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Endomorphism rings of torsion-free modules over a complete discret valuation ring.

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ENDOMORPHISM RINGS OF TORSION-FREE MODULES OVER A COMPLETE DISCRETE VALUATION RING

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Introduction

Our object in this paper is to characterize all the rings which can occur as endomorphism rings of reduced torsion-free modules over a complete discrete valuation ring. A complete characterization has been given by Liebert [7] but, with a view to possible applications, we approach the problem from a module theoretic position making use of the concept of a basic submodule. In particular we shall show how all endomorphism rings of such modules can be obtained as extensions of an ideal which is determined by a single basic submodule (*cf.* Pierce's characterization of endomorphism rings of abelian p-groups [9]). We remark that, like Liebert in [7], we also require a certain topological property to achieve the desired characterization.

We close this introduction by establishing some terminology and notation. Throughout the paper R will denote a complete discrete valuation ring, i.e. a commutative principal ideal domain with unique prime element p. If X is any torsion-free R-module, then X may be topologized by taking the submodules $p^nX (n = 0, 1, 2, \ldots)$ as a basis of neighbourhoods of zero. This topology is Hausdorff precisely if $\bigcap p^nX = 0$; since X is torsion-free this is equivalent to X being reduced. The completion of a reduced torsion-free R-module X in this p-adic topology will be denoted by $\hat{X}$.

1. *Some properties of the endomorphism ring of a reduced torsion-free R-module*

In this section we present, without proof, some well-known properties of modules over a complete discrete valuation ring R and their endomorphism rings. The proofs may be found in any of the standard texts, e.g. Fuchs [3], [4] or Kaplansky [6].

LEMMA 1.1. *If G is a reduced torsion-free R-module with a basic submodule B then*

(i) *G is a pure submodule of the p-adic completion $\hat{B}$ of this basic submodule*;

(ii) *any endomorphism of G extends uniquely to an endomorphism of $\hat{B}$.*

In view of Lemma 1.1 we may, and do, regard endomorphisms of a reduced torsion-free R-module G as endomorphisms of the completion of any fixed basic submodule B of G.

If $E(\hat{B})$ denotes the ring of R-endomorphisms of $\hat{B}$ then we define

$$I(B) = \{\phi \in E(\hat{B}) | \hat{B}\phi \leqslant B\}.$$

It is clear that $I(B)$ is a left ideal of $E(\hat{B})$; it will play a crucial role in our characterization of endomorphism rings of reduced torsion-free R-modules.

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Recall that Szele's finite topology [11] on the endomorphism ring of a R-module X has as a subbase of neighbourhoods of zero the left ideals

$$U_x = \{\phi \in E(X) | x\phi = 0\}$$

for all $x \in X$.

LEMMA 1.2. *If G is a torsion-free R-module with a basic submodule B then*

(i) $I(B) \leqslant E(G)$;

(ii) *$E(G)$ is a p-pure subalgebra of $E(\hat{B})$;*

(iii) *$E(G)$ is complete in its finite topology.*

We remark that since endomorphic images of complete R-modules are complete and a basic submodule B only contains complete modules of finite rank, it is clear that every endomorphism in $I(B)$ has finite rank. However if $G \neq B$, there will be endomorphisms of G with finite rank which are not in $I(B)$ and so $I(B)$ is not the socle of $E(G)$ in this case [*cf.* Liebert [7]). We conclude this section by recalling that any torsion-free R-module G is both fully transitive and separable.

2. *A Galois connection*

Let B be a free R-module of infinite rank and let Σ be a p-pure unital sub-R-algebra of $E(\hat{B})$ which contains $I(B)$. Set $G(\Sigma) = \{b\phi | \phi \in \Sigma, b \in B\}$.

LEMMA 2.1. *If $x \in G(\Sigma)$ then there exists ψ in Σ and b in B such that $\hat{B} = \langle b \rangle \oplus K$ and $b\psi = x$, $K\psi = 0$.*

Proof. Choose any element b of p-height zero in B. Clearly $\hat{B} = \langle b \rangle \oplus K$ for some K. Now if $x = a\phi$, $a \in B$, $\phi \in \Sigma$ let δ be the endomorphism of $\hat{B}$ sending b to a and annihilating K. Clearly $\delta \in I(B)$. Set $\psi = \delta\phi$, then ψ is in Σ since Σ is a subring and ψ has the required properties.

