u u_x^2 + 2\omega u^2) dx. \quad (4)$$

If $\omega \neq 0$ the invariance group of the Hamiltonian is the Virasoro group, $\text{Vir} = \text{Diff}(\mathbb{S}^1) \times \mathbb{R}$ and the central extension of the corresponding Virasoro algebra is proportional to ω .^{10-12,19,25,30} Thus CH has various conformal properties.²¹ It is also completely integrable, possesses bi-Hamiltonian form and infinite sequence of conservation laws.^{4,8,9,22,32}

The soliton solution has the form

$$q(x, t) = \int_0^\infty \delta(x - X(\xi, t)) P(\xi, t) d\xi, \quad (5)$$

where $X(\xi, t)$ and $P(\xi, t)$ are quantities well defined in terms of the scattering data^{7,8,10} ($q(x, 0) > 0$ is assumed, otherwise wave breaking occurs⁶). From (5) one can easily compute $u = (1 - \partial^2)^{-1}(q - \omega)$,

$$u(x, t) = \frac{1}{2} \int_0^\infty e^{-|x - X(\xi, t)|} P(\xi, t) d\xi - \omega. \quad (6)$$

Substitution of (5) and (6) into the equation (1) and using the fact that

$$f(x)\delta'(x - x_0) = f(x_0)\delta'(x - x_0) - f'(x_0)\delta(x - x_0)$$

we derive a system of integral equations for X and P :

$$X_t(\xi, t) = \int G(X(\xi, t) - X(\underline{\xi}, t)) P(\underline{\xi}, t) d\underline{\xi} - \omega, \quad (7)$$

$$P_t(\xi, t) = - \int G'(X(\xi, t) - X(\underline{\xi}, t)) P(\xi, t) P(\underline{\xi}, t) d\underline{\xi}, \quad (8)$$

where $G(x) \equiv \frac{1}{2}e^{-|x|}$. From (5) and (6) the Hamiltonian H_1 can be expressed as

$$H_1(X, P) = \frac{1}{2} \int G(X(\xi_1, t) - X(\xi_2, t)) P(\xi_1, t) P(\xi_2, t) d\xi_1 d\xi_2 - \omega \int P(\xi, t) d\xi$$

and the equations (7) and (8) as

$$X_t(\xi, t) = \frac{\delta H}{\delta P(\xi, t)}, \quad P_t(\xi, t) = - \frac{\delta H}{\delta X(\xi, t)}, \quad (9)$$

i.e. these equations are Hamiltonian, with respect to the canonical Poisson bracket

$$\{A, B\}_c = \int \left(\frac{\delta A}{\delta X(\xi, t)} \frac{\delta B}{\delta P(\xi, t)} - \frac{\delta B}{\delta X(\xi, t)} \frac{\delta A}{\delta P(\xi, t)} \right) d\xi. \quad (10)$$

and the canonical variables are $X(\xi, t)$, $P(\xi, t)$:

$$\{X(\xi_1, t), P(\xi_2, t)\}_c = \delta(\xi_1 - \xi_2), \quad (11)$$

$$\{P(\xi_1, t), P(\xi_2, t)\}_c = \{X(\xi_1, t), X(\xi_2, t)\}_c = 0. \quad (12)$$

Now we can show that (5) is a momentum map that produces the Poisson brackets, given by the Hamiltonian operator J_1 . To do this we will use the canonical Poisson brackets (11), (12) to compute $\{q(x_1), q(x_2)\}_c$.

Indeed

$$\begin{aligned}
\{q(x_1, t), q(x_2, t)\}_c &= \\
&\left\{ \int_0^\infty \delta(x_1 - X(\xi_1, t)) P(\xi_1, t) d\xi_1, \int_0^\infty \delta(x_2 - X(\xi_2, t)) P(\xi_2, t) d\xi_2 \right\}_c = \\
&- \int_0^\infty \int_0^\infty \{X(\xi_1, t), P(\xi_2, t)\}_c \delta'(x_1 - X(\xi_1, t)) P(\xi_1, t) \delta(x_2 - X(\xi_2, t)) d\xi_1 d\xi_2 \\
&- \int_0^\infty \int_0^\infty \{P(\xi_1, t), X(\xi_2, t)\}_c \delta(x_1 - X(\xi_1, t)) \delta'(x_2 - X(\xi_2, t)) P(\xi_2, t) d\xi_1 d\xi_2 \\
&= -\delta'(x_1 - x_2) \int_0^\infty P(\xi_2, t) \delta(x_2 - X(\xi_2, t)) d\xi_2 + \\
&\qquad\qquad\qquad \delta'(x_2 - x_1) \int_0^\infty P(\xi_1, t) \delta(x_1 - X(\xi_1, t)) d\xi_1 \\
&= -q(x_2, t) \delta'(x_1 - x_2) + q(x_1, t) \delta'(x_2 - x_1) \\
&= -\left(q(x_1, t) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} q(x_1, t) \right) \delta(x_1 - x_2) = J_1(x_1) \delta(x_1 - x_2).
\end{aligned}$$

