Two soliton interactions of BD.I multicomponent NLS equations and their gauge equivalent

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Two soliton interactions of BD.I multicomponent NLS equations and their gauge equivalent

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Abstract. Using the dressing Zakharov-Shabat method we re-derive the effects of the two-soliton interactions for the MNLS equations related to the **BD.I**-type symmetric spaces. Next we generalize this analysis for the Heisenberg ferromagnet type equations, gauge equivalent to MNLS.

Keywords: Multicomponent nonlinear Schrödinger equations, gauge equivalence, soliton solutions, reduction group
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INTRODUCTION

The multicomponent nonlinear Schrödinger (MNLS) equations related to symmetric spaces [26, 4] and their gauge equivalent multicomponent Heisenberg ferromagnets (MHF) systems have been extensively studied during the last decades. A number of important results such as the spectral theory of their Lax operators [29, 27, 5] and its equivalence to a Riemann-Hilbert problem [32, 30, 25, 18, 13, 7, 8, 23, 16], their Hamiltonian structures and the theory of the relevant recursion operators are well known by now [9, 10, 12, 15, 24, 19].

An important and still unsolved problem which we will address below is the analysis of the soliton interactions for these MNLS. In [11] we used one the versions of the dressing Zakharov-Shabat method to derive explicitly the two-soliton solution of **BD.I**-type MNLS equations:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^{\,\dagger},\vec{q})\vec{q} - (\vec{q},s_0\vec{q})s_0\vec{q}^{\,*} = 0, \tag{1}$$

and evaluated their asymptotics for $t \to \pm\infty$.

For $n = 3$ one can recover MNLS equations describing BEC with spin $F = 1$ while $n = 5$ one recovers similar equations describing BEC with spin $F = 2$ [21, 17, 28]. The Hamiltonians for the MNLS equations (1) are given by

$$H_{\rm MNLS} = \int_{-\infty}^{\infty} dx \left((\partial_x \vec{q}^{\,\dagger}, \partial_x \vec{q}) - (\vec{q}^{\,\dagger},\vec{q})^2 + \frac{1}{2}(\vec{q}^{\,\dagger}, s_0\vec{q}^{\,*})(\vec{q}^{\,T}, s_0\vec{q}) \right), \tag{2}$$

The classical results of Zakharov and Shabat about soliton interactions [35] were generalized for the vector nonlinear Schrödinger equation by Manakov [26]. For detailed exposition see the monographs [32, 3] and also [1, 23, 31]. However the soliton interactions for the MNLS related to symmetric spaces [4] remain an open problem. Below, for the class of **BD.I** symmetric spaces we provide a tool for solving it.

The Zakharov Shabat approach consisted in calculating the asymptotics of generic N-soliton solution of NLS for $t \to \pm\infty$ and establishing the pure elastic character of the generic soliton interactions [35]. By generic here we mean N-soliton solution whose parameters $\lambda_k^{\pm} = \mu_k \pm i\nu_k$ are such that $\mu_k \neq \mu_j$ for $k \neq j$. The pure elastic character of the soliton interactions is demonstrated by the fact that for $t \to \pm\infty$ the generic N-soliton solution splits into sum of N one soliton solutions each preserving its amplitude $2\nu_k$ and velocity μ_k. The only effect of the interaction consists in shifting the center of mass and the initial phase of the solitons. These shifts can be expressed in terms of $\lambda_k^{\pm}$ only; for detailed exposition for the scalar NLS eq. see [3].

It is well known also, that the Lax representation $[L(\lambda), M(\lambda)] = 0$ is invariant with respect to the gauge group action [3]. The first nontrivial example is the gauge equivalence between the nonlinear Schrödinger (NLS) equation and the Heisenberg feromagnet (HF) equation.

Our aim in this paper is to rederive the result in [11] using an alternative version of the dressing method. Here we follow the classical approach of Zakharov, Shabat and Manakov developed first for the scalar NLS equation [35] and generalize it for the **BD.I**-type MNLS. In the next Section we generalize these results for the gauge equivalent HF-type systems:

$$i\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(\mathrm{ad}_S^{-1}\frac{\partial S}{\partial x}\right) = 0 \tag{3}$$

where $\mathrm{ad}_S X = [S, X]$, $S^3 = S$ and as a result (see [6])

$$\mathrm{ad}_S^{-1} X = \frac{1}{4}\left(5\mathrm{ad}_S - \mathrm{ad}_S^3\right) X.$$

In the conclusions we outline possible extensions of these results.

PRELIMINARIES

It is well known that the MNLS (1) allows Lax representation. The inverse scattering problem for the corresponding Lax operator can be reduced to a Riemann-Hilbert problem, see [32, 33] for the general case and [11, 12, 14, 17, 22] for the **BD.I**-type MNLS.