LEMMA 2.2. *The module $G(\Sigma)$ is a p-pure submodule of B containing B.*

Proof. We show firstly that $G(\Sigma)$ is a submodule. Let z, w be in $G(\Sigma)$ and suppose $z = b_1\phi_1$, $w = b_2\phi_2$ where $\phi_i \in \Sigma$ and $b_i \in B$ $(i = 1, 2)$. Since $\hat{B}$ is fully transitive and separable we can find an endomorphism δ of $\hat{B}$ with $b_1\delta = b_2$ or $b_2\delta = b_1$ such that δ is in $I(B)$. Say $b_1\delta = b_2$. Then $w = b_1\delta\phi_2$ and since Σ is a subalgebra $\delta\phi_2$ is in Σ. Hence $z - w = b_1(\phi_1 - \delta\phi_2) \in G(\Sigma)$. If $r \in R$ then for any x in $G(\Sigma)$ it is clear that xr is in $G(\Sigma)$. Thus $G(\Sigma)$ is a submodule of $\hat{B}$ and it clearly contains B.

Finally, to show that $G(\Sigma)$ is p-pure, suppose $y \in G(\Sigma) \cap p^k\hat{B}$. Let $y = p^kx$. Now applying Lemma 2.1 to y we get $y = b\psi = p^kx$ for some $b \in B$, $\psi \in \Sigma$. Define an endomorphism ζ of $\hat{B}$ be setting $b\zeta = x$, $K\zeta = 0$. Clearly $\psi = p^k\zeta \in \Sigma \cap p^kE(\hat{B}) = p^k\Sigma$. It follows by torsion-freeness that $\zeta \in \Sigma$, so that $x \in G(\Sigma)$. Thus $y \in p^kG(\Sigma)$ and this completes the proof.

3. *The first characterization*

Before proceeding to the characterization we introduce a further topological concept. For an arbitrary ring Σ, let $\Phi(\Sigma)$ denote the set of minimal idempotents of Σ. If $N_\pi = \{\eta \in \Sigma | \pi\eta = 0\}$ then we can define a topology on Σ by taking as a basis of neighbourhoods of zero, the sets N_π $(\pi \in \Phi(\Sigma))$. For an arbitrary ring there is no reason to suppose that this definition would yield a topology. However, in the case where $I(B) \leqslant \Sigma \leqslant E(\hat{B})$, where B is a free R-module of infinite rank, we do get a basis for a topology τ on Σ. (See Fuchs [4; pp. 221–222]). Since a reduced torsion-free R-module G is separable we deduce from Fuchs [4; Theorem 107.2] that $E(G)$ is complete in the τ-topology and that τ coincides with the Szele finite topology. Thus we may simply refer to τ as the finite topology.

We are now in a position to prove the main part of our characterization.

THEOREM 3.1. *Let B be a free R-module of infinite rank and let Σ be a p-pure subalgebra of $E(\hat{B})$ which contains $I(B)$. Then, if Σ is complete in its finite topology, $\Sigma = E(G(\Sigma))$.*

Proof. Since $G(\Sigma)$ is clearly a faithful right Σ-module we can identify $\Sigma \leqslant E(G(\Sigma))$. Let $\zeta \in E(G(\Sigma))$ and let $\{\pi_i | i \in D\}$ be the set of finite idempotents $\Phi(\Sigma)$ of Σ, ordered so that $i \leqslant j$ if and only if $\operatorname{Im}\pi_i \leqslant \operatorname{Im}\pi_j$. Our aim is to construct a net $\{\zeta_i\}$ $(i \in D)$ in Σ such that $\pi_i(\zeta - \zeta_i) = 0$ for then the net $\{\zeta_i\}$ is Cauchy with limit ζ. Hence $\zeta \in \Sigma$. So it suffices to construct a net with the above properties.