Now it is straightforward to check, using (2), that (1) can be written in a Hamiltonian form as

$$q_t = \{q, H_1\}_c,$$

with the Poisson bracket, generated by J_1 :

$$\begin{aligned}
\{A, B\}_c &= \int \frac{\delta A}{\delta q(x)} J_1(x) \frac{\delta B}{\delta q(x)} dx \\
&= - \int q(x) \left(\frac{\delta A}{\delta q(x)} \frac{\partial}{\partial x} \frac{\delta B}{\delta q(x)} - \frac{\delta B}{\delta q(x)} \frac{\partial}{\partial x} \frac{\delta A}{\delta q(x)} \right) dx. \quad (13)
\end{aligned}$$

A singular momentum map of type (5) is used¹⁸ for the construction of peakon, filament and sheet singular solutions for higher dimensional EPDiff equations.

The parallel with the geometric interpretation of the integrable $SO(3)$ top can be made explicit by a discretization of CH equation based on Fourier modes expansion.²³ Since the Virasoro algebra is an infinite-dimensional algebra, the obtained equation represents an 'integrable top' with infinitely many momentum components.

If we compare (6) and (7) we have

$$X_t(\xi, t) = u(X(\xi, t), t), \quad (14)$$

i.e. $X(\xi, t)$ is explicitly the diffeomorphism related to the geodesic curve,^{10,27,30} i.e. $X(x, t)$ is an one-parameter curve of diffeomorphisms of \mathbb{R} (or, with periodic boundary conditions, of the circle \mathbb{S}^1), depending on a parameter t and associated with a right-invariant metric given by the Hamiltonian H_1 .

For the peakon solutions ($\omega = 0$) the dependence on the scattering data is also known. For completeness and comparison we mention the analogous results for this case. The N -peakon solution has the form^{2,4}

$$u(x, t) = \frac{1}{2} \sum_{i=1}^N p_i(t) \exp(-|x - x_i(t)|), \quad (15)$$

provided p_i and x_i evolve according to the following system of ordinary differential equations:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad (16)$$

where the Hamiltonian is $H = \frac{1}{4} \sum_{i,j=1}^N p_i p_j \exp(-|x_i - x_j|)$. Now one can see immediately the analogy between $X(\xi, t)$ and $x_i(t)$; $P(\xi, t)$ and $p_i(t)$ due to the fact that the N -soliton solution with the limit $\omega \rightarrow 0$ converges to the N -peakon solution.³

2. n -dimensional EPDiff equations

Let us consider motion in \mathbb{R}^n with a velocity field $\mathbf{u}(\mathbf{x}, t): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and define a momentum variable $\mathbf{m} = Q\mathbf{u}$ for some (inertia) operator Q (for CH generalizations Q is the Helmholtz operator $Q = 1 - \partial_i \partial_i = 1 - \Delta$, where $\partial_i = \frac{\partial}{\partial x^i}$). The kinetic energy defines a Lagrangian

$$L[\mathbf{u}] = \frac{1}{2} \int \mathbf{m} \cdot \mathbf{u} \, d^n \mathbf{x}. \quad (17)$$

Since the velocity $\mathbf{u} = u^i \partial_i$ is a vector field, $\mathbf{m} = m_i dx^i \otimes d^n \mathbf{x}$ is a $n + 1$ -form density, we have a natural bilinear form

$$\langle \mathbf{m}, \mathbf{u} \rangle = \int \mathbf{m} \cdot \mathbf{u} \, d^n \mathbf{x}. \quad (18)$$

The Euler-Poincaré equation for the geodesic motion is^{18,19}

$$\frac{d}{dt} \frac{\delta L}{\delta \mathbf{u}} + \text{ad}_{\mathbf{u}}^* \frac{\delta L}{\delta \mathbf{u}} = 0, \quad \mathbf{u} = G * \mathbf{m}, \quad (19)$$

where G is the Green function for the operator Q . The corresponding Hamiltonian is

$$H[\mathbf{m}] = \langle \mathbf{m}, \mathbf{u} \rangle - L[\mathbf{u}] = \frac{1}{2} \int \mathbf{m} \cdot G * \mathbf{m} \, d^n \mathbf{x}, \quad (20)$$

and the equation in Hamiltonian form ($\mathbf{u} = \frac{\delta H}{\delta \mathbf{m}}$) is

$$\frac{\partial \mathbf{m}}{\partial t} = -\text{ad}_{\frac{\delta H}{\delta \mathbf{m}}}^* \mathbf{m}. \quad (21)$$

