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A Note on Elongations of Abelian Groups

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Throughout this note the word group will refer to an additively written abelian group. For notation and any unexplained terminology we refer to Fuchs [2].

If G is a p -group then for each positive integer n , $p^n G$ is a subgroup of G . We set $p^\omega G = \bigcap p^n G$; $p^\omega G$ is the subgroup of elements of infinite p -height. A reduced p -group G is said to be an ω -elongation of A by B if there exist reduced p -groups A and B such that

- (i) $p^\omega G \cong B \neq 0$
- (ii) $G/p^\omega G \cong A$.

We call a pair of p -groups (A, B) uniquely ω -elongating if

- (i) there exists a p -group G such that $p^\omega G \cong B$
- (ii) $B \neq 0$
- (iii) $G/p^\omega G \cong A$
- (iv) any two such groups are isomorphic.

It follows from the work of Crawley [1] and Hill and Megibben [4] that if A is a direct sum of cyclic p -groups then (A, B) is uniquely ω -elongating for any B for which ω -elongations exist.

The converse problem, viz., if (A, B) is a uniquely ω -elongating pair of p -groups, is A a direct sum of cyclic groups? , is a long standing problem. We consider here a particular case of this problem.

§1. Definitions and preliminary results.

DEFINITION: For an abelian p -group G we define the injective corank of G to be the rank of D/G where D is a minimal divisible group containing G . (Note this rank is independent of the choice of D by a result of Kulikov [6].)

DEFINITION: If H is a subgroup of the p -group G then H is said to be $p^\omega G$ -high (or simply high) if H is maximal subject to being a subgroup disjoint from $p^\omega G$.

DEFINITION: If G is a p -group such that all $p^\omega G$ -high subgroups are direct sums of cyclic groups then G is said to be a Σ -group.

Proposition 1.1. (Irwin, Peercy and Walker [5]): If one high subgroup of G is a direct sum of cyclic groups, then all high subgroups are isomorphic and G is a Σ -group.

Proposition 1.2. (Megibben [7]): If D is a minimal divisible group containing K and H is a group without elements of infinite height which has a pure subgroup X such that $H/X \cong D/K$, then there exists a group G such that

- (i) $G' = K$
- (ii) $G/G' \cong H$
- (iii) X is a high subgroup of G , where G' is the subgroup of elements of infinite height in G .

The next result gives necessary and sufficient conditions for ω -elongations to exist.

Theorem 1.3. (Pierce [8]): Let A and B be reduced abelian p -groups such that $p^\omega A = 0$. Then there is a group G such that $p^\omega G \cong B$, $G/p^\omega G \cong A$ if and only if

$$\dim (p^n A[p]) \geq \dim (B/pB) \quad \text{for all } n < \omega .$$

§2. Elongations of p -groups.

Theorem 2.1: Let A and B be reduced p -groups such that ω -elongations of A by B exist. Then, if the final rank of A equals the injective corank of B and A is not a direct sum of cyclic groups, the pair (A, B) is not uniquely ω -elongating.

Proof:

Since A is not a direct sum of cyclic groups it contains a pure subgroup X such that X is pure, dense, the rank of A/X equals the final rank of A and X is not a direct sum of cyclic groups. (If not by a result of Hill [3], A would itself be a direct sum of cyclic groups; cf. Warfield [9].)

If C is a lower basic subgroup of A then $\text{rk. } (A/C) = \text{final rk. } A = \text{rk. } (D/B)$ where D is a minimal divisible group containing B . Then by Prop. 1.2. there exists a p -group G with $p^\omega G \cong B$, $G/p^\omega G \cong A$ and C is high in G . Since C is a direct sum of cyclic groups, it follows from Prop. 1.1. that G is a Σ -group.

However $\text{rk. } (A/X) = \text{final rk. } A = \text{rk. } (D/B)$ and so by Prop. 1.2. there exists a p-group H with $p^\omega H \cong B$, $H/p^\omega H \cong A$ and X is high in H. Since X is not a direct sum of cyclic groups it is clear from Prop. 1.1. that H is not a Σ -group. Thus $H \not\cong G$, i.e. the pair (A, B) is not uniquely ω -elongating.

Corollary 2.2.: If A is not a direct sum of cyclic groups and ω -elongations of A by B exist then there is a cardinal m such that ω -elongations of A by $\bigoplus_m B$ exist but these are not unique.

Proof:

By Th. 1.3. $\text{Dim } B/pB \leq \text{final rk. } A$. Then if m equals the final rank of A we have $\text{Dim } \left(\bigoplus_m B / p \left(\bigoplus_m B \right) \right) = \text{final rk. } A$. So by Th. 1.3. ω -elongations of A by $\bigoplus_m B$ exist. Moreover injective corank $\bigoplus_m B = \text{final rk. } A$ and so the result follows from 2.1.

We remark in conclusion that the above may be easily generalised to elongations of modules over a complete discrete valuation ring.

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