Now if π_i is a finite idempotent in Σ then $\hat{B}\pi_i = G_i$ is a finite rank direct summand of $\hat{B}$ which is contained in $G(\Sigma)$. Let $G_i = \langle g_1 \rangle \oplus \ldots \oplus \langle g_k \rangle$. Now for each $j = 1, 2, \ldots k$, $g_j\zeta \in G(\Sigma)$ and so $g_j\zeta = b_j\phi_j$ for some $b_j \in B$, $\phi_j \in \Sigma$. Define $\chi_j : \hat{B} \to \hat{B}$ by $g_j\chi_j = b_j$ and χ_j annihilates the complement of $\langle g_j \rangle$. Clearly χ_j is in $I(B)$ for each j. Set

$$\xi = \sum_{j=1}^{k} \chi_j \phi_j;$$

ξ is in Σ and if we define $\zeta_i = \pi_i\xi$ then ζ_i is also an element of Σ. But then

$$\hat{B}\pi_i(\zeta - \zeta_i) = G_i(\zeta - \zeta_i) = G_i(\zeta - \xi) = 0.$$

Thus $\pi_i(\zeta - \zeta_i) = 0$. Hence the net $\{\zeta_i\}(i \in D)$ has the required property and so the result follows.

Combining Lemma 1.2 and the above result we obtain our first characterization:

THEOREM 3.2. *A ring Σ is the endomorphism ring of a reduced torsion-free R-module if and only if there exists a free R-module B such that Σ is isomorphic to a p-pure subring of $E(\hat{B})$ containing $I(B)$ and Σ is complete in its finite topology.*

4. *A ring-theoretic characterization*

From the viewpoint of constructing examples the condition in Theorem 3.1 that $I(B) \leqslant \Sigma \leqslant E(\hat{B})$ is reasonably satisfactory but from the viewpoint of abstract ring theory the left ideal $I(B)$ is totally unnatural. We can however give ring-theoretic conditions which ensure that we are working only with subrings of the endomorphism ring of a complete module $\hat{B}$ which contain $I(B)$.

For this section we shall suppose simply that E is a unital associative ring.

LEMMA 4.1. *If E is a ring with idempotents e and f such that $eE \cong fE$ as right E-modules, then eEf is a free eEe-module on one generator. Moreover $eEfE = eE$.*

Proof. See Liebert [8; Lemma 2.3].

LEMMA 4.2. *Let E be a ring with a minimal idempotent e such that*

(i) *for any minimal idempotent f of E, fEf is a complete discrete valuation ring;*

(ii) *if f and g are any two minimal idempotents then* $fE \cong gE$ *as right E-modules;*

(iii) *E contains a set of minimal orthogonal idempotents* $\{e_j | j \in J\}$ *and a set of nilpotent elements* $\{\varepsilon_{jk} | j, k \in J, j \neq k\}$ *such that*

(a) $e_i \varepsilon_{jk} = \delta_{ij} e_k$ *(δ_{ij} a Kronecker delta),*

(b) $eE \leqslant \left(\bigoplus_{j \in J} eEe_j\right)^{\wedge}$, *the completion being with respect to the p-adic topology where p is the generator of the Jacobson radical of eEe;*

(iv) $\bigcap_{j \in J} \operatorname{ann}_r(e_j) = 0$ *where* $\operatorname{ann}_r(e_j) = \{f \in E | e_j f = 0\}$.

Then there exists a free R-module F over a complete discrete valuation ring R such that $I(F) \leqslant E \leqslant E(\hat{F})$ *where* $\hat{F}$ *is the p-adic completion of the R-module F.*

Proof. Let $R = eEe$ then by (i) R is a complete discrete valuation ring. Set $F = \bigoplus_{j \in J} eEe_j$ Clearly F is a free R-module since by Lemma 4.1 and (ii) eEe_j is a free eEe-module of rank 1. Let $\hat{F}$ be the p-adic completion of F. Condition (iii) (b) shows us that E acts as a set of homomorphisms from F to $\hat{F}$ for by Lemma 4.1 $eEe_jE = eE$. Condition (iv) ensures that this action is faithful and so we may regard E as a subring of $E(\hat{F})$.