The left Lie algebra of vector fields is $[\mathbf{u}, \mathbf{v}] = -(u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p$. For an arbitrary vector field \mathbf{v} one can write¹⁹

$$\begin{aligned} \langle \text{ad}_{\mathbf{u}}^* \mathbf{m}, \mathbf{v} \rangle &= \langle \mathbf{m}, \text{ad}_{\mathbf{u}} \mathbf{v} \rangle = \langle \mathbf{m}, [\mathbf{u}, \mathbf{v}] \rangle \\ &= -\langle m_l dx^l \otimes d^n \mathbf{x}, (u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p \rangle \\ &= -\int m_p (u^k(\partial_k v^p) - v^k(\partial_k u^p)) d^n \mathbf{x} \\ &= -\int (\partial_k (m_p u^k v^p) - (\partial_k m_p) u^k v^p - m_p v^p (\partial_k u^k) - m_p v^k (\partial_k u^p)) d^n \mathbf{x} \\ &= \int v^p (u^k (\partial_k m_p) + m_p (\partial_k u^k) + m_k (\partial_p u^k)) d^n \mathbf{x} \\ &= \langle ((\mathbf{u} \cdot \nabla) m_p + \mathbf{m} \cdot \partial_p \mathbf{u} + m_p \text{div} \mathbf{u}) dx^p \otimes d^n \mathbf{x}, \mathbf{v} \rangle, \end{aligned}$$

and therefore (21) has the form

$$\frac{\partial m_p}{\partial t} + (\mathbf{u} \cdot \nabla) m_p + \mathbf{m} \cdot \partial_p \mathbf{u} + m_p \text{div} \mathbf{u} = 0. \quad (22)$$

Let us now define an one-parametric group of diffeomorphisms of \mathbb{R}^n , with elements that satisfy

$$\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{x}, t), t), \quad \mathbf{X}(\mathbf{x}, 0) = \mathbf{x}. \quad (23)$$

Due to the invariance of the Hamiltonian under the action of the group there is a momentum conservation law:

$$m_i(\mathbf{X}(\mathbf{x}, t), t) \partial_j X^i(\mathbf{x}, t) \det \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right) = m_j(\mathbf{x}, 0), \quad (24)$$

where $\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right)_{ij} = \frac{\partial X^i}{\partial x^j}$ is the Jacobian matrix.

The Lie-Poisson bracket is

$$\begin{aligned} \{A, B\}(\mathbf{m}) &= \left\langle \mathbf{m}, \left[\frac{\delta A}{\delta \mathbf{m}}, \frac{\delta B}{\delta \mathbf{m}} \right] \right\rangle \\ &= -\int m_i \left(\frac{\delta A}{\delta m_k} \partial_k \frac{\delta B}{\delta m_i} - \frac{\delta B}{\delta m_k} \partial_k \frac{\delta A}{\delta m_i} \right) d^n \mathbf{x}. \end{aligned} \quad (25)$$

When $n = 1$ clearly (25) gives (13) and the algebra, associated with the bracket is the algebra of vector fields on the circle. This algebra admits a generalization with a central extension, which is the famous Virasoro algebra.^{10-12,19,25,30} In two dimensions, $n = 2$, the algebra, associated with the bracket is the algebra of vector fields on a torus.^{1,16,33} This algebra also admits central extensions.^{16,20}

3. Reduction to the subgroup of volume-preserving diffeomorphisms

In the case of volume-preserving diffeomorphisms we consider vector fields, further restricted by the condition $\operatorname{div} \mathbf{u} = 0$. Let us restrict ourselves to the three-dimensional case ($n = 3$) and let us assume that $\mathbf{m} = (1 - \Delta)\mathbf{u}$, so that $\operatorname{div} \mathbf{m} = 0$ as well. According to the Helmholtz decomposition theorem for vector fields, \mathbf{m} can be determined only by the quantity $\boldsymbol{\Omega} = \nabla \times \mathbf{m}$. Therefore we can write the Lie-Poisson brackets (25) in terms of $\boldsymbol{\Omega}$. Indeed, one can compute that

$$\frac{\delta A}{\delta \mathbf{m}} = \nabla \times \frac{\delta A}{\delta \boldsymbol{\Omega}} \quad (26)$$