The MNLS equation (1) possesses Lax representation $[L, M] = 0$ as follows

$$L\psi(x,t,\lambda) \equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \tag{4}$$

$$M\psi(x,t,\lambda) \equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \tag{5}$$

$$V_1(x,t) = Q(x,t), \qquad V_0(x,t) = i\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right]. \tag{6}$$

where

$$Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{q}^* & 0 & s_0\vec{q} \\ 0 & \vec{q}^\dagger s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1, 0, \ldots 0, -1). \tag{7}$$

Below we use the following definition of orthogonality: $X \in so(2r+1)$ if $X + S_0 X^T S_0 = 0$ where

$$S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (E_{kn})_{ij} = \delta_{ik}\delta_{nj} \tag{8}$$

The soliton solutions can be derived by an appropriate modification [17] of the Zakharov-Shabat dressing method [34]. Skipping the details we provide the dressing factor for the N-soliton solution:

$$u(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{A_k(x,t)}{\lambda - \lambda_k^+} + \frac{B_k(x,t)}{\lambda - \lambda_k^-} \right). \tag{9}$$

The Lax operator L has a $\mathbb{Z}_2$-symmetry due to the fact that $Q(x,t) = Q^\dagger(x,t)$. One of the consequences of this symmetry is that the poles of the dressing factors must satisfy $\lambda_k^- = (\lambda_k^-)^*$; we also assume that $\lambda_k = \mu_k + i\nu_k$ is located in $\mathbb{C}_+$ – the upper half of the complex λ-plane, i.e. $\nu_k > 0$.

The residues of u admit the following decomposition

$$A_k(x,t) = X_k(x,t) F_k^T(x,t), \qquad B_k(x,t) = Y_k(x,t) G_k^T(x,t),$$

where all matrices involved for simplicity are supposed to be of rank 1 [33, 15]. For the pure solitonic case the factors F_k and G_k can be expressed by the trivial fundamental solutions $\chi_0^\pm(x,t,\lambda) = e^{-i\lambda(x+\lambda t)J}$, corresponding to vanishing potential of L, as follows

$$F_k^T(x,t) = F_{k,0}^T [\chi_0^+(x,t,\lambda_k^+)]^{-1}, \qquad G_k^T(x,t) = G_{k,0}^T [\chi_0^-(x,t,\lambda_k^-)]^{-1}.$$

The constant vectors $2r+1$-component $F_{k,0}$ and $G_{k,0}$ obey the algebraic relations

$$F_{k,0}^T S_0 F_{k,0} = 0, \qquad G_{k,0}^T S_0 G_{k,0} = 0.$$

The other two types of vectors $X_k(x,t)$ and $Y_k(x,t)$ are solutions to the algebraic system

$$\begin{aligned} S_0 F_k &= \sum_{l \neq k} \frac{X_l F_l^T S_0 F_k}{\lambda_l^+ - \lambda_k^+} + \sum_l \frac{Y_l G_l^T S_0 F_k}{\lambda_l^- - \lambda_k^+}, \\ S_0 G_k &= \sum_l \frac{X_l F_l^T S_0 G_k}{\lambda_l^+ - \lambda_k^-} + \sum_{l \neq k} \frac{Y_l G_l^T S_0 G_k}{\lambda_l^- - \lambda_k^-}. \end{aligned} \tag{10}$$

The corresponding N-soliton solution can be recovered from $u(x,t,\lambda)$ using the relation

$$\begin{aligned} Q_{\mathrm{Ns}} &= \lim_{\lambda \to \infty} \lambda (J - uJu^{-1}(x,t,\lambda)) \\ &= \left[J, \sum_{k=1}^{N} A_k + B_k \right]. \end{aligned} \tag{11}$$

We also introduce the following more convenient parametrization for F_k and G_k:

$$\begin{aligned} F_k(x,t) = S_0|n_k(x,t)\rangle = \left(\begin{array}{c} e^{-z_k+i\phi_k} \\ -\sqrt{2}s_0\vec{v}_{0k} \\ e^{z_k-i\phi_k} \end{array}\right), \\ G_k(x,t) = |n_k^*(x,t)\rangle = \left(\begin{array}{c} e^{z_k+i\phi_k} \\ \sqrt{2}\vec{v}_{0k}^{\,*} \\ e^{-z_k-i\phi_k} \end{array}\right), \end{aligned} \qquad (12)$$

where $\vec{v}_{0k}$ are constant $2r-1$-component polarization vectors and

$$\begin{aligned} z_j = \nu_j(x+2\mu_j t) + \xi_{0j}, \qquad \phi_j = \mu_j x + (\mu_j^2 - \nu_j^2)t + \delta_{0j}, \\ \langle n_j^T(x,t)|S_0|n_j(x,t)\rangle = 0, \qquad \text{or} \qquad (\vec{v}_{0,j}s_0\vec{v}_{0,j}) = 1. \end{aligned} \qquad (13)$$

For $|X_k\rangle$ and $|Y_k\rangle$ one can derive a set of algebraic equations (see [11, 33]) which can be easily solved. In particular for $N=2$ we get:

$$\begin{aligned} |X_1\rangle &= \frac{1}{Z}\left(\frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_2\rangle - \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}S_0|n_1^*\rangle + \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}S_0|n_2^*\rangle\right), \\ |X_2\rangle &= \frac{1}{Z}\left(-\frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_1\rangle + \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}S_0|n_1^*\rangle - \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}S_0|n_2^*\rangle\right), \\ |Y_1\rangle &= \frac{1}{Z}\left(\frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}|n_1\rangle - \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}|n_2\rangle - \frac{f_{12}}{\lambda_1^+ - \lambda_2^+}S_0|n_2^*\rangle\right), \\ |Y_2\rangle &= \frac{1}{Z}\left(-\frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}|n_1\rangle + \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}|n_2\rangle + \frac{f_{12}}{\lambda_2^+ - \lambda_1^+}S_0|n_1^*\rangle\right), \end{aligned} \qquad (14)$$

where

$$\begin{aligned} Z(x,t) = \left(\frac{|f_{12}|^2}{|\lambda_2^+ - \lambda_1^+|^2} - \frac{\kappa_{12}\kappa_{21}}{|\lambda_2^+ - \lambda_1^-|^2} + \frac{\kappa_{11}\kappa_{22}}{4\nu_1\nu_2}\right), \\ \kappa_{ij}(x,t) = \langle n_i^\dagger|n_j\rangle, \qquad f_{ij}(x,t) = f_{ji}(x,t) = \langle n_i|S_0|n_j\rangle. \end{aligned} \qquad (15)$$