Now if $\phi \in I(F)$ then ϕ has finite rank and so there is a finite set $e_1, \ldots, e_k$ such that $\phi = e_1\phi + \ldots, + e_k\phi$. Moreover for each $j = 1, \ldots, k$ there is an $n(j)$ in the index set J such that $e_j\phi$ has image contained in $\bigoplus_{k=1}^{n(j)} eEe_k$. Let $N = \max\{n(j) | j = 1, \ldots, k\}$. Then

$$(eEe_j)\phi \in \bigoplus_{k=1}^{N} eEe_k$$

and so $e_j\phi$ is a finite sum of multiples of the nilpotent elements ε_{jk} and e_j. Thus it is an element of E and hence ϕ is in E. Thus $I(F) \leqslant E$ as required. The next result contains the required characterization.

THEOREM 4.3. *A ring E is isomorphic to the endomorphism ring of a reduced torsion-free module over a complete discrete valuation ring if and only if E has a minimal idempotent e and*

(i) *for any minimal idempotent f of E, fEf is a complete discrete valuation ring;*

(ii) *if f and g are any two minimal idempotents then* $fE \cong gE$ *as right E-modules;*

(iii) *E contains a set of minimal idempotents* $\{e_j | j \in J\}$ *and a set of nilpotent elements* $\{\varepsilon_{jk} | j, k \in J, j \neq k\}$ *such that*

(a) $e_i\varepsilon_{jk} = \delta_{ij}e_k$, *($\delta_{ij}$ a Kronecker delta),*

(b) $eE \leqslant \left(\bigoplus_{j \in J} eEe_j\right)^{\wedge}$, *the completion being with respect to the p-adic topology where p is the generator of the Jacobson radical of eEe;*

(iv) *E is complete in its finite topology;*

(v) *if I denotes the ideal of finite endomorphisms of* $\bigoplus_{j \in J} eEe_j$, *then if* $\zeta \in E$ *and* $\zeta I \leqslant p^k I$ *then* $\zeta \in p^k E$ *(p, again, is the generator of the Jacobson radical of eEe).*

Proof. Since completeness of the finite topology includes the Hausdorff condition then (iv) above contains part (iv) of Lemma 4.2. In view of Theorem 3.2 it will suffice to show that condition (v) above is equivalent to the p-purity of E in $E(\hat{F})$. The proof is completed by the following result.

LEMMA 4.4. *If F is a free module over a complete discrete valuation ring and $I(F) \leqslant E \leqslant E(\hat{F})$ then E is p-pure in $E(\hat{F})$ if and only if, for any ζ in E, $\zeta I(F) \leqslant p^k I(F)$ implies $\zeta \in p^k E$.*

Proof. Let $\zeta \in p^k E(\hat{F}) \cap E$, then $\zeta I(F) = p^k \chi I(F)$ for some $\chi \in E(\hat{F})$. Since $I(F)$ is a left ideal of $E(\hat{F})$ we have $\zeta I(F) \leqslant p^k I(F)$ and so by assumption $\zeta \in p^k E$. Thus $p^k E(\hat{F}) \cap E = p^k E$. Hence E is p-pure in $E(\hat{F})$. Conversely suppose E is p-pure and $\zeta I(F) \leqslant p^k I(F)$. Since $I(F)$ contains all the projections of $\hat{F}$ onto summands of F it then follows that $\hat{F}\zeta \leqslant p^k \hat{F}$. If we now define $\psi : \hat{F} \to \hat{F}$ by $x\psi = p^{-k}(x\zeta)$ then the torsion-freeness of $\hat{F}$ ensures that ψ is well-defined. Clearly $p^k \psi = \zeta$. Thus $\zeta = p^k \psi \in E \cap p^k E(\hat{F}) = p^k E$. So $\zeta \in p^k E$.

5. *Further topological considerations*

In Section 4 we developed some ring-theoretic approaches which removed the condition $I(B) \leqslant \Sigma \leqslant E(\hat{B})$ from Theorem 3.2. To achieve this we made use of a p-adic topology and in this section we show how a further use of topology can yield yet another characterization. (The situation is similar to, but simpler than, the corresponding problem for endomorphism rings of p-groups, Pierce [9].)