Thus

$$\nabla \cdot \frac{\delta A}{\delta \mathbf{m}} = \nabla \cdot \left(\nabla \times \frac{\delta A}{\delta \boldsymbol{\Omega}} \right) = 0, \quad (27)$$

i.e. the vector fields $\frac{\delta A}{\delta \mathbf{m}}$ are divergence-free. Therefore

$$m_i \frac{\delta A}{\delta m_k} \partial_k \frac{\delta B}{\delta m_i} = \partial_k \left(m_i \frac{\delta A}{\delta m_k} \frac{\delta B}{\delta m_i} \right) - (\partial_k m_i) \frac{\delta A}{\delta m_k} \frac{\delta B}{\delta m_i}$$

and from (25) we obtain

$$\begin{aligned} \{A, B\} &= \int (\partial_k m_i) \left(\frac{\delta A}{\delta m_k} \frac{\delta B}{\delta m_i} - \frac{\delta B}{\delta m_k} \frac{\delta A}{\delta m_i} \right) d^3 \mathbf{x} \\ &= \int \boldsymbol{\Omega} \cdot \left(\frac{\delta A}{\delta \mathbf{m}} \times \frac{\delta B}{\delta \mathbf{m}} \right) d^3 \mathbf{x} \\ &= \int \boldsymbol{\Omega} \cdot \left(\left(\nabla \times \frac{\delta A}{\delta \boldsymbol{\Omega}} \right) \times \left(\nabla \times \frac{\delta B}{\delta \boldsymbol{\Omega}} \right) \right) d^3 \mathbf{x}. \end{aligned} \quad (28)$$

This is the well known Poisson bracket used in fluid mechanics.^{1,17,26,29,31,34} The curl of the equation (22) gives the following equation for $\boldsymbol{\Omega}$:¹⁹

$$\boldsymbol{\Omega}_t + (\mathbf{u} \cdot \nabla) \boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \nabla) \mathbf{u} = 0, \quad \boldsymbol{\Omega} = (1 - \Delta)(\nabla \times \mathbf{u}). \quad (29)$$

Note that \mathbf{u} can be expressed through $\boldsymbol{\Omega}$:

$$\mathbf{u}[\boldsymbol{\Omega}] = (1 - \Delta)^{-1} \left(\nabla \times \int \frac{\boldsymbol{\Omega}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \right),$$

thus

$$H[\boldsymbol{\Omega}] = \frac{1}{2} \int \mathbf{u}[\boldsymbol{\Omega}] \cdot (1 - \Delta) \mathbf{u}[\boldsymbol{\Omega}] d^3 \mathbf{x}.$$

The vector $\boldsymbol{\Omega}$ is always perpendicular to \mathbf{u} . The further reduction to an equation in $n = 2$ dimensions is straightforward. Introducing a (scalar) stream function $\psi(x^1, x^2)$ we have

$$\mathbf{u}(x^1, x^2) = (-\partial_2\psi, \partial_1\psi, 0) = \mathbf{e}_3 \times \nabla\psi, \quad (30)$$

$$\boldsymbol{\Omega}(x^1, x^2) = (1 - \Delta)(\nabla \times \mathbf{u}) = (1 - \Delta)\Delta\psi\mathbf{e}_3, \quad (31)$$

where \mathbf{e}_3 is the unit vector in the direction of x^3 . Since $(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u} = \Omega\partial_3\mathbf{u} = 0$, (29) leads to the equation

$$\boldsymbol{\Omega}_t + (\mathbf{u} \cdot \nabla)\boldsymbol{\Omega} = 0,$$

which produces a scalar equation for the stream function ψ due to (30) and (31), or alternatively for $\Omega \equiv \boldsymbol{\Omega} \cdot \mathbf{e}_3$. The Poisson bracket that one can find from (28) is

$$\begin{aligned} \{A, B\} &= \int \Omega \left(\partial_1 \left(\frac{\delta A}{\delta \Omega} \right) \partial_2 \left(\frac{\delta B}{\delta \Omega} \right) - \partial_2 \left(\frac{\delta A}{\delta \Omega} \right) \partial_1 \left(\frac{\delta B}{\delta \Omega} \right) \right) d^2\mathbf{x} \\ &= \int \boldsymbol{\Omega} \cdot \left(\nabla \left(\frac{\delta A}{\delta \Omega} \right) \times \nabla \left(\frac{\delta B}{\delta \Omega} \right) \right) d^2\mathbf{x} \end{aligned} \quad (32)$$

and the Hamiltonian

$$H = \frac{1}{2} \int \nabla\psi \cdot (1 - \Delta)\nabla\psi d^2\mathbf{x}.$$

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