Inserting this result into eq. (11) we obtain the following expression for the 2-soliton solution of the MNLS [11]:

$$\begin{aligned} Q_{2s}(x,t) &= [J, A_1 + A_2 + B_1 + B_2] = \frac{1}{Z}[J, C(x,t) - S_0C^T(x,t)S_0], \\ C(x,t) &= \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}|n_1\rangle\langle n_1^\dagger| - \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}|n_1\rangle\langle n_2^\dagger| - \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}|n_2\rangle\langle n_1^\dagger| \\ &\quad + \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}|n_2\rangle\langle n_2^\dagger| - \frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_1\rangle\langle n_2|S_0 - \frac{f_{12}}{\lambda_1^+ - \lambda_2^+}S_0|n_2^*\rangle\langle n_1^\dagger|. \end{aligned} \qquad (16)$$

Similarly one can derive the N-soliton solutions.

Next we can calculate the asymptotics of the 2-soliton solution [11] along the trajectory of the first soliton. To this end we keep $z_1(x,t)$ fixed and let $\tau = z_2 - z_1$ tend to $\pm\infty$. This is possible if $\mu_1 \neq \mu_2$, i.e the two solitons have different velocities. For definiteness we assume that $\mu_2 > \mu_1$.

Therefore it will be enough to insert the asymptotic values of the matrix elements of $\mathcal{M}$ for $\tau \to \pm\infty$ and keep only the leading terms:

$$\begin{aligned} \kappa_{22} &\simeq e^{\pm 2\tau} \exp(\pm \nu_2 z_1/\nu_1) + 2\mathcal{C}_1, \\ \kappa_{12} &= e^{\pm\tau} \exp(\pm(1+\nu_2/\nu_1)z_1 \pm i(\phi_1 - \phi_2)) + \mathcal{O}(1), \\ \kappa_{21} &= e^{\pm\tau} \exp(\pm(1+\nu_2/\nu_1)z_1 \mp i(\phi_1 - \phi_2)) + \mathcal{O}(1), \\ f_{12} &= e^{\pm\tau} \exp(\mp(1-\nu_2/\nu_1)z_1 \pm i(\phi_1 - \phi_2)) + \mathcal{O}(1), \end{aligned} \tag{17}$$

After somewhat lengthy calculations we get:

$$\begin{aligned} \lim_{\tau\to\infty} \vec{q}_{2\mathrm{s}}(x,t;z_1,z_2;\phi_1,\phi_2) &= \vec{q}_{1\mathrm{s}}(x,t;z_1+r_+,\phi_1-\alpha_+), \\ \lim_{\tau\to-\infty} \vec{q}_{2\mathrm{s}}(x,t;z_1,z_2;\phi_1,\phi_2) &= \vec{q}_{1\mathrm{s}}(x,t;z_1-r_+,\phi_1+\alpha_+), \end{aligned} \tag{18}$$

where $\vec{q}_{1\mathrm{s}}$ is the one-soliton solution

$$\vec{q}_{1\mathrm{s}}(x,t;z_1,\phi_1) = -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1)} \left(e^{-z_1} s_0 |\vec{v}_{01}\rangle + e^{z_1} |\vec{v}_{01}^*\rangle\right)}{\cosh(2z_1) + (\vec{v}_{01}^\dagger, \vec{v}_{01})}, \tag{19}$$

and the shifts of its arguments

$$r_+ = \ln\left|\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}\right|, \qquad \alpha_+ = \arg\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}. \tag{20}$$

are expressed in terms of the discrete eigenvalues $\lambda_j^\pm$ only.

THE TWO SOLITON INTERACTIONS REVISITED

We will re-derive the above results by using an alternative version of the dressing method [32, 33, 34]. Here we will apply another version of the dressing method, namely we will do the dressing in two steps, each time adding just one pair of eigenvalues $\lambda_j^\pm$ to the discrete spectrum of L. Obviously the two-soliton solution can be obtained in two different ways:

$$u_{2\mathrm{s;A}} = u_{2,1}(x,t,\lambda)u_1(x,t,\lambda), \qquad u_{2\mathrm{s;B}} = u_{1,2}(x,t,\lambda)u_2(x,t,\lambda), \tag{21}$$

where

$$\begin{aligned} u_j(x,t,\lambda) &= \mathbb{1} + (c_j(\lambda)-1)P_j + (c_j^{-1}(\lambda)-1)\bar{P}_j, & c_j(\lambda) &= \frac{\lambda-\lambda_j^+}{\lambda-\lambda_j^-}, \\ P_j(x,t) &= \frac{|n_j\rangle\langle n_j^\dagger|}{\langle n_j^\dagger|n_j\rangle}, & \bar{P}_j &= S_0 P_j^T S_0, \end{aligned} \tag{22}$$

and

$$u_{2,1}(x,t,\lambda) = \mathbb{1} + (c_2(\lambda)-1)\mathbf{P}_{2,1} + (c_2^{-1}(\lambda)-1)\bar{\mathbf{P}}_{2,1},$$
$$u_{1,2}(x,t,\lambda) = \mathbb{1} + (c_1(\lambda)-1)\mathbf{P}_{1,2} + (c_1^{-1}(\lambda)-1)\bar{\mathbf{P}}_{1,2}, \qquad (23)$$