Again let R denote a complete discrete valuation ring and set, for μ an infinite ordinal, $B = \bigoplus_{j < \mu} Re_j$. Let π_j denote the projection of B onto Re_j and for any $\beta < \mu$ let δ_β denote the projection from B onto $\bigoplus_{j < \beta} Re_j$. If $\hat{B}$ is the p-adic completion of B then π_j and δ_β extend to endomorphisms of $\hat{B}$. As before $I(B) = \{\phi \in E(\hat{B}) | \hat{B}\phi \leqslant B\}$.

LEMMA 5.1. *If $\zeta \in I(B)$ then* $\lim_{\beta \to \mu} \delta_\beta \zeta = \zeta$ *where the limit is taken with respect to the finite topology on $I(B)$.*

Proof. Let q be a finite idempotent in $I(B)$. Then there is a $\gamma < \mu$ such that $q\delta_\gamma = q$. But then $q(\zeta - \delta_\beta \zeta) = q\delta_\beta(\zeta - \delta_\beta \zeta) = 0$ for all $\beta \geqslant \gamma$. Thus ζ is the limit of $\{\delta_\beta \zeta\}$.

LEMMA 5.2. *Let Σ be a ring which contains $I(B)$ as a faithful left ideal. Suppose that the mapping $\phi \to \zeta\phi$, where $\phi \in I(B)$, $\zeta \in \Sigma$ is continuous in the finite topology of $I(B)$ for all $\zeta \in \Sigma$. Then there is a ring isomorphism of Σ into $E(\hat{B})$ which is the identity on $I(B)$.*

Proof. Define a map $\lambda : \Sigma \to E(\hat{B})$ by setting, for $\zeta \in \Sigma$ and $x \in \hat{B}$,

$$x\lambda_\zeta = \sum_{i < \mu} x(\zeta \pi_i).$$

It is easy to check that λ_ζ is a well defined endomorphism of $\hat{B}$. We need to show that λ is a ring isomorphism from Σ into $E(\hat{B})$ which acts as the identity on $I(B)$.

Now if $\zeta \in I(B)$ and $x \in \hat{B}$, then

$$x\lambda_\zeta = \sum_{i<\mu} x(\zeta\pi_i) = \sum_{i<\mu} (x\zeta)\pi_i = x\zeta.$$

Thus λ acts as the identity on $I(B)$. The distributive law in Σ implies that $\lambda_{\zeta-\eta} = \lambda_\zeta - \lambda_\eta$ for any ζ, η in Σ and so to show that λ is a ring homomorphism, we need only show that $\lambda_{\zeta\eta} = \lambda_\zeta\lambda_\eta$. Since B is dense in $\hat{B}$ in the p-adic topology, it will suffice to show $\lambda_{\zeta\eta}$ and $\lambda_\zeta\lambda_\eta$ agree on B.

If $x \in B$, then it follows from Lemma 5.1 that

$$x\eta\pi_j = \lim_{\beta\to\mu} x\delta_\beta(\eta\pi_j) = \lim_{\beta\to\mu} \sum_{i<\beta} x\pi_i(\eta\pi_j).$$

Thus on B,

$$\eta\pi_j = \lim_{\beta\to\mu} \sum_{i<\beta} \pi_i(\eta\pi_j).$$

Since multiplication on the left by elements of Σ is continuous we deduce that on B,

$$\zeta\eta\pi_j = \lim_{\beta\to\mu} \zeta\pi_i\eta\pi_j = \sum_{i<\mu} \zeta\pi_i\eta\pi_j$$

for any $\zeta \in \Sigma$. It is then easy to check that $x\lambda_{\zeta\eta} = x\lambda_\zeta\lambda_\eta$ for any $x \in B$. Hence we conclude that λ is a ring homomorphism.

Finally suppose $\lambda_\zeta = 0$ for some $\zeta \in \Sigma$. Then if $\eta \in I(B)$, $\zeta\eta$ is also in $I(B)$ and so $\zeta\eta = \lambda_{\zeta\eta} = \lambda_\zeta\lambda_\eta = 0$. Since $I(B)$ is a faithful ideal, we deduce that $\zeta = 0$. Thus λ is a ring isomorphism and this completes the proof.

