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Brendan Goldsmith

Technological University Dublin, [brendan.goldsmith@tudublin.ie](mailto:brendan.goldsmith@tudublin.ie)

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## A topological approach to a problem of Nunke

By

B. GOLDSMITH

**1. Introduction.** If  $P$  denotes the Specker group i.e. the direct product of countably many infinite cyclic groups, then the following question was raised by Nunke [6]: "If  $P$  has a subgroup  $A$  such that  $P/A \cong \mathbb{Q}$ , the additive group of rationals, is  $A \cong P$ ?"

The question has been answered negatively by Meijer [4]. However the problem arose from essentially homological work of Nunke and Meijer's solution was also homological. We present here a solution to the problem which uses a topological idea introduced by Lady [3] and avoids all homological machinery.

The word group will be understood to refer to an additively written abelian group and we refer to the standard works [1] and [2] of Fuchs for notation and terminology.

**2. The Main Result.** A Specker group  $P$  may be topologized by using the product topology obtained by considering each component as being discretely topologized. We refer to this topology on  $P$  simply as the product topology.  $P$  may also be topologized by choosing as a basis of neighbourhoods of 0 the subgroups  $nP$  ( $n \in \mathbb{Z}$ ,  $n \neq 0$ ). This is the familiar  $\mathbb{Z}$ -adic topology (see [1] page 30). We will make use of a topology, which, following Lady [3], we call the strong topology. This is the topology on  $P$  which is the supremum of the  $\mathbb{Z}$ -adic and product topologies on  $P$ .

The result we require can be stated as

**Theorem 1** (Meijer). *Let  $G$  be a torsion-free group of at most countable rank and let  $X$  be an extension of a Specker group  $P$  by  $G$ . Then  $X \cong P$  if and only if  $G$  is free of finite rank.*

We use the following results:

**Lemma 2.** *If  $Y \leq P$  and  $Y \cong P$  then there are integers  $k_n$  and elements  $a_n$  of  $P$  ( $n = 1, 2, 3, \dots$ ) such that*

$$P = \prod_{n=1}^{\infty} \langle a_n \rangle \quad \text{and} \quad Y = \prod_{n=1}^{\infty} \langle k_n a_n \rangle$$

Proof. Suppose  $Y = \prod_{n=1}^{\infty} \langle y_n \rangle$ . Define elements  $w_n$  of  $P$  by

$$\begin{aligned} w_1 &= (y_1, 0, 0, \dots), \\ w_2 &= (0, y_2, y_3, y_4, \dots), \\ w_3 &= (0, 0, y_3, y_4, \dots), \\ &\vdots \\ w_n &= (0, 0, \dots, 0, y_n, y_{n+1}, \dots) \quad \text{etc.} \end{aligned}$$

Since, for every positive integer  $j$ , the  $j$ 'th components of almost all the  $w_n$  vanish, the infinite sum  $\sum_{n=1}^{\infty} s_n w_n$  ( $s_n$  integers) makes sense and so the subgroup  $W$  consisting of all such infinite sums is a product in the sense defined by Fuchs [2] § 95.

Claim  $W = Y$ . For if  $w \in W$  then  $w = \sum_{n=1}^{\infty} s_n w_n$  ( $s_n \in \mathbb{Z}$ ) and so

$$w = (s_1 y_1, s_2 y_2, (s_2 + s_3) y_3, \dots, (s_2 + s_3 + \dots + s_n) y_n, \dots) \in Y.$$

Thus  $W \leq Y$ . While conversely if  $y \in Y$  then  $y = (t_1 y_1, t_2 y_2, \dots, t_n y_n, \dots)$  for integers  $t_n$ , and so  $y = \sum_{n=1}^{\infty} s_n w_n$  where the  $s_n$  are given by

$$s_1 = t_1, \quad s_2 = t_2, \quad s_3 = t_3 - s_2, \quad s_4 = t_4 - s_3, \dots \quad \text{etc.}$$

Thus  $y \in W$  and so  $W = Y$  as claimed.

Thus  $Y$  is a product in the sense of Fuchs [2] § 95 and the result follows from [2] Lemma 95.1.

**Proposition 3** (Nunke). *Every epimorphic image of a Specker group  $P$  is the direct sum of a cotorsion group and a direct product of at most countably many infinite cyclic groups.*

Proof. See Nunke [5] or Fuchs [2] 95.2.