Here

$$P_{2,1}(x,t) = \frac{|\mathbf{n}_2\rangle\langle\mathbf{n}_2^\dagger|}{\langle\mathbf{n}_2^\dagger|\mathbf{n}_2\rangle}, \qquad P_{1,2}(x,t) = \frac{|\mathbf{n}_1\rangle\langle\mathbf{n}_1^\dagger|}{\langle\mathbf{n}_1^\dagger|\mathbf{n}_1\rangle},$$
$$|\mathbf{n}_1\rangle = u_2(x,t,\lambda_1^+)|n_1\rangle, \qquad |\mathbf{n}_2\rangle = u_1(x,t,\lambda_2^+)|n_2\rangle, \qquad (24)$$

where

$$|\mathbf{n}_1\rangle = |n_1\rangle + (c_2(\lambda_1^+)-1)\frac{\kappa_{21}}{\kappa_{22}}|n_2\rangle + (c_2^{-1}(\lambda_1^+)-1)\frac{f_{12}}{\kappa_{22}}S_0|n_2^*\rangle,$$
$$|\mathbf{n}_2\rangle = |n_2\rangle + (c_1(\lambda_2^+)-1)\frac{\kappa_{12}}{\kappa_{11}}|n_1\rangle + (c_1^{-1}(\lambda_2^+)-1)\frac{f_{12}}{\kappa_{11}}S_0|n_1^*\rangle, \qquad (25)$$
$$\langle\mathbf{n}_2^\dagger|\mathbf{n}_2\rangle = \frac{4\nu_1\nu_2}{\kappa_{11}}Z, \qquad \langle\mathbf{n}_1^\dagger|\mathbf{n}_1\rangle = \frac{4\nu_1\nu_2}{\kappa_{22}}Z.$$

The limits for $\tau \to \pm\infty$ are given by:

$$u_j(x,t,\lambda) = \exp(\ln c_j(\lambda)(P_j(x,t) - \bar{P}_j(x,t)),$$
$$C_j(\lambda) = \lim_{x\to\infty} u_j(x,t,\lambda) = \begin{pmatrix} c_j(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & c_j^{-1}(\lambda) \end{pmatrix}, \qquad (26)$$
$$\lim_{x\to-\infty} u_j(x,t,\lambda) = C_j^{-1}(\lambda),$$

After all these calculations one is able to show that in fact $u_{2\mathrm{s;A}} = u_{2\mathrm{s;B}}$, i.e. the result of dressing is independent on the order in which one adds up the pairs of eigenvalues to the discrete spectrum of L. The corresponding two-soliton solution is given by:

$$\begin{aligned} Q_{2\mathrm{s}}(x,t) &= \lim_{\lambda\to\infty} \lambda\,(J - u_{2\mathrm{s;B}} J u_{2\mathrm{s;B}}(x,t,\lambda)) \\ &= (\lambda_1^- - \lambda_1^+)[J, P_1 - \bar{P}_1] + (\lambda_1^- - \lambda_1^+)[J, \mathbf{P}_2 - \bar{\mathbf{P}}_2] \\ &= (\lambda_1^- - \lambda_1^+)[J, \mathbf{P}_1 - \bar{\mathbf{P}}_1] + (\lambda_1^- - \lambda_1^+)[J, P_2 - \bar{P}_2]. \end{aligned} \qquad (27)$$

These two results are behind the nonlinear superposition principle for the Bäcklund transformations [2, 19].

Let us now calculate the limits of $u_{2\mathrm{s;B}}(x,t,\lambda)$ for z_1 fixed and t tending to $+\infty$ and $-\infty$. In this way we will find out what is the effect of the second soliton on the asymptotic behavior of the first one. One can easily check that:

$$z_2 = \frac{\nu_2}{\nu_1}z_1 + 2\nu_2(\mu_2 - \mu_1)t + \frac{\xi_{01}\nu_1 - \xi_{02}\nu_2}{\nu_1}. \qquad (28)$$

Therefore, since $\mu_2 > \mu_1$ and $\nu_2 > 0$, z_2 and t tend simultaneously to $+\infty$ (resp., to $-\infty$). To be more explicit we slightly change the notation for the one-soliton dressing factor and write it down showing explicitly the relevant soliton parameters:

$$u_j(x,t,\lambda) = u_{1s}(\lambda; z_j, \phi_j, \vec{v}_{0j}, \lambda_j^+). \tag{29}$$

In this configuration the second soliton moves with velocity μ_2 and for $t \to -\infty$ (resp. for $t \to -\infty$) is behind (resp. outstands) the slower first one. The corresponding asymptotic values for the dressing factors are:

$$\begin{aligned} \lim_{\tau\to-\infty} u_{2s;B} &= u_{1s}(\lambda; z_1 - r_+, \phi_1 + \alpha_+, \vec{v}_{01}, \lambda_1^+) C_2^{-1}(\lambda), \\ \lim_{\tau\to\infty} u_{2s;B} &= u_{1s}(\lambda; z_1 + r_+, \phi_1 - \alpha_+, \vec{v}_{01}, \lambda_1^+) C_2(\lambda), \end{aligned} \tag{30}$$

where r_+ and α_+ are given by eq. (20). Inserting these limits into eq. (27) and taking into account that $C_2(\lambda)$ commutes with J we get:

$$\begin{aligned} \lim_{\tau\to-\infty} Q_{2s} &= \lim_{\tau\to-\infty} \lim_{\lambda\to\infty} \lambda \left(J - u_{2s;B} J u_{2s;B}(x,t,\lambda)\right) \\ &= Q_{1s}(z_1 - r_+, \phi_1 + \alpha_+; \vec{v}_{01}) \\ \lim_{\tau\to\infty} Q_{2s} &= Q_{1s}(z_1 + r_+, \phi_1 - \alpha_+; \vec{v}_{01}). \end{aligned} \tag{31}$$

Thus we have rederived the eqs. (18) and established that soliton interactions for the **BD.I**-type MNLS are very much like the the ones for the scalar NLS. The effect of the interaction is just shift of the relative center of mass and relative phase. The polarization vector $\vec{v}_{01}$ does not change its direction.