We may combine the above results with the results of §3 to obtain the following characterization.

THEOREM 5.3. *Let R be a complete discrete valuation ring and B a free R-module of infinite rank. Let Σ be a ring which contains $I(B)$ as a faithful left ideal. Assume the following conditions hold:—*

(a) *left multiplication by elements of Σ is a continuous homomorphism of $I(B)$ in the finite topology on $I(B)$;*

(b) *if $\zeta \in \Sigma$ and $\zeta I(B) \leqslant p^k I(B)$ then $\zeta \in p^k\Sigma$;*

(c) *Σ is complete in its finite topology.*

Then there is a reduced R-module G having B as a basic submodule and a ring isomorphism λ of Σ onto $E(G)$ such that λ is the identity on $I(B)$.

Remark. It is clear that $I(B)$ is in fact the ideal of finite endomorphism of the free R-module B and so we could extend Theorem 5.3. to give a more ring-theoretic flavour to the result. This could be achieved by using techniques similar to Liebert [**8**]. We leave the formulation of this result to the reader.

6. *An example*

To conclude this paper we use our first characterization to obtain a module G over a complete discrete valuation ring R such that $E(G)$ is a split extension of the ring R by an ideal consisting only of finite rank endomorphisms. Corresponding results for p-groups have been established by Pierce, Beaumont and Corner [1], [10] and [2]; the present author has obtained results of this type in [5] also.

Suppose B is a free R-module of countably infinite rank and G is a maximal pure submodule of $\hat{B}$ containing B (i.e. the quotient $\hat{B}/G$ is isomorphic to Q, the field of fractions of R). Further suppose G does not contain an isomorphic copy of $\hat{B}$. (The existence of such modules was established in [5].) Let

$$I = I(G) = \{\phi \in E(G) | \hat{B}\hat{\phi} \leqslant G\}$$

where $\hat{\phi}$ denotes the unique extension of ϕ to $\hat{B}$.

Observe that every endomorphism in I has finite rank by the choice of G. We show that $R \oplus I$ is the endomorphism ring of some R-module (indeed of G itself). Clearly $I(B) \leqslant R \oplus I \leqslant E(\hat{B})$ while if $r+\theta = p^k\chi$ for some $\chi \in E(\hat{B})$, $\theta \in I$, $r \in R$, then $p^k | r$ for otherwise θ would be a unit. It follows easily that $\theta \in p^k I$ and so $R \oplus I$ is p-pure in $E(\hat{B})$.

Finally if $\{\zeta_i\}$ is a Cauchy net in $R \oplus I$ we show that it has a limit in $R \oplus I$. Let $\{\pi_j\}$ $(j \in J)$ denote, as usual, the projections of $\hat{B}$ onto the rank one basis elements of B. Then $\pi_j(\zeta_i - \zeta_k) = 0$ for all i, k sufficiently large. Let $\pi_j\zeta^j$ denote the common value of $\pi_j\zeta_k$ for k sufficiently large. Then $\zeta = \sum_{j \in J} \pi_j\zeta^j$ is a well defined element of $E(\hat{B})$. It is clearly the limit of the Cauchy net and so it remains to show that $\zeta \in R \oplus I$.

Now from the maximality of G, $\hat{B}$ is the pure submodule generated by G and x, for some $x \in \hat{B}$. Since ζ is the limit of the net we have that there is a ζ_x in the Cauchy net such that $x(\zeta - \zeta_x) = 0$. Since ζ_x belongs to $R \oplus I$ we have $\zeta_x = r + \theta$ for some $r \in R$, $\theta \in I$. But then $(\zeta - r)$ maps x into G and since G is invariant under $(\zeta - r)$ we must have $\hat{B}(\zeta - r) \leqslant G$. Thus $\zeta \in R \oplus I$.

It follows rather easily that there exist R-modules which are essentially indecomposable, i.e. in any direct decomposition one of the summands has finite rank.

We remark in conclusion that Liebert in [7] handled discrete valuation rings which, unlike ours, were not necessarily commutative. However it is quite easy to make the modifications necessary to extend all the above results to the non-commutative case. We leave this exercise to the reader.

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