Proof of Theorem 1. Suppose  $X$  is isomorphic to  $P$ , then applying Proposition 3 to  $G$  where  $X/P \cong G$ , we see that  $G$  is a direct sum of a cotorsion group and a product of infinite cyclic groups. Since  $G$  has at most countable rank it is clear that the product of infinite cyclic groups must be a finite product. Moreover since a reduced torsion-free cotorsion group must contain a summand isomorphic to the  $p$ -adic integers, for some  $p$ , it is clear that the cotorsion summand of  $G$  must be divisible. Thus  $G = D \oplus F$  where  $D$  is divisible and  $F$  is a finite product of infinite cyclic groups i.e.  $F$  is free of finite rank. It remains to show that  $D = 0$ .

Choose a subgroup  $Y$  of  $X$  such that  $Y/P \cong F$ . Clearly  $Y \cong P$  and  $X/Y \cong D$ . Now applying Lemma 2 we may write

$$X = \prod_{n=1}^{\infty} \langle a_n \rangle \quad \text{and} \quad Y = \prod_{n=1}^{\infty} \langle k_n a_n \rangle.$$

We topologize  $X$  with the strong topology. (We remark that we are only concerned with this presentation of  $X$  and we don't need to show that the product topology on  $X$  is independent of the presentation although this is indeed true.) The proof is completed by the following result:

**Lemma 4.**  $Y$  is closed and dense in  $X$  in the strong topology on  $X$ .

*Proof.* (i)  $Y$  is dense. If  $x$  is in  $X$  then a basis for the neighbourhoods of  $x$  in the strong topology is  $x + m \prod_{n \notin J} \langle a_n \rangle$ , where  $J$  is a finite subset of  $\{1, 2, \dots\}$  and  $m \in \mathbb{Z} \setminus \{0\}$ . To show density it suffices to show that for any finite subset  $J$  and any integer  $m \neq 0$ ,

$$Y \cap (x + m \prod_{n \notin J} \langle a_n \rangle) \text{ is non-null.}$$

Since  $Y$  is dense in  $X$  in the  $\mathbb{Z}$ -adic topology, we may write for any non-zero integer  $m$ ,  $x = mx' + y$ , some  $x' \in X$ ,  $y \in Y$ . Say  $x' = (\dots, r_i a_i, \dots)$ . Then set  $x' = u + v$  where  $u = (\dots, r_i a_i, \dots)_{i \notin J}$ , and  $v = (\dots, r_i a_i, \dots)_{i \in J}$ . Then  $x - mu = mv + y$ . But  $Y$  is pure in  $X$  and  $k_n a_n \in Y \cap k_n X$ , so  $a_n \in Y$  each  $n$ . Since  $v$  is just a finite sum of multiples of elements of  $Y$ ,  $v$  is in  $Y$ . Thus  $x - mu \in Y$  i.e.

$$Y \cap (x + m \prod_{n \notin J} \langle a_n \rangle) \text{ is non-null.}$$

(ii)  $Y$  is closed in  $X$ . Since the strong topology is the supremum of the product and  $\mathbb{Z}$ -adic topologies, it is clearly sufficient to show  $Y$  is closed in the product topology on  $X$ . If  $\bar{Y}$  denotes the closure of  $Y$  in the product topology on  $X$  then we see that  $x \in \bar{Y}$  if, and only if, for each  $n = (1, 2, \dots)$  there is a  $u^n$  in  $Y$  such that  $x_i = u_i^n$  for all  $i < n$ . (Here the subscript  $i$  denotes the  $i$ -th component.)

So if  $x \in \bar{Y}$  then there is a  $u^2$  in  $Y$  such that

$$x_1 = u_1^2 = r_1 k_1 a_1 \quad \text{some } r_1 \in \mathbb{Z}.$$

Similarly  $x_2 = u_2^3 = r_2 k_2 a_2$  some  $r_2 \in \mathbb{Z}$  etc.

But then  $x = (r_1 k_1 a_1, r_2 k_2 a_2, \dots, r_i k_i a_i, \dots)$  is clearly in  $Y$ . Thus  $Y$  is closed as required.

We can easily deduce

**Corollary 5.** If  $X$  is a pure subgroup of the Specker group  $P$  such that  $P/X$  is of at most countable rank, then  $X \cong P$  if and only if  $P/X \cong \mathbb{Z}^n$  for some non-negative integer  $n$ .

*Remark.* It is hoped that this paper, along with Lady's paper [3], will illustrate how useful simple topological techniques may be in areas which have hitherto used homological machinery.

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Anschrift des Autors:

B. Goldsmith  
College of Technology  
Kevin Street  
Dublin 8, Ireland  
and  
School of Theoretical Physics  
Dublin Institute for Advanced Studies  
Dublin 4, Ireland