Obviously, we can repeat the calculation using $u_{2s;A}(x,t,\lambda)$ and considering the limit for fixed z_2. In this way we establish that the second soliton experiences opposite shifts in the relative center of mass and relative phase.

THE GAUGE EQUIVALENCE

Here we start with a brief description of the Zakharov-Shabat system in pole gauge [33]

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv i\frac{d\tilde{\psi}}{dx} - \lambda S(x,t)\tilde{\psi}(x,t,\lambda) = 0, \tag{32}$$

where

$$S(x,t) = \mathrm{Ad}_g \cdot J = g^{-1} J g(x,t), \qquad J = \mathrm{diag}\,(1,0,\dots,0,-1), \tag{33}$$

i.e. $S^3 = S$ and the gauge group elements $g(x,t) = \psi(x,t,\lambda = 0)$ and satisfy the equation:

$$\left(i\frac{d}{dx} + Q(x,t)\right) g(x,t) = 0.$$

The second Lax operator takes the form:

$$\tilde{M}\tilde{\psi}(x,t,\lambda) \equiv i\frac{d\tilde{\psi}}{dt} - \left(i\lambda\, \mathrm{ad}_S^{-1} S_x + \lambda^2 S\right)\tilde{\psi}(x,t,\lambda) = 0. \tag{34}$$

Thus the compatibility condition $[\tilde{L}(\lambda), \tilde{M}(\lambda)] = 0$ gives the multicomponent HF type (3) related to **BD.I**-type symmetric spaces.

The direct scattering problem for the Lax operator (32) is based on the Jost solutions and the scattering matrix $T(\lambda)$:

$$\lim_{x\to\infty} \tilde{\psi}(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \tilde{\phi}(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \quad \tilde{T}(\lambda) = (\tilde{\psi}(x,\lambda))^{-1}\tilde{\phi}(x,\lambda). \tag{35}$$

The fundamental analytic solutions(FAS) $\tilde{\chi}^{\pm}(x,\lambda)$ of $\tilde{L}(\lambda)$ are related to the Jost solutions by [32, 18]

$$\tilde{\chi}^{\pm}(x,\lambda) = \tilde{\phi}(x,\lambda)\tilde{S}^{\pm}(\lambda) = \tilde{\psi}(x,\lambda)\tilde{T}^{\mp}(\lambda)\tilde{D}^{\pm}(\lambda), \tag{36}$$

where $\tilde{T}^{\pm}(\lambda)$ and $\tilde{S}^{\pm}(\lambda)$, $\tilde{D}^{\pm}(\lambda)$ are elements of the corresponding Lie group and are factors in the generalized Gauss decomposition of the scattering matrix: $\tilde{T}(\lambda) = \tilde{T}^{-}(\lambda)\tilde{D}^{+}(\lambda)\hat{\tilde{S}}^{+}(\lambda) = \tilde{T}^{+}(\lambda)\tilde{D}^{-}(\lambda)\hat{\tilde{S}}^{-}(\lambda)$. Here the superscript "+" (resp "−") stays for denoting upper-triangular (resp. lower-triangular) matrices for the Gauss factors $\tilde{S}^{\pm}(\lambda)$ and $\tilde{T}^{\pm}(\lambda)$ while the matrix elements of the block-diagonal matrices $\tilde{D}^{\pm}(\lambda)$ are analytic functions of λ for $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ respectively.

On the real axis $\tilde{\chi}^{+}(x,\lambda)$ and $\tilde{\chi}^{-}(x,\lambda)$ are related by $\tilde{\chi}^{+}(x,\lambda) = \tilde{\chi}^{-}(x,\lambda)\tilde{G}_0(\lambda)$, $\tilde{G}_0(\lambda) = \hat{\tilde{S}}^{-}(\lambda)\tilde{S}^{+}(\lambda)$, and the function $\tilde{G}_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of (32) [32, 18].

The FAS $\tilde{\psi}$ for the MNLS systems on symmetric spaces of **BD**.**I**-type are related to the FAS ψ for the corresponding gauge equivalent MHF systems as follows:

$$\tilde{\psi}(x,\lambda) = g(x)\psi(x,\lambda)g_{+}^{-1}, \qquad \tilde{\phi}(x,\lambda) = g(x)\psi(x,\lambda)g_{-}^{-1}, \tag{37}$$

where

$$g(x) = \phi(x,\lambda=0), \qquad g_{\pm} = \lim_{x\to\pm\infty} g(x)$$

We request that $g_{\pm}$ be diagonal matrices. Then, for the corresponding set of scattering data for the MHF systems and their gauge equivalent MNLS ones we get:

$$\begin{aligned} \tilde{T}(\lambda) &= g_{+}T(\lambda)g_{-}^{-1}, \qquad & \tilde{T}^{\pm}(\lambda) &= g_{+}T^{\pm}(\lambda)g_{+}^{-1}, \\ \tilde{S}^{\pm}(\lambda) &= g_{-}S^{\pm}(\lambda)g_{-}^{-1}, \qquad & \tilde{D}^{\pm}(\lambda) &= g_{+}D^{\pm}(\lambda)g_{-}^{-1}. \end{aligned} \tag{38}$$

Similar formulas hold for the renormalised FAS:

$$\tilde{\chi}^{\pm}(x,\lambda) = g(x)\chi^{\pm}(\lambda)g_{-}^{-1}. \tag{39}$$

The dressing method proposed by Zakharov and Shabat [34] can be naturally extended also to the gauge equivalent systems. It allows one starting from a FAS $\tilde{\xi}^{\pm}_{(0)}(x,\lambda)$ of $\tilde{L}$ with potential $S_{(0)}$ to construct a new singular solution $\tilde{\xi}^{\pm}_{(1)}(x,\lambda)$ with singularities located at prescribed positions $\lambda_1^{\pm}$. Then the new solutions $\tilde{\xi}^{\pm}_{(1)}(x,\lambda)$ will correspond to a potential $S_{(1)}$ of L with two discrete eigenvalues $\lambda_1^{\pm}$. It is related to the regular one by the dressing factors $\tilde{u}(x,\lambda)$:

$$\tilde{\xi}^{\pm}_{(1)}(x,\lambda) = \tilde{u}(x,\lambda)\tilde{\xi}^{\pm}_{(0)}(x,\lambda)\tilde{u}_{-}^{-1}(\lambda), \qquad \tilde{u}_{-}(\lambda) = \lim_{x\to-\infty}\tilde{u}(x,\lambda), \tag{40}$$

The one-soliton gauge equivalent dressing factors $\tilde{u}(x,\lambda)$ are related to those for the 'canonical' gauge $u(x,\lambda)$ by

$$\begin{aligned}\tilde{u}(x,\lambda) &= u^{-1}(x,\lambda=0)u(x,\lambda)g_{(0)}\\ &= \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(0)} - 1\right)P_1 + \left(\frac{c_1(0)}{c_1(\lambda)} - 1\right)\bar{P}_1,\end{aligned} \tag{41}$$

where $P_1(x,t)$ and $\bar{P}_1(x,t)$ are the same rank 1 projectors used above. The dressing factors of the gauge equivalent MHF equations in the pure solitonic case satisfies the equation:

$$i\frac{du}{dx} - \lambda S_{1s}(x,t)\tilde{u}(x,t,\lambda) + \lambda\tilde{u}(x,t,\lambda)J = 0, \tag{42}$$

and as a consequence the projectors $\tilde{P}_{\pm 1}$ satisfy the equations:

$$\begin{aligned}&i\frac{dP_1}{dx} + \lambda_1^- P_1 J - \lambda_1^- S_{1s}(x,t)P_1 = 0,\\ &i\frac{d\bar{P}_1}{dx} + \lambda_1^+ \bar{P}_1 J - \lambda_1^+ S_{1s}(x,t)\bar{P}_1 = 0,\end{aligned} \tag{43}$$

The "dressed" one-soliton potential can be obtained by:

$$S_{1s}(x,t) = J + i\frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+\lambda_1^-}\frac{d}{dx}(P_1(x,t) - \bar{P}_1(x,t)). \tag{44}$$

The dressing factor can be written also in the form:

$$\tilde{u}(x,t,\lambda) = \exp\left[\ln\left(\frac{c_1(\lambda)}{c_1(0)}\right)p(x,t)\right], \tag{45}$$

where $p(x,t) = P_1 - \bar{P}_1 \in \mathfrak{g}$ and consequently $\tilde{u}(x,t,\lambda)$ belongs to the corresponding orthogonal group.

One can construct an N-soliton dressing factor $\tilde{u}_{\rm Ns}(x,t,\lambda)$ in analogy with eq. (9). Then the corresponding N-soliton solution of MHF equation will be given by:

$$S_{\rm Ns}(x,t) = \lim_{\lambda\to 0}\left(\frac{i}{\lambda}\frac{d\tilde{u}_{\rm Ns}}{dx}\tilde{u}_{\rm Ns}^{-1} + \tilde{u}_{\rm Ns}J\tilde{u}_{\rm Ns}^{-1}(x,t,\lambda)\right). \tag{46}$$

TWO-SOLITON INTERACTIONS AND GAUGE EQUIVALENCE

For the pure solitonic case $g(x,t) = u(x,t,\lambda=0)$. Using the explicit expressions for the asymptotics of the two-soliton dressing factors of MNLS it is not difficult to derive the corresponding asymptotics for the two-soliton solutions of the gauge equivalent HF equation.

We first write down the 2-soliton dressing factor for the MHF equation as follows:

$$\begin{aligned}\tilde{u}_{2s;B} &= u_{2s;B}^{-1}(x,t,\lambda=0)u_{2s;B}(x,t,\lambda)\\ &= u_2^{-1}(x,t,\lambda=0)u_{1,2}^{-1}(x,t,\lambda=0)u_{1,2}(x,t,\lambda)u_2(x,t,\lambda)\\ &= u_2^{-1}(x,t,\lambda=0)\left(1\!\!1+\left(\frac{c_1(\lambda)}{c_1(0)}-1\right)\mathbf{P}_{1,2}+\left(\frac{c_1(0)}{c_1(\lambda)}-1\right)\bar{\mathbf{P}}_{1,2}\right)u_2(x,t,\lambda).\end{aligned} \tag{47}$$

Next we make use of eq. (30) and obtain the following result for the large time asymptotics of $\tilde{u}_{2s;B}$ with fixed z_1:

$$\begin{aligned}\lim_{\tau\to-\infty}\tilde{u}_{2s;B} &= C_{20}\tilde{u}_{1s}(\lambda;z_1-r_+,\phi+\alpha_+,\vec{v}_{01},\lambda_1^+)C_2^{-1}(\lambda),\\ \lim_{\tau\to\infty}\tilde{u}_{2s;B} &= C_{20}^{-1}\tilde{u}_{1s}(\lambda;z_1+r_+,\phi-\alpha_+,\vec{v}_{01},\lambda_1^+)C_2(\lambda),\end{aligned} \tag{48}$$

where

$$C_{20}=C_2(\lambda=0)=\operatorname{diag}\left(\frac{\lambda_2^+}{\lambda_2^-},1\!\!1,\frac{\lambda_2^-}{\lambda_2^+}\right), \tag{49}$$

and like in eq. (30) above we use notation with soliton parameters:

$$\begin{aligned}\tilde{u}_{1s}(\lambda;z_1,\phi,\vec{v}_{01},\lambda_1^+) &= u_{1s}^{-1}(x,t,\lambda=0)u_{1s}(x,t,\lambda)\\ &= 1\!\!1+\left(\frac{c_1(\lambda)}{c_1(0)}-1\right)P_1+\left(\frac{c_1(0)}{c_1(\lambda)}-1\right)\bar{P}_1.\end{aligned} \tag{50}$$

Next we insert eq. (48) into eq. (44) and get:

$$\begin{aligned}\lim_{\tau\to-\infty}S_{2s}(x,t) &= C_{20}S_{1s}(z_1-r_+,\phi_1+\alpha_+,\vec{v}_{01})C_{20}^{-1}\\ &= S_{1s}(z_1-r_+,\phi_1+\tilde{\alpha}_+,\vec{v}_{01}),\\ \lim_{\tau\to\infty}S_{2s}(x,t) &= C_{20}^{-1}S_{1s}(z_1+r_+,\phi_1-\alpha_+,\vec{v}_{01})C_{20}\\ &= S_{1s}(z_1+r_+,\phi_1-\tilde{\alpha}_+,\vec{v}_{01}),\end{aligned} \tag{51}$$

where

$$\tilde{\alpha}_+=\arg\left(\frac{\lambda_1^+-\lambda_2^+}{\lambda_1^+-\lambda_2^-}\frac{\lambda_2^-}{\lambda_2^+}\right)=\alpha_+-2\arg\lambda_2^+. \tag{52}$$

Thus we have shown that the gauge transformation affects the soliton interaction by changing the relative phase shift. The relative center of mass shift is gauge independent.

CONCLUSIONS

Using the explicit form of the N soliton solution for the scalar NLS they calculated explicitly their asymptotics along the soliton trajectories in the generic case, when the solitons move with different velocities. The important result consists in the following: i) the N-soliton interactions are purely elastic and always split into sequences of elementary 2-soliton interactions; ii) the effect of each 2-soliton interaction consists in shifts of the relative center of mass and relative phases of each of the solitons; iii) there are no non-trivial 3-soliton interactions. Finally, the soliton interaction for the gauge equivalent HF equations has the same character. The only effect of the gauge transformation is to modify the phase shifts of the solitons.

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REFERENCES

1. M. Ablowitz, J., Prinari B. and Trubatch A. D., "Discrete and continuous nonlinear Schrödinger systems" Cambridge Univ. Press, Cambridge, (2004).
2. E. V. Doktorov, S. B. Leble. A dressing method in mathematical physics. Mathematical physics study **28**. Springer Verlag, Berlin (2007).
3. L. D. Faddeev and L. A. Takhtadjan, "Hamiltonian Approach in the Theory of Solitons", Springer Verlag, Berlin, (1987).
4. A. P. Fordy, and P. P. Kulish. Nonlinear Schrödinger equations and simple Lie algebras. *Commun. Math. Phys.* **89**, 427–443 (1983).
5. V. S. Gerdjikov. On the spectral theory of the integro–differential operator Λ, generating nonlinear evolution equations. *Lett. Math. Phys.* **6,** n. 6, 315–324, (1982).
6. V. S. Gerdjikov. The Zakharov–Shabat dressing method and the representation theory of the semisimple Lie algebras. *Phys. Lett. A,* **126A,** n. 3, 184–188, (1987).
7. V. S. Gerdjikov. Generalised Fourier transforms for the soliton equations. Gauge covariant formulation. *Inverse Problems* **2**, no. 1, 51–74, (1986).
8. V. S. Gerdjikov. The Generalized Zakharov–Shabat System and the Soliton Perturbations. *Theor. Math. Phys.* **99**, No. 2, 292–299 (1994).
9. V. S. Gerdjikov. Algebraic and Analytic Aspects of N-wave Type Equations. **nlin.SI/0206014**; *Contemporary Mathematics* **301**, 35-68 (2002).
10. V. S. Gerdjikov. *Basic Aspects of Soliton Theory.* Eds.: I. M. Mladenov, A. C. Hirshfeld. "Geometry, Integrability and Quantization", pp. 78-125; Softex, Sofia 2005. **nlin.SI/0604004**
11. V. S. Gerdjikov. *Bose-Einstein Condensates and spectral properties of multicomponent nonlinear Schrödinger equations.* Discrete and Continuous Dynamical Systems B (In press) **arXiv: 1001.0164 [nlin.SI]**
12. V. S. Gerdjikov, D. J. Kaup, N. A. Kostov, T. I. Valchev. *On classification of soliton solutions of multicomponent nonlinear evolution equations. J. Phys. A: Math. Theor.* **41** 315213 (2008) (36pp).
13. V. S. Gerdjikov, G. G. Grahovski, *On N-wave and NLS type systems: generating operators and the gauge group action: the* $so(5)$ *case,* Proc. of IM of NAS of Ukraine **50**, 388–395 (2004).
14. V. S. Gerdjikov, G. G. Grahovski, *On the multi-component NLS type systems and their gauge equivalent: Examples and reductions*, In "Global Analysis and Applied Mathematics", Eds: K. Taş, D. Krupka, O. Krupkova, D. Baleanu, AIP Conference Proceedings **729** (2004), pp.162-169.
15. V. S. Gerdjikov, G. G. Grahovski and N. A. Kostov, *"On the multi-component NLS type equations on symmetric spaces and their reductions",* Theor. Math. Phys. **144**, (2005), 1147–1156.

16. V. S. Gerdjikov, G. G. Grahovski, R. I. Ivanov, N. A. Kostov, *N-wave interactions related to simple Lie algebras.* $\mathbb{Z}_2$*- reductions and soliton solutions* Inverse Problems **17**, 999-1015 (2001).
17. V. S. Gerdjikov, N. A. Kostov, T. I. Valchev. Solutions of multi-component NLS models and Spinor Bose-Einstein condensates *Physica D* **238**, 1306-1310 (2009); **ArXiv:0802.4398 [nlin.SI]**.
18. V. S. Gerdjikov, P. P. Kulish, *The generating operator for the* $n \times n$ *linear system*, Physica D, **3**, 549–564, (1981);
V. S. Gerdjikov, *Generalized Fourier transforms for the soliton equations. Gauge covariant formulation*, Inverse Problems **2**, 51, (1986).
19. V. S. Gerdjikov, G. Vilasi, A. B. Yanovski. *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods* Lecture Notes in Physics **748**, Springer Verlag, Berlin, Heidelberg, New York (2008). ISBN: 978-3-540-77054-1.
20. Helgasson S. *Differential geometry, Lie groups and symmetric spaces*, Academic Press, (1978).
21. J. Ieda, T. Miyakawa, and M. Wadati. Exact analysis of soliton dymamics in spinor Bose-Einstein condesates. *Phys. Rev Lett.* **93**, 194102 (2004).
22. R. Ivanov. On the dressing method for the generalised ZakharovŰShabat system. *Nuclear Physics B* **694 [PM]** 509Ű524 (2004).
23. T. Kanna and M. Lakshmanan, *"Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons",* Phys. Rev. E **67**, (2003), 046617.
24. Nikolay Kostov, Vladimir Gerdjikov. Reductions of multicomponent mKdV equations on symmetric spaces of **DIII**-type. *SIGMA* **4** (2008), paper 029, 30 pages; **ArXiv:0803.1651**.
25. P. P. Kulish, E. K. Sklyanin. $O(N)$-invariant nonlinear Schrodinger equation - a new completely integrable system. *Phys. Lett.* **84A**, 349-352 (1981).
26. S. V. Manakov, *"On the theory of two-dimensional stationary self-focusing of electromagnetic waves"*, Zh. Eksp. Teor. Fiz [Sov.Phys. JETP], **65** [**38**], 505–516, (1973) [(1974)], [248–253].
27. Mikhailov A V. The Reduction Problem and the Inverse Scattering Problem. *Physica D*, **3D**, no. 1/2, 73–117 (1981).
28. H. E. Nistazakis, D.J. Frantzeskakis, P.G. Kevrekidis, B.A. Malomed, and R. Carretero-Gonzt'alez. Bright-Dark Soliton Complexes in Spinor Bose-Einstein Condensates. *Phys. Rev. A* **77**, 033612 (2008).
29. A. B. Shabat. Inverse-scattering problem for a system of differential equations. *Functional Analysis and Its Applications*, **9**, 244–247 (1975).
— An inverse scattering problem. *Diff. Equations*, **15** 1299–1307 (1979).
30. S I Svinolupov. Second-order evolution equations with symmetries. Russian Mathematical Surveys **40**, 241-242 (1985).
S. I. Svinolupov and V. V. Sokolov. Vector-matrix generalizations of classical integrable equations *Theor. Math. Phys.* **100**, 214-218, (1994).
31. T. Tsuchida, *"N-soliton collision in the Manakov model",* Prog. Theor. Phys. **111** (2004), 151-182.
32. Zakharov V E., Manakov S V., Novikov S P., Pitaevskii L I. *Theory of solitons. The inverse scattering method*, Plenum, N.Y. (1984).
33. V. E. Zakharov, and A. V. Mikhailov. On The Integrability of Classical Spinor Models in Two-dimensional Space-time *Comm. Math. Phys.* **74**, 21–40 (1980).
34. V. E. Zakharov, A. B. Shabat. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I. *Functional Analysis and Its Applications*, **8**, 226–235 (1974).
V. E. Zakharov, A. B. Shabat. Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. *Functional Analysis and Its Applications*, **13**, 166–174 (1979).
35. V. E. Zakharov and A. B. Shabat, *"Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media",* Soviet Physics-JETP, **34**, (1972), 